The Funk-Radon transform and spherical tomography

Michael Quellmalz

Chemnitz University of Technology
Faculty of Mathematics

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Content

Funk–Radon transform

[Funk 1911]

- Sphere $\mathbb{S}^{d-1} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$
- Funk–Radon transform

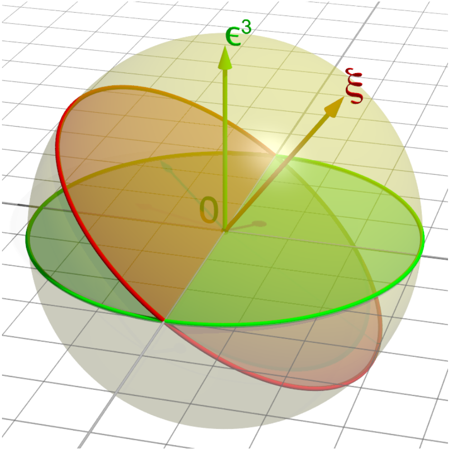
$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}\boldsymbol{\eta}\rangle = 0} f(\boldsymbol{\eta})\,\mathrm{d}\lambda(\boldsymbol{\eta})$$

(integrals of f along all great circles)

Goal

Reconstruct the function f from the integrals $\mathcal{F}f$

- Solved for even functions $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$



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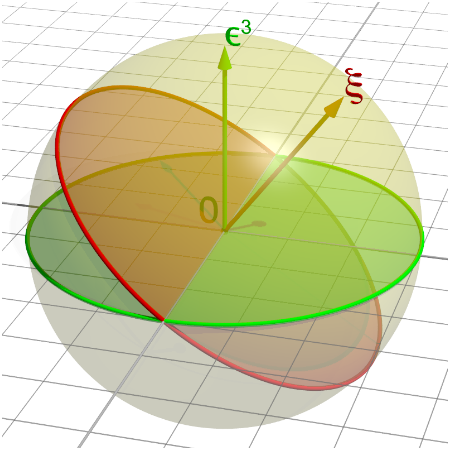

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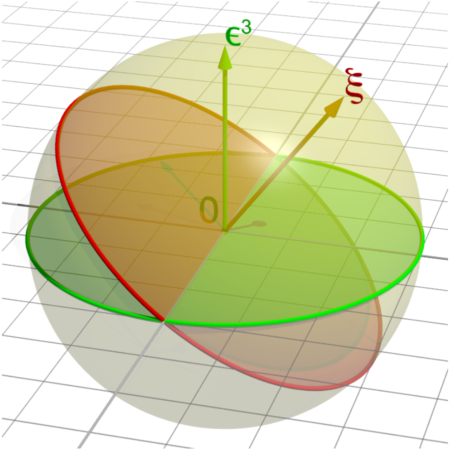

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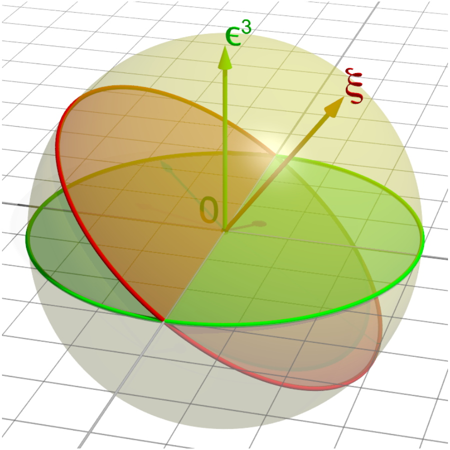

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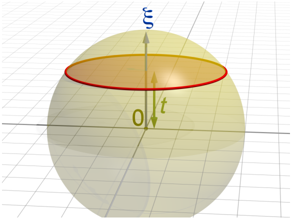
Circular means on the sphere

- $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$
- **Mean operator** integrates f along all hyperplane sections:

$$\mathcal{M}f(\boldsymbol{\xi},t) = \int_{\langle \boldsymbol{\xi},\boldsymbol{\eta}\rangle = t} f(\boldsymbol{\eta})\,\mathrm{d}\lambda(\boldsymbol{\eta}), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}, t \in [-1,1]$$



- The inversion of $\mathcal{M}$ is overdetermined
  e.g. $\mathcal{M}f(\xi,1) = f(\xi)$
- Reconstruct f knowing $\mathcal{M}f$ on a submanifold of $\mathbb{S}^{d-1} \times [-1,1]$

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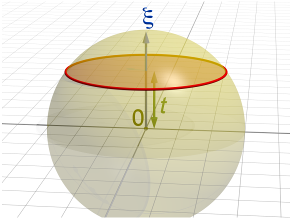

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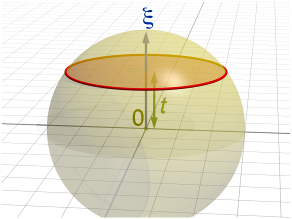

Singular value decomposition [Berens, Butzer & Pawelke 1961]

- Y_n^k spherical harmonic of degree n
- $P_{n,d}$ Legendre (ultraspherical) polynomial of degree n in dimension d, orthogonal polynomial on $[-1, 1]$ w.r.t. the weight $(1 - t^2)^{\frac{d-3}{2}}$

Then

$$\mathcal{M}Y_n^k(\boldsymbol{\xi}, t) = Y_n^k(\boldsymbol{\xi}) P_{n,d}(t).$$

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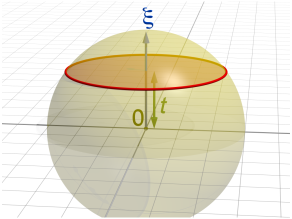

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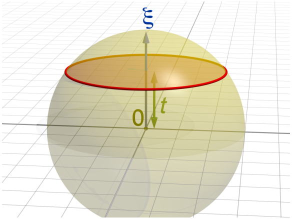

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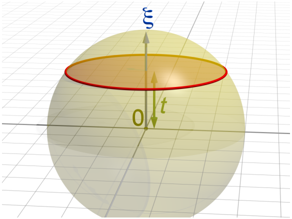

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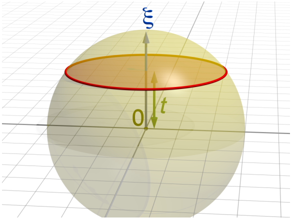

Theorem "John's equation" [Hielscher, Q.]

Let $f \in C^2(\mathbb{S}^{d-1})$. Denote by $\Delta^\bullet_{\boldsymbol{\xi}}$ the Laplace–Beltrami operator w.r.t. $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. Then, for $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $t \in (-1,1)$, the mean operator $\mathcal{M}f$ satisfies

$$\Delta^\bullet_{\boldsymbol{\xi}} \mathcal{M}f(\boldsymbol{\xi},t) = \left((1-t^2)\,\frac{\partial^2}{\partial t^2} - (d-1)\,t\,\frac{\partial}{\partial t} \right) \mathcal{M}f(\boldsymbol{\xi},t).$$

Sobolev spaces

- Sobolev space $H^s(\mathbb{S}^{d-1})$ of order $s \ge 0$ comprises functions $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ with norm

$$\|f\|^2_{H^s(\mathbb{S}^{d-1})} = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\langle f, Y_n^k\rangle\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s}$$

- Sobolev norm in $H^{s,t}(\mathbb{S}^{d-1} \times [-1,1])$ for $s,t \ge 0$

$$\|g\|^2_{H^{s,t}(\mathbb{S}^{d-1}\times[-1,1])} = \sum_{n,l=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\langle g, Y_n^k P_{l,d}\rangle\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s} \left(l + \tfrac{d-2}{2}\right)^{2l}$$

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Injectivity sets of $\mathcal{M}$

Theorem [Hielscher, Q.]

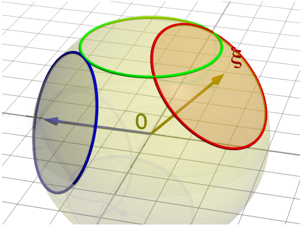
Let $D \subset \mathbb{S}^{d-1} \times [-1,1]$, $g_0 \colon D \to \mathbb{C}$, and let $s > \frac{d-1}{2}$. The following are equivalent:

1. The problem
$$\mathcal{M}\big|_D f = g_0$$
has a unique solution $f \in H^s(\mathbb{S}^{d-1})$.
2. The John-type differential equation
$$\Delta_{\boldsymbol{\xi}}^{\bullet} g(\boldsymbol{\xi}, t) = \left((1-t^2)\,\frac{\partial^2}{\partial t^2} - (d-1)\,t\,\frac{\partial}{\partial t} \right) g(\boldsymbol{\xi}, t).$$
with boundary condition $g\big|_D = g_0$ has a unique solution
$$g \in H^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d).$$

Circles with fixed radius

For fixed $z \in [-1, 1]$, compute

$$\mathcal{T}_z f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z} f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}$$



Eigenvalue decomposition

$$\mathcal{T}_z Y_n^k = P_{n,d}(z) \, Y_n^k$$

"Freak theorem" [Schneider 1969]

The set of values z for which $\mathcal{T}_z$ is **not** injective is countable and dense in $[-1, 1]$.

This is because $\mathcal{T}_z$ is injective if and only if $P_{n,d}(z) = 0 \; \forall n \in \mathbb{N}_0$.

Explicit algorithm to determine if $\mathcal{T}_z$ is injective for given z [Rubin 2000]

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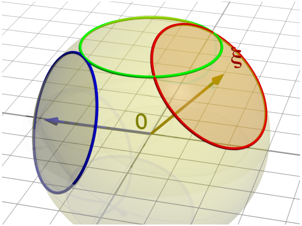

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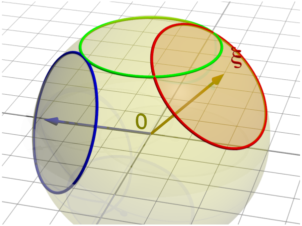

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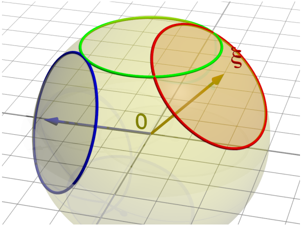

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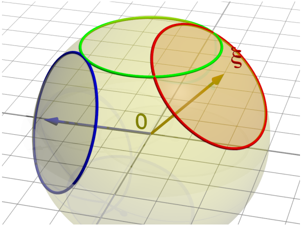

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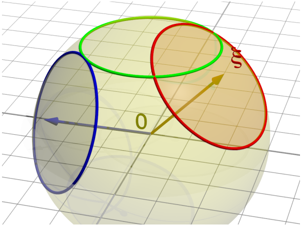

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Vertical slices

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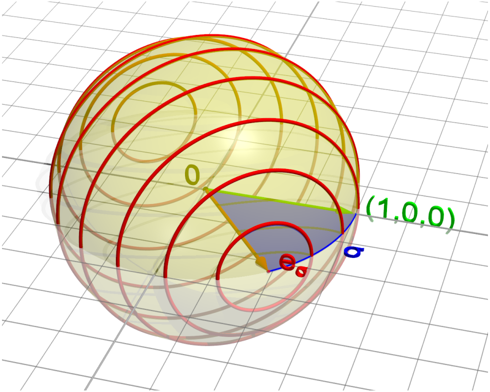

- **Circles perpendicular to the equator**
- Injective for symmetric functions $f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3)$
- Orthogonal projection onto equatorial plane ↗ Radon transform in $\mathbb{R}^2$ [Gindikin, Reeds & Shepp 1994]
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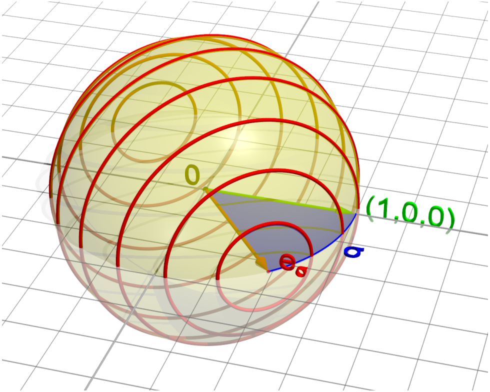

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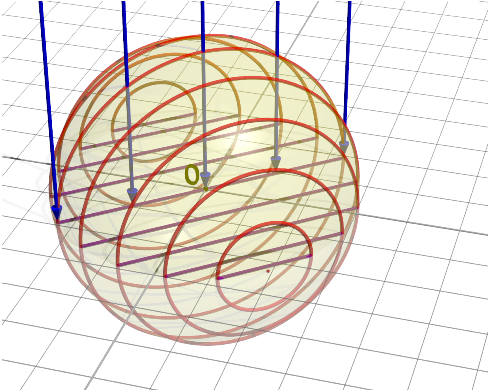

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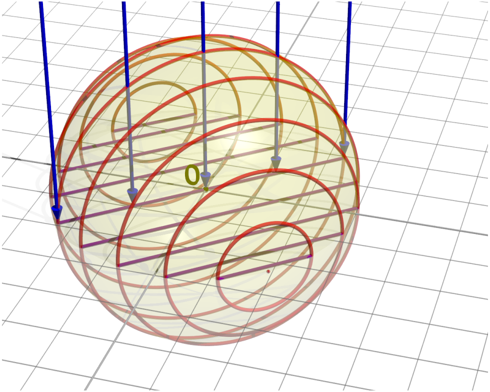

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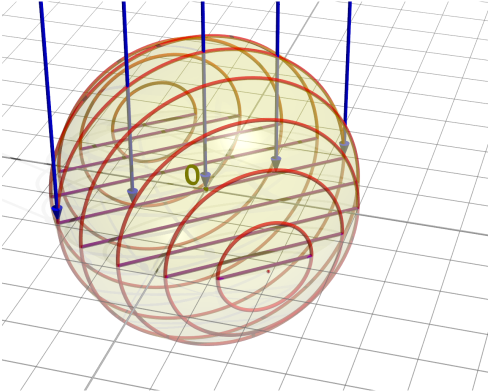

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Planes through a fixed point

[Salman 2016]

Consider an arbitrary point inside the sphere:

$$(0,\dots,0,z), \qquad 0 \le z < 1$$

Plane section through $(0,\dots,0,z)$ is

$$\{\eta \in \mathbb{S}^{d-1} : \langle \xi, \eta \rangle = z\xi_d\}.$$

Definition

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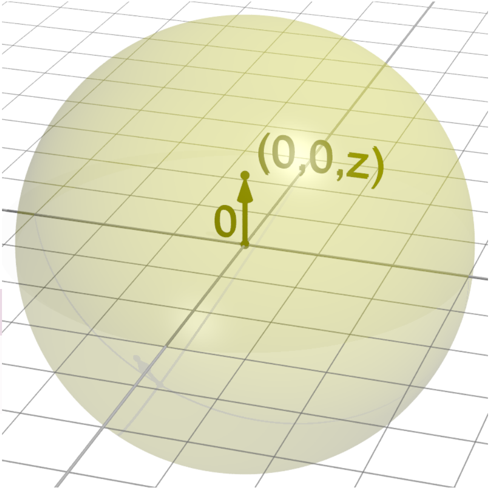
$z = 0$: Funk–Radon transform



Planes through a fixed point

[Salman 2016]

Consider an arbitrary point inside the sphere:

$$(0, \dots, 0, z), \qquad 0 \le z < 1$$

Plane section through $(0, \dots, 0, z)$ is

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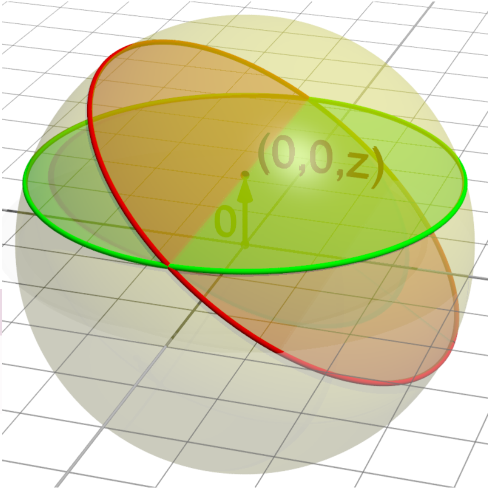

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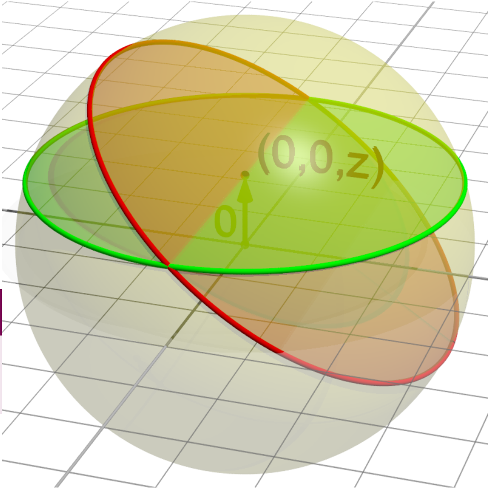

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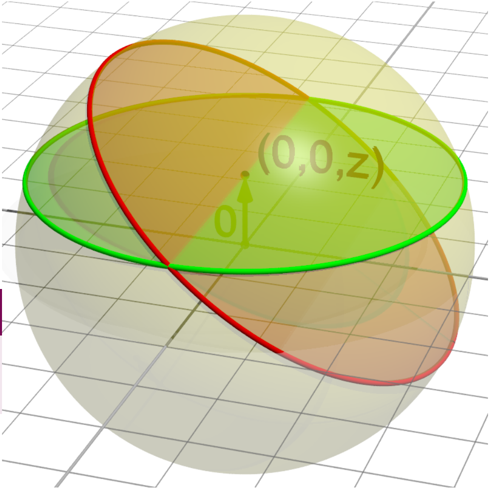

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Nullspace of $\mathcal{U}_z$ [Q. 2018]

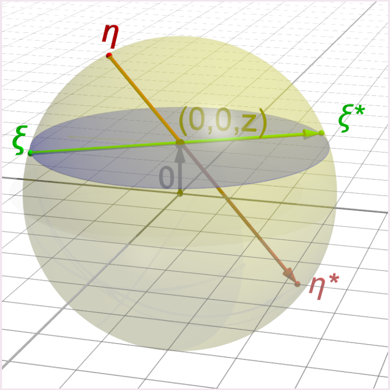
For $\xi \in \mathbb{S}^{d-1}$, we define $\xi^* \in \mathbb{S}^{d-1}$ as the point reflection of the sphere about the point $(0, \dots, 0, z)$.

Let $f \in L^2(\mathbb{S}^{d-1})$. Then

$$\mathcal{U}_z f = 0$$

if and only if for almost every $\xi \in \mathbb{S}^{d-1}$

$$f(\xi) = -\frac{1 - z^2}{1 + z^2 - 2z\eta_d} f(\xi^*).$$



Reconstruction is unique for two center points [Agranovsky & Rubin 2019]

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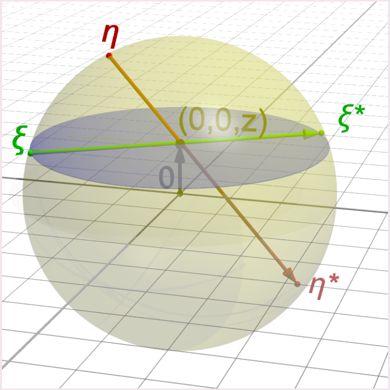

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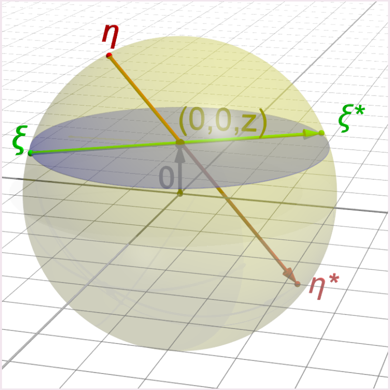

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Circles through the north pole

[Abouelaz & Daher 1993]

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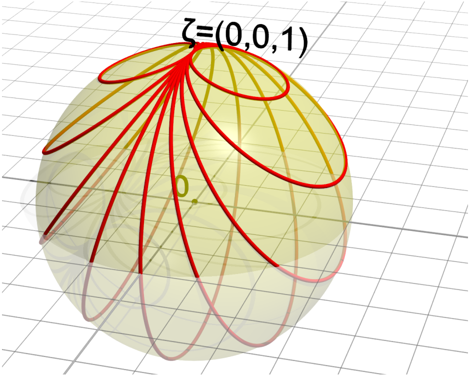

- Stereographic projection turns circles into lines in the plane
  ↗ Radon transform in equatorial plane $\mathbb{R}^{d-1}$
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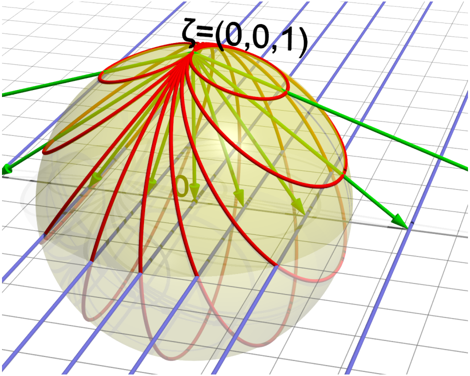

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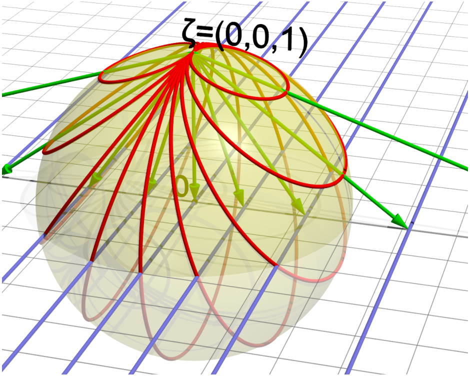

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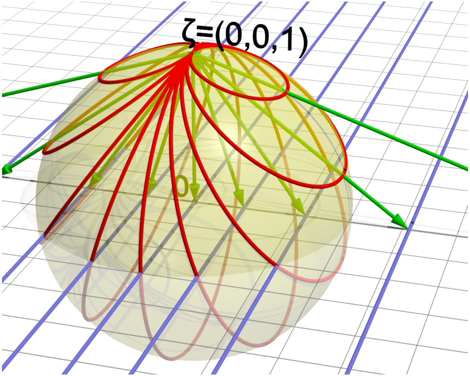

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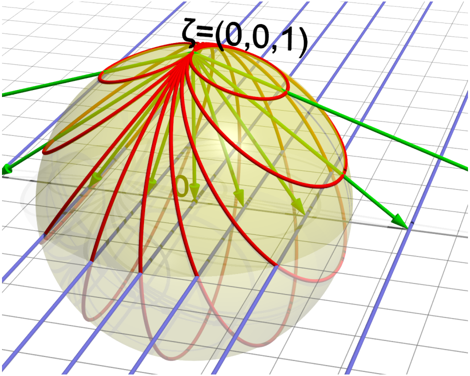

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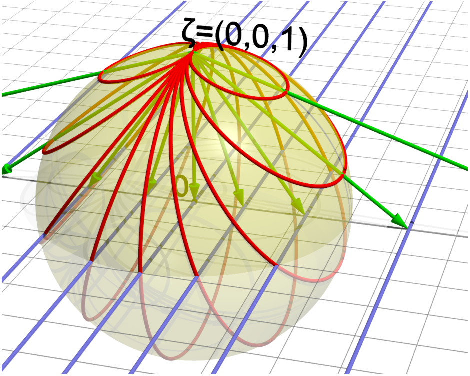

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mean operator	$\mathcal{M}f(\boldsymbol{\xi},t)$	✓	$\subset H^{d/2-1,0}_{\mathrm{mix}}$	✓
Funk–Radon transform	$\mathcal{M}f(\boldsymbol{\xi},0)$	$f(\boldsymbol{\xi})=f(-\boldsymbol{\xi})$	$=H^{\frac{d-2}{2}}_{\mathrm{even}}$	✓
spherical section transform	$\mathcal{M}f(\boldsymbol{\xi},z)$, $z\in[-1,1]$ fixed	✓ if $P_{n,d}(z)\neq 0$ $\forall n\in\mathbb{N}_0$	$\subset H^{\frac{d-2}{2}}$	✓
vertical slice transform	$\mathcal{M}f((\begin{smallmatrix}\boldsymbol{\sigma}\\0\end{smallmatrix}),t)$, $\boldsymbol{\sigma}\in\mathbb{S}^{d-2}$	$f(\boldsymbol{\xi}',\xi_d)=f(\boldsymbol{\xi}',-\xi_d)$	$\subset H^{0,\frac{d-2}{2}-\frac{1}{4}}_{\mathrm{mix}}$	✓
sections through fixed point	$\mathcal{M}f(\boldsymbol{\xi},z\xi_d)$, $z\in(-1,1)$ fixed	f even w.r.t. some reflection in $z\boldsymbol{\epsilon}^d$	$=\widetilde{H}_z^{\frac{d-2}{2}}$	✗
spherical slice transform	$\mathcal{M}f(\boldsymbol{\xi},\xi_d)$	✓ for $f\in L^\infty(\mathbb{S}^{d-1})$		✗

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Thank you for your attention