Reconstructing Functions on the Sphere from Circular Means

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Content

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Funk–Radon transform

[Funk 1911]

- Sphere $\mathbb{S}^{d-1} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$
- **Funk–Radon transform**

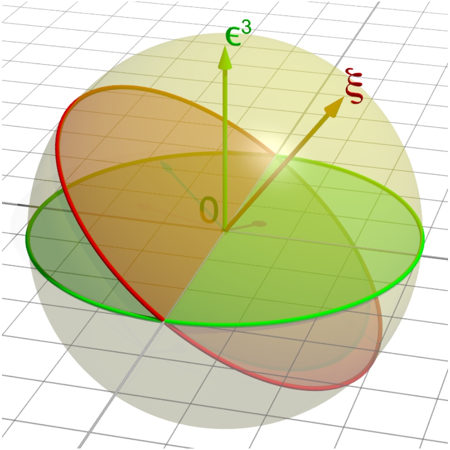
$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0} f(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta})$$

(integrals of f along all great circles)

Goal

Reconstruct the function f from integrals $\mathcal{F}f$

- Possible for even functions $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$



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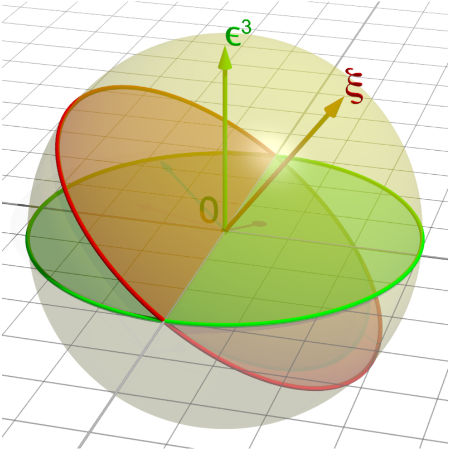

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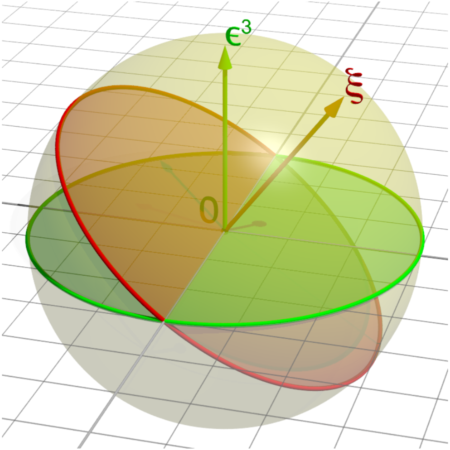

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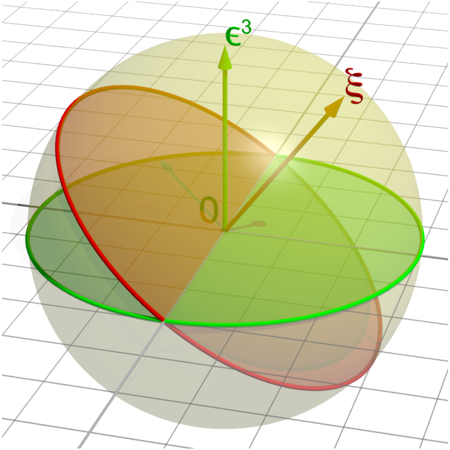

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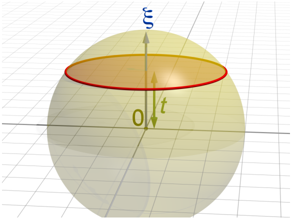

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$$\mathcal{M}f(\boldsymbol{\xi}, t) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = t} f(\boldsymbol{\eta}) \,\mathrm{d}\lambda(\boldsymbol{\eta}), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}, t \in [-1, 1]$$

- integral $\mathrm{d}\lambda$ is normalized to one
- The inversion of $\mathcal{M}$ is overdetermined
  e.g. $\mathcal{M}f(\xi, 1) = f(\xi)$
- Reconstruct f knowing $\mathcal{M}f$ on a submanifold of $\mathbb{S}^{d-1} \times [-1, 1]$

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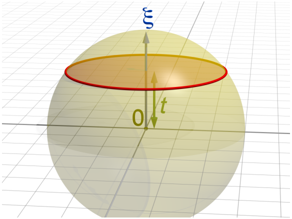

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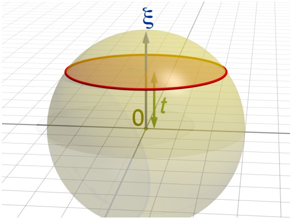

Singular value decomposition [Berens, Butzer & Pawelke 1961]

- Y_n^k spherical harmonic of degree n
- $P_{n,d}$ Legendre (ultraspherical) polynomial of degree n in dimension d, orthogonal polynomial on $[-1, 1]$ w.r.t. the weight $(1 - t^2)^{\frac{d-3}{2}}$

Then

$$\mathcal{M}Y_n^k(\boldsymbol{\xi}, t) = Y_n^k(\boldsymbol{\xi}) P_{n,d}(t).$$

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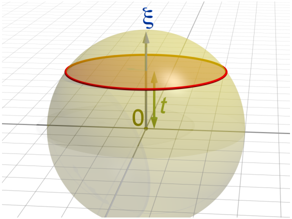

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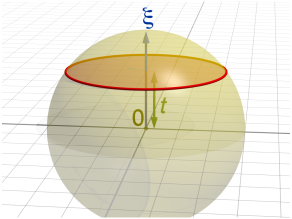

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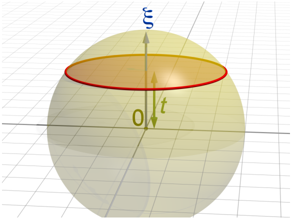

Theorem “Euler–Poisson–Darboux equation”

Let $f \in C^2(\mathbb{S}^{d-1})$. Denote by $\Delta_{\boldsymbol{\xi}}^{\bullet}$ the Laplace–Beltrami operator w.r.t. $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. Then, for $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $t \in (-1, 1)$, the mean operator $\mathcal{M}f$ satisfies

$$\Delta_{\boldsymbol{\xi}}^{\bullet} \mathcal{M}f(\boldsymbol{\xi}, t) = \left((1 - t^2) \frac{\partial^2}{\partial t^2} - (d - 1)\, t \frac{\partial}{\partial t} \right) \mathcal{M}f(\boldsymbol{\xi}, t).$$

Sobolev spaces

- Sobolev space $H^s(\mathbb{S}^{d-1})$ with smoothness index $s \in \mathbb{R}$ is the completion of the space of smooth functions $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ with the norm

$$\|f\|^2_{H^s(\mathbb{S}^{d-1})} = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\langle f, Y_n^k \rangle\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s}$$

- Sobolev norm in $H^{s,r}(\mathbb{S}^{d-1} \times [-1,1])$ for $s, r \in \mathbb{R}$

$$\|g\|^2_{H^{s,r}(\mathbb{S}^{d-1} \times [-1,1]; w_d)} = \sum_{n,l=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\langle g, Y_n^k \widetilde{P}_{l,d} \rangle\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s} \left(l + \tfrac{d-2}{2}\right)^{2r}$$

$Y_n^k(\xi)\, \widetilde{P}_{l,d}(t)$ form orthonormal basis in $L^2(\mathbb{S}^{d-1} \times [-1,1]; w_d)$ with weight

$$w_d(\xi, t) = (1 - t^2)^{\frac{d-3}{2}}$$

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Sobolev estimate of $\mathcal{M}$

Theorem

Let $s \in \mathbb{R}$. The mean operator $\mathcal{M}$ on the sphere $\mathbb{S}^{d-1}$ extends to a bounded linear operator

$$\mathcal{M} \colon H^s(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d).$$

Injectivity sets of the mean operator $\mathcal{M}$

Theorem [Hielscher, Q.]

Let $D \subset \mathbb{S}^{d-1} \times [-1,1]$, $g_0 \colon D \to \mathbb{C}$, and let $s > \frac{d-1}{2}$. The following are equivalent:

1. The problem
$$\mathcal{M}\big|_D f = g_0$$
has a unique solution $f \in H^s(\mathbb{S}^{d-1})$.
2. The Euler–Poisson–Darboux differential equation
$$\Delta^\bullet_{\boldsymbol{\xi}} g(\boldsymbol{\xi}, t) = \left((1-t^2) \frac{\partial^2}{\partial t^2} - (d-1)\, t \, \frac{\partial}{\partial t} \right) g(\boldsymbol{\xi}, t).$$
with boundary condition $g\big|_D = g_0$ has a unique solution
$$g \in H^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d).$$

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Circles with fixed radius

For fixed $z \in [-1, 1]$, compute

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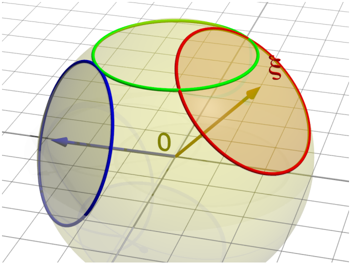

Eigenvalue decomposition

$$\mathcal{T}_z Y_n^k = P_{n,d}(z) \, Y_n^k$$

"Freak theorem" [Schneider 1969]

The set of values z for which $\mathcal{T}_z$ is **not** injective is countable and dense in $[-1, 1]$.

This is because $\mathcal{T}_z$ is injective if and only if $P_{n,d}(z) = 0 \, \forall n \in \mathbb{N}_0$.

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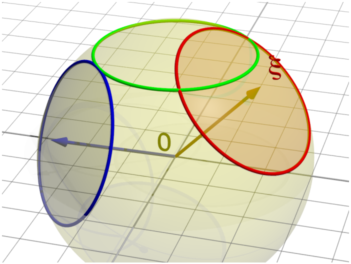

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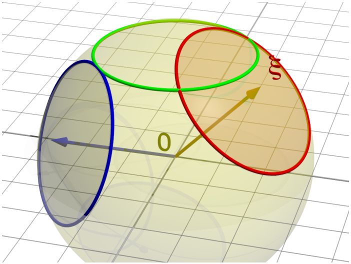

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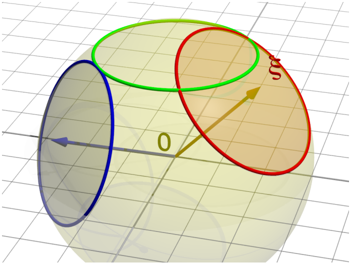

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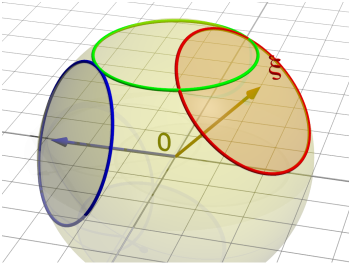

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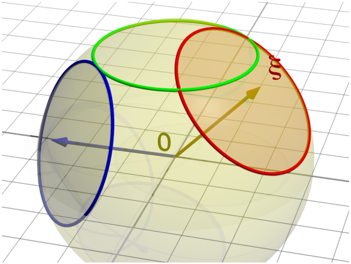

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Vertical slices

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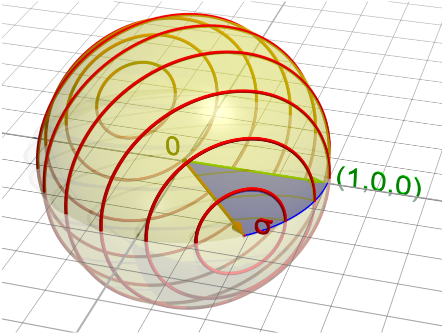

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- Injective for symmetric functions $f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3)$
- Orthogonal projection onto equatorial plane ↗ Radon transform in $\mathbb{R}^2$
  [Gindikin, Reeds & Shepp 1994]
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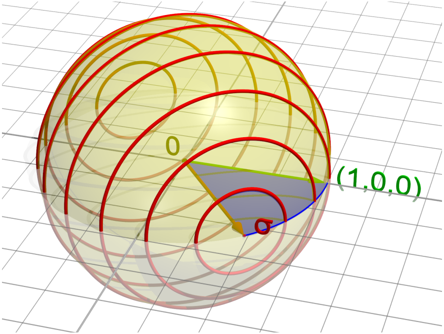

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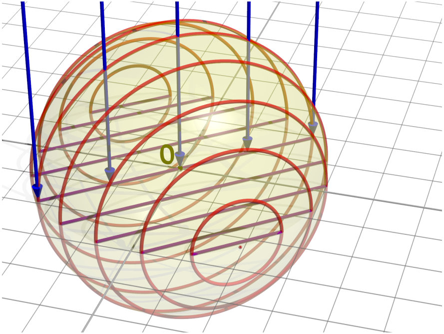

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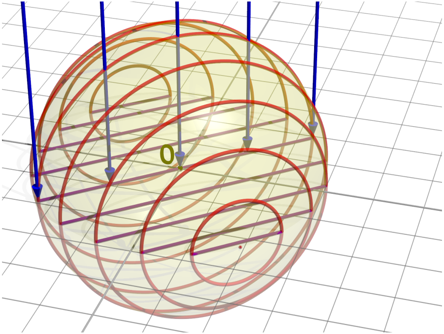

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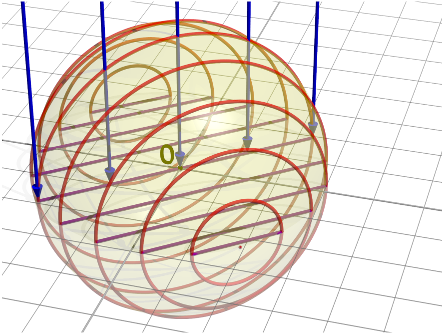

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Planes through a fixed point

[Salman 2016]

Consider an arbitrary point inside the sphere:

$$(0, \ldots, 0, z), \qquad 0 \le z < 1$$

Plane section through $(0, \ldots, 0, z)$ is

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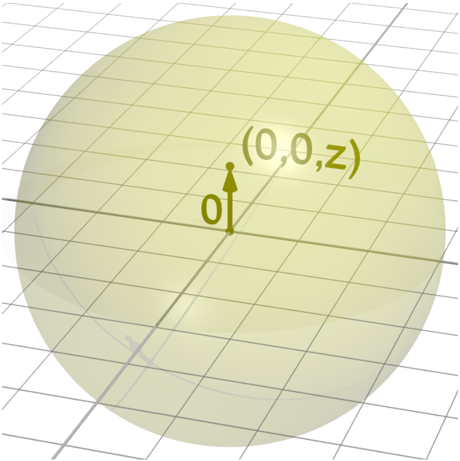

Planes through a fixed point

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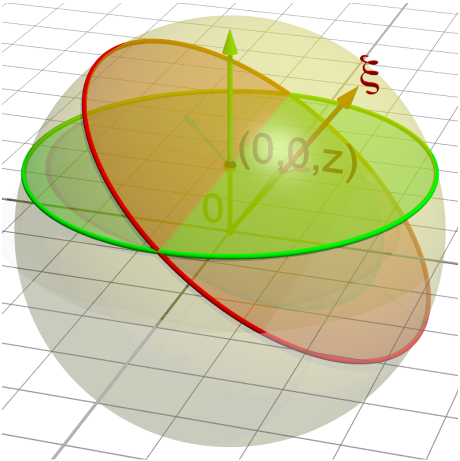

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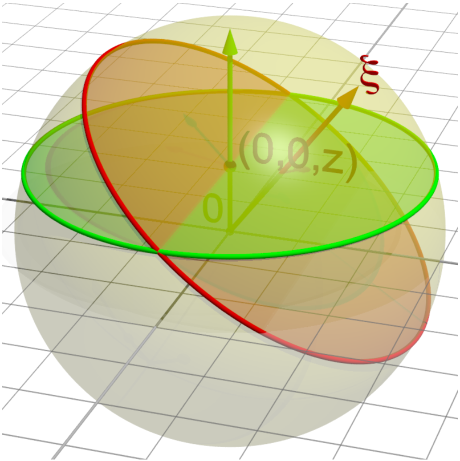

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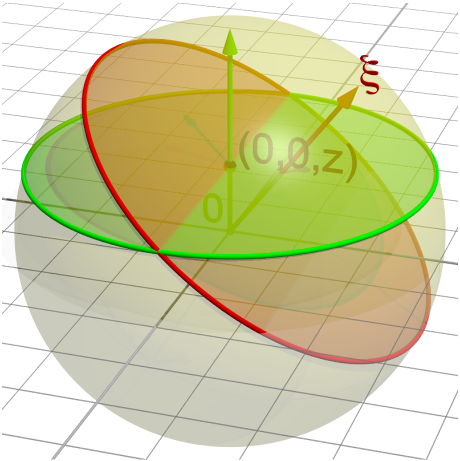

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Connection with the Funk–Radon transform

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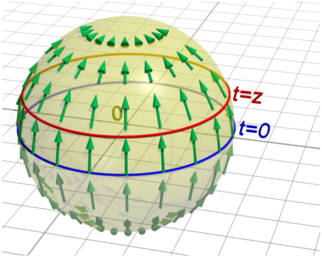
$$h(\boldsymbol{\xi}) = \pi^{-1}\left(\sqrt{\frac{1+z}{1-z}}\,\pi(\boldsymbol{\xi})\right), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}$$



that consists of

1. Stereographic projection $\pi\colon \mathbb{S}^{d-1} \to \mathbb{R}^{d-1}$
2. Uniform scaling $\mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$, $\boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
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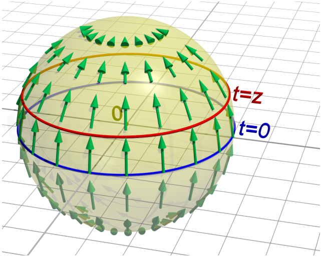

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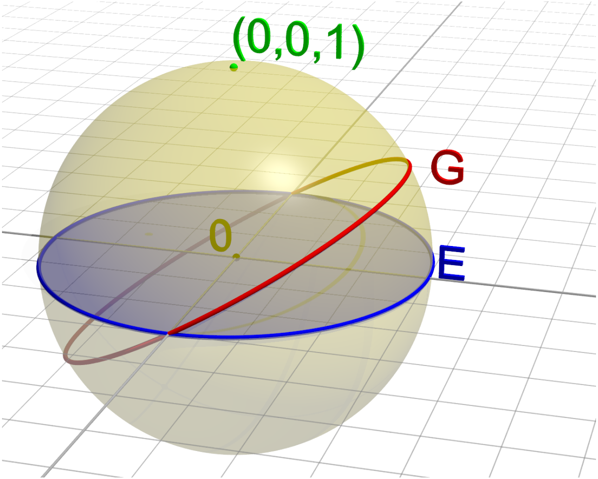
- G ... great circle of $\mathbb{S}^2$
- E ... equator of $\mathbb{S}^2$

- G intersects E in two antipodal points (or is identical to E)

- $\pi(E) = E$
- $\pi(G)$ is circle or line in $\mathbb{R}^2$ and intersects $\pi(E)$ in two antipodal points



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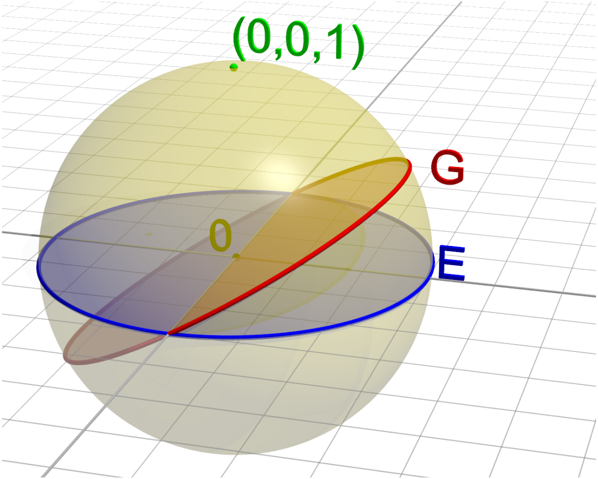

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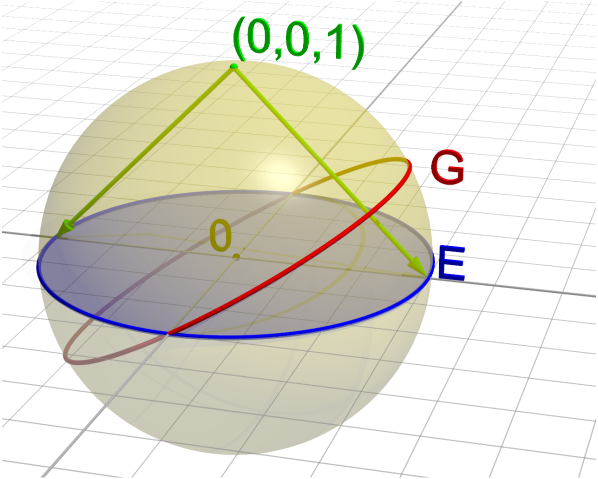

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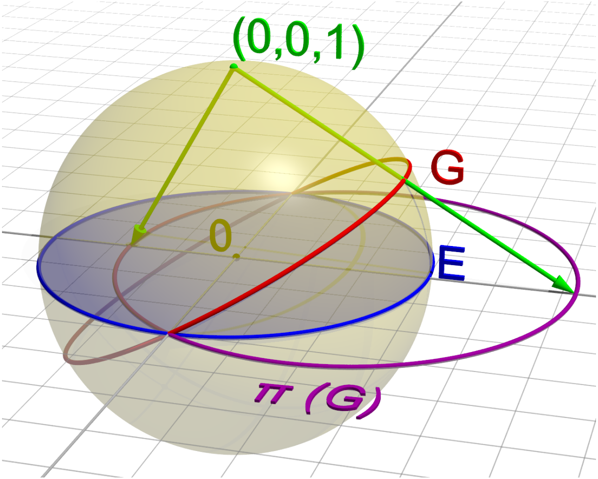

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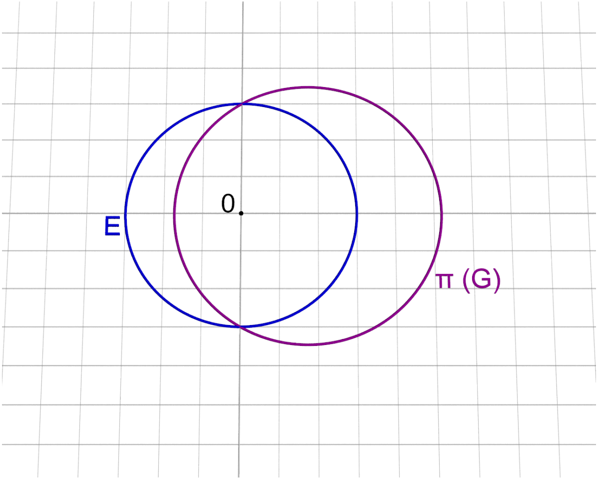

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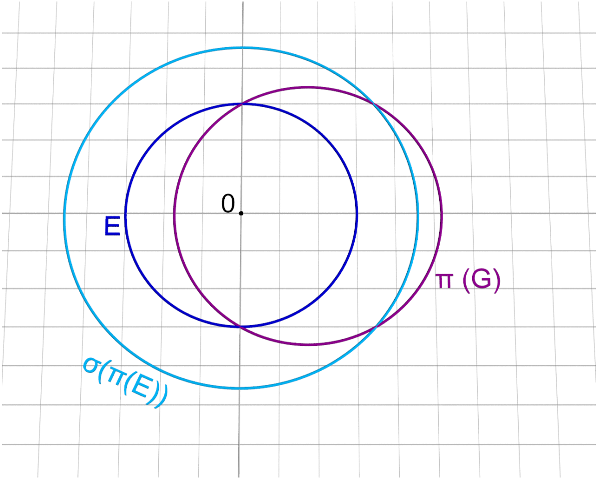

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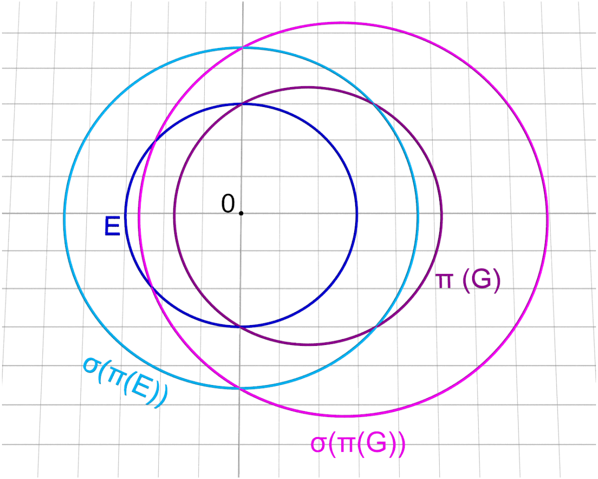

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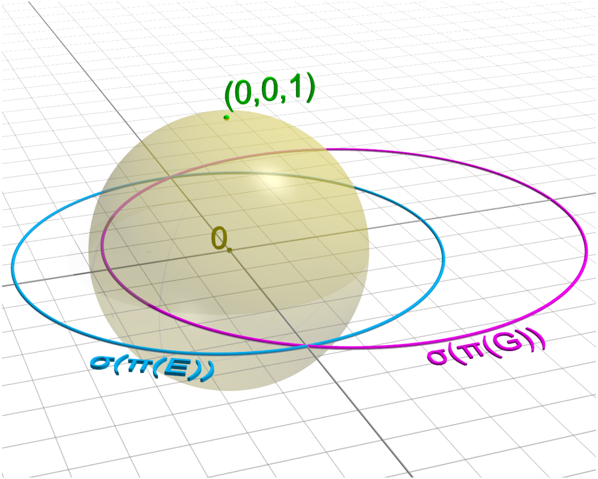

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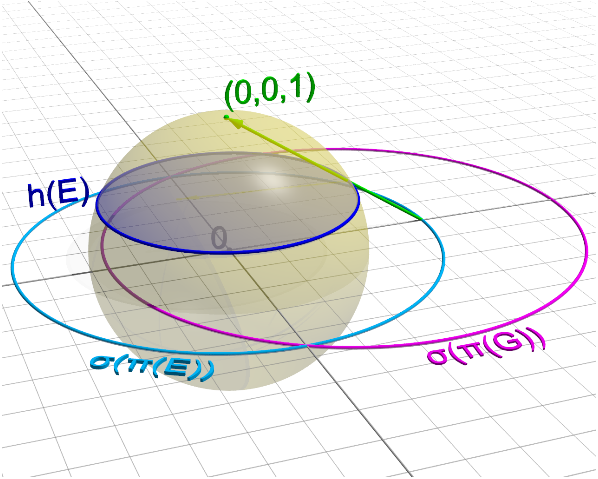

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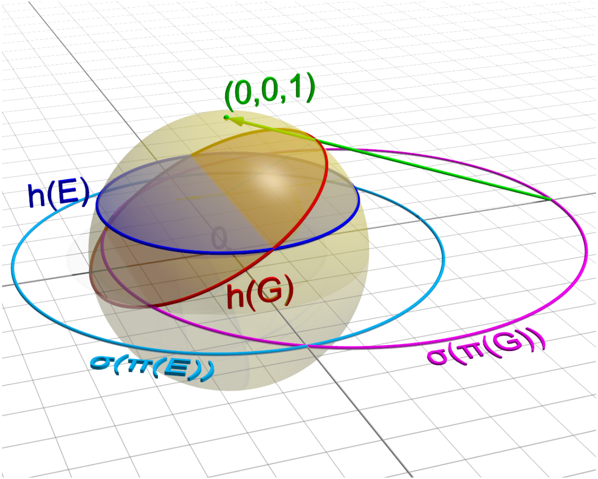

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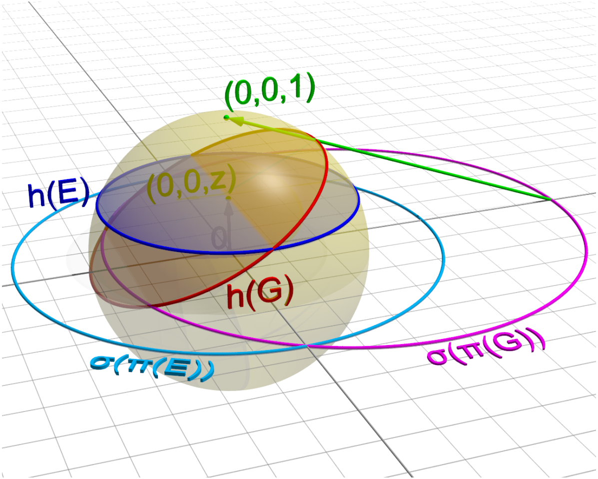

Nullspace of $\mathcal{U}_z$ [Q. 2018]

For $\xi \in \mathbb{S}^{d-1}$, we define $\xi^* \in \mathbb{S}^{d-1}$ as the point reflection of the sphere about the point $(0, \dots, 0, z)$.

Let $f \in L^2(\mathbb{S}^{d-1})$. Then

$$\mathcal{U}_z f = 0$$

if and only if for almost every $\xi \in \mathbb{S}^{d-1}$

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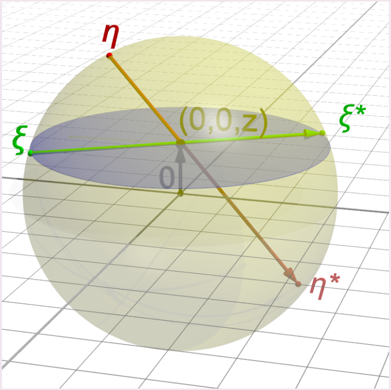

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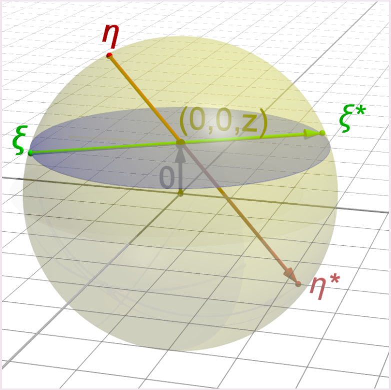

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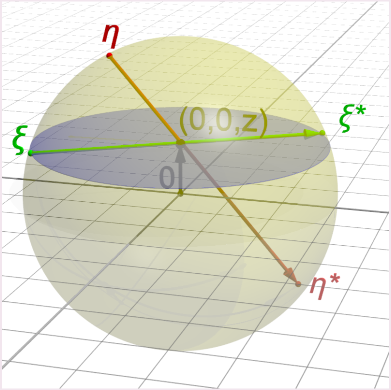

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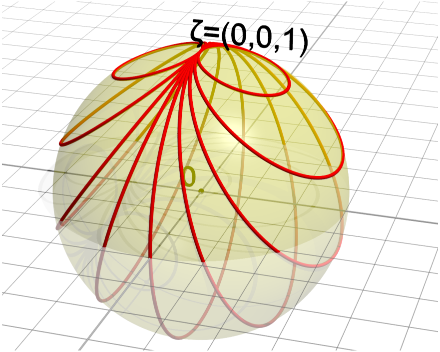

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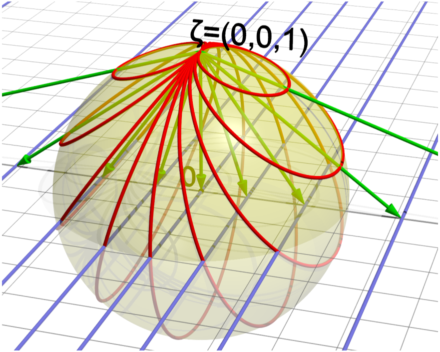

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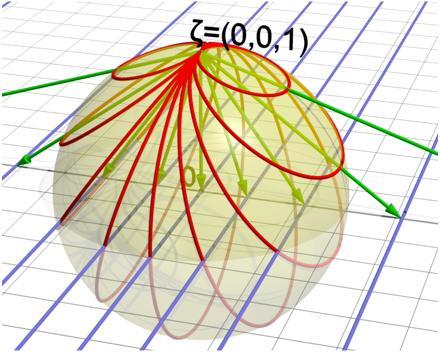

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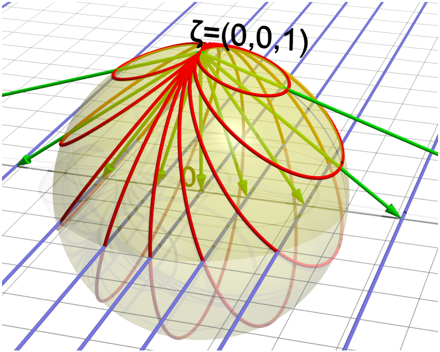

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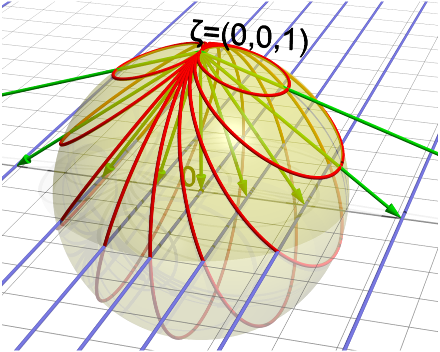

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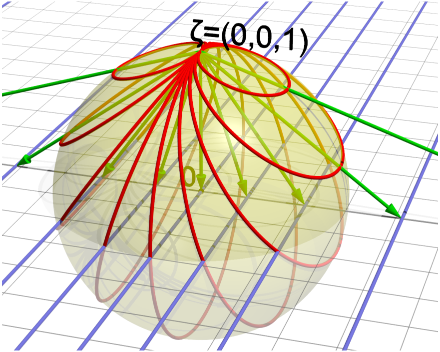

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Name	Definition	Injectivity	Range	SVD
mean operator	$\mathcal{M}f(\boldsymbol{\xi}, t)$	✓	$\subset H^{d/2-1,0}_{\text{even}}$	✓
Funk–Radon	$\mathcal{M}f(\boldsymbol{\xi}, 0)$	$f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$	$= H^{\frac{d-2}{2}}_{\text{even}}$	✓
spherical section transform	$\mathcal{M}f(\boldsymbol{\xi}, z)$, $z \in [-1,1]$ fixed	✓ if $P_{n,d}(z) \neq 0 \; \forall n \in \mathbb{N}_0$	$\subset H^{\frac{d-2}{2}}$	✓
vertical slices	$\mathcal{M}f(\binom{\boldsymbol{\sigma}}{0}, t)$, $\boldsymbol{\sigma} \in \mathbb{S}^{d-2}$	$f(\boldsymbol{\xi}', \xi_d) = f(\boldsymbol{\xi}', -\xi_d)$	$\subset H^{0,\frac{d-2}{2}-\frac{1}{4}}_{\text{even}}$	✓
sections through fixed point	$\mathcal{M}f(\boldsymbol{\xi}, z\xi_d)$, $z \in (-1,1)$ fixed	f even w.r.t. some reflection in $z\boldsymbol{\epsilon}^d$	$= \widetilde{H}_z^{\frac{d-2}{2}}$	✗
sections through North Pole	$\mathcal{M}f(\boldsymbol{\xi}, \xi_d)$	✓ for $f \in L^\infty(\mathbb{S}^{d-1})$		✗

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