A generalization of the Funk–Radon transform

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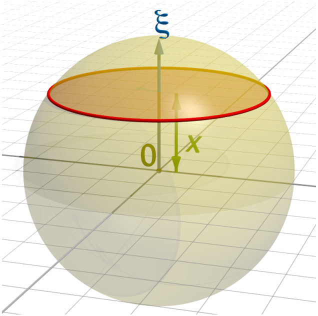
- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f : \mathbb{S}^2 \to \mathbb{C}$



- Circles on the sphere are intersections of the sphere with planes:

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x\},$$

for $\boldsymbol{\xi} \in \mathbb{S}^2, \ x \in [-1, 1]$

Definition

Define the **mean operator**

$$\mathcal{S} : C(\mathbb{S}^2) \to C(\mathbb{S}^2 \times [-1, 1]),$$

$$\mathcal{S}f(\boldsymbol{\xi}, x) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x} f(\boldsymbol{\eta}) \, \mathrm{d}\lambda(\boldsymbol{\eta})$$

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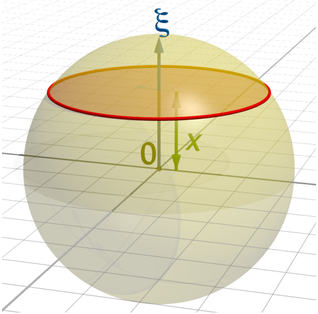

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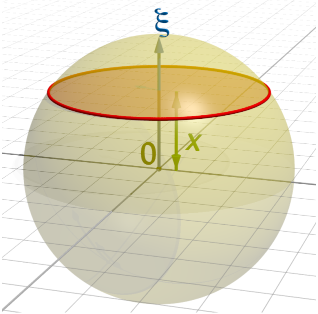

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Funk–Radon transform

[Funk, 1913]

- Restriction to all great circles
- **Funk–Radon transform** (a.k.a. Funk transform or spherical Radon transform)

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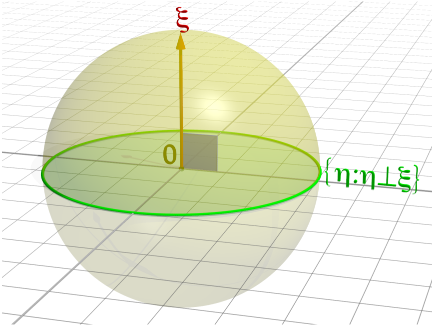

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   (Knowing the mean values of f on certain circles, can we reconstruct f?)
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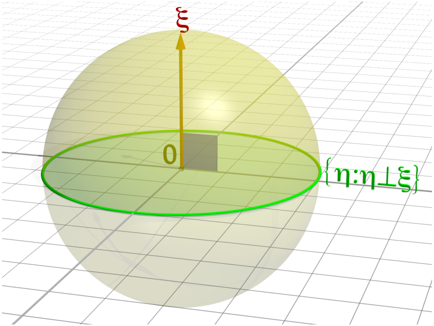

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Fourier series

- Write $f \in L^2(\mathbb{S}^2)$ in terms of the **spherical harmonics** Y_n^k of degree n

$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \hat{f}(n,k) Y_n^k$$

Eingenvalue decomposition [Minkowski, 1904]

For spherical harmonics Y_n^k

$$\mathcal{F} Y_n^k(\xi) = P_n(0) Y_n^k(\xi), \qquad P_n(0) = \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ even}, \\ 0, & n \text{ odd} \end{cases}$$

P_n – Legendre polynomial of degree n

- Every even function $f \in L^2(\mathbb{S}^2)$ can be reconstructed from $\mathcal{F} f$

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Sobolev spaces

- For $s \geq 0$, the **Sobolev space** $H^s(\mathbb{S}^2)$ is the completion of the space of polynomials $f : \mathbb{S}^2 \to \mathbb{C}$ with the norm

$$\|f\|_s^2 := \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left|\hat{f}(n,k)\right|^2 \left(n + \frac{1}{2}\right)^{2s}.$$

Theorem [Strichartz, 1981]

The Funk–Radon transform is bijective

$$\mathcal{F} \colon H^s_{\mathrm{even}}(\mathbb{S}^2) \to H^{s+\frac{1}{2}}_{\mathrm{even}}(\mathbb{S}^2)$$

- $\mathcal{F}$ is smoothing of degree $\frac{1}{2}$

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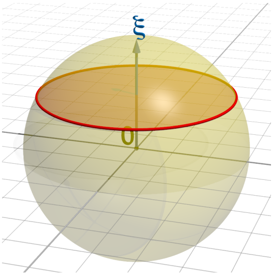
- **Circles with fixed radius**
- For fixed $x_0 \in [-1, 1]$, compute

$$\mathcal{S}_{x_0} f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x_0} f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}$$

- Eigenvalue decomposition yields

$$\mathcal{S}_{x_0} Y_n^k = P_n(x_0) Y_n^k$$



Freak theorem [Schneider, 1969]

The set of values x_0 for which $\mathcal{S}_{x_0}$ is not injective is countable and dense in $[-1, 1]$.

There is an algorithm to determine whether $\mathcal{S}_{x_0}$ is injective for a given x_0 [Rubin, 2000]

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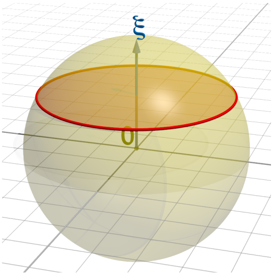

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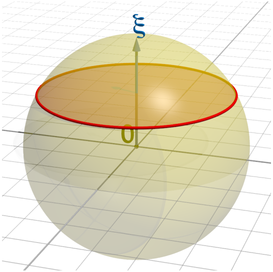

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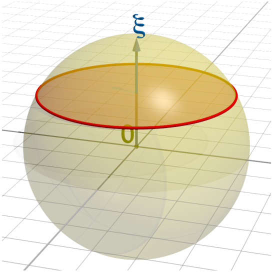

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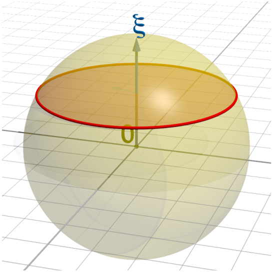

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Spherical slice transform

[Abouelaz & Daher, 1993]

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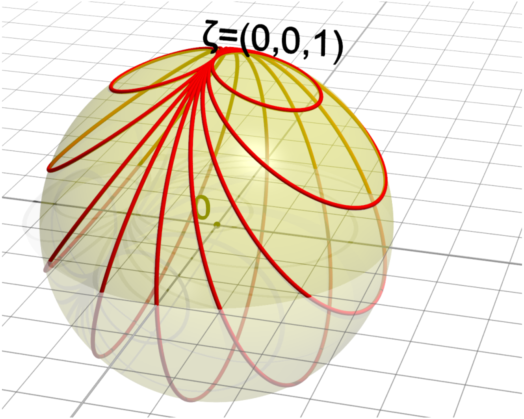

- Circles through the North pole
- Stereographic projection turns circles into lines in the plane
  ↗ Radon transform in $\mathbb{R}^2$
- Injective if f is differentiable and vanishes at $(0,0,1)$ [Helgason, 1999]
- Injective for all bounded functions [Rubin, 2015]

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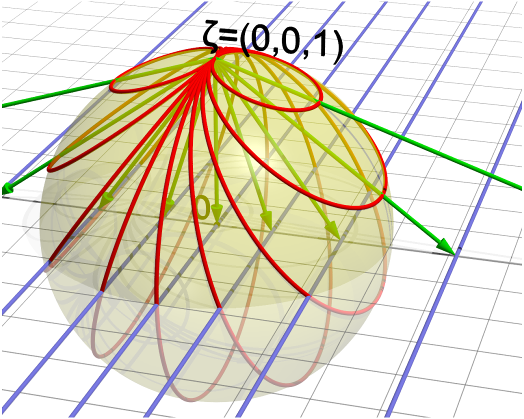

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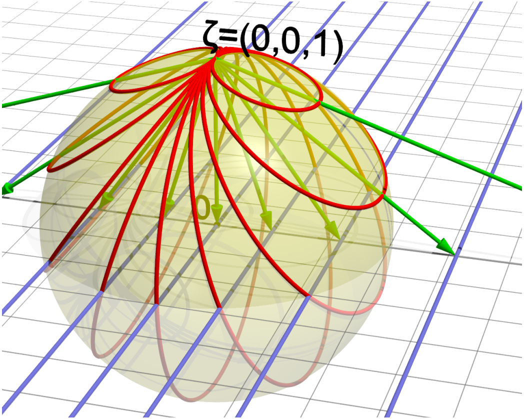

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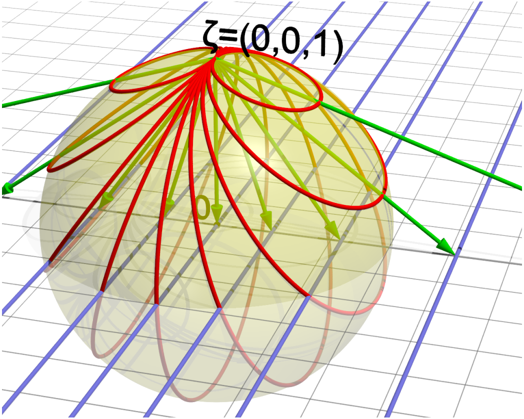

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Vertical slices

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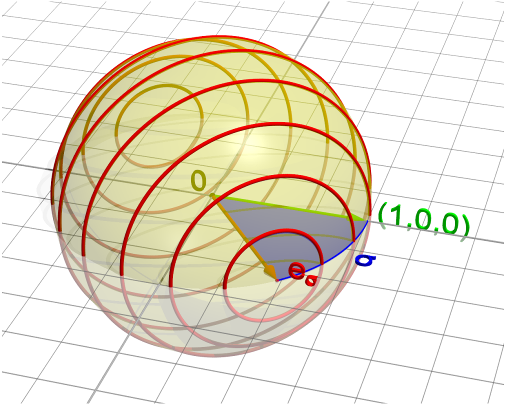

- Circles perpendicular to the equator
- Injective for symmetric functions $f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3)$
- Proof1: Orthogonal projection onto equatorial plane [Gindikin, Reeds & Shepp, 1994]
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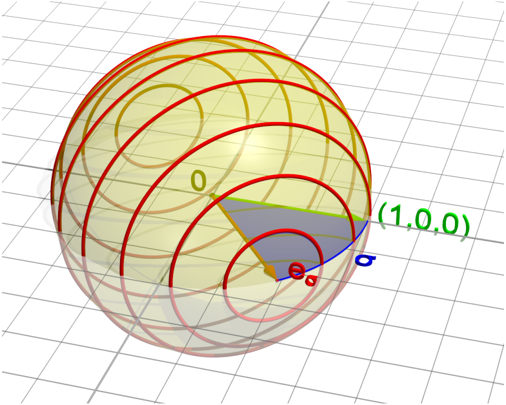

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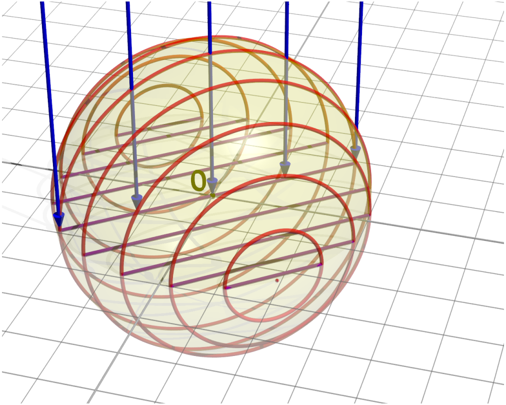

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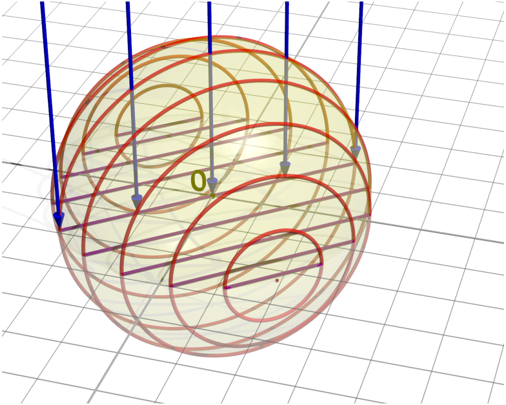

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Planes through a fixed point

[Salman, 2015]

Replace 0 by an arbitrary point

$$\zeta = (0, 0, z)^\top, \qquad 0 \le z < 1$$

inside the sphere.

Circle through ζ is

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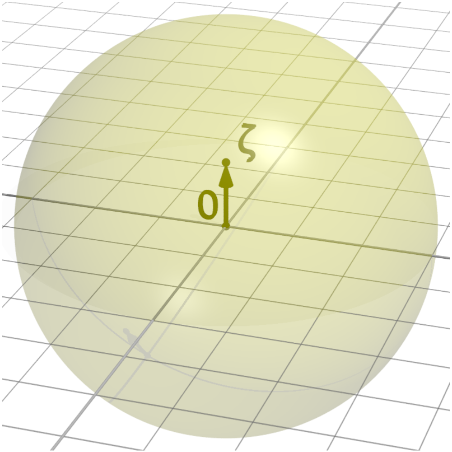

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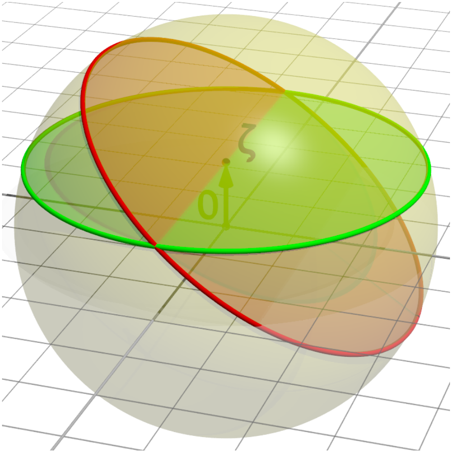

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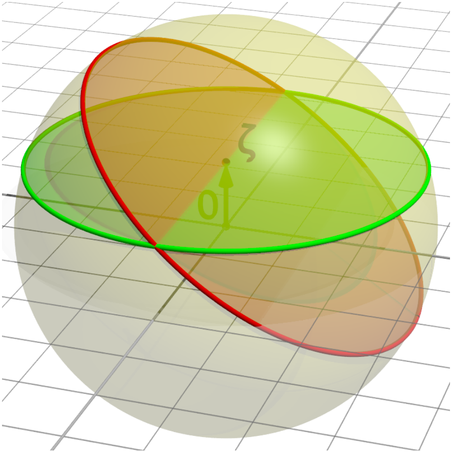

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From great circles to small circles

Definition

Define the map

$$h\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad h = \pi^{-1} \circ \sigma \circ \pi$$

consisting of

1. Stereographic projection $\pi\colon \mathbb{S}^2 \to \mathbb{R}^2$
2. Scaling in the plane $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
3. Inverse stereographic projection $\pi^{-1}\colon \mathbb{R}^2 \to \mathbb{S}^2$

We show that

h maps great circles to small circles through $\zeta = (0,0,z)^\top$.

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1. Stereographic projection $\pi\colon \mathbb{S}^2 \to \mathbb{R}^2$
2. Scaling in the plane $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
3. Inverse stereographic projection $\pi^{-1}\colon \mathbb{R}^2 \to \mathbb{S}^2$

We show that

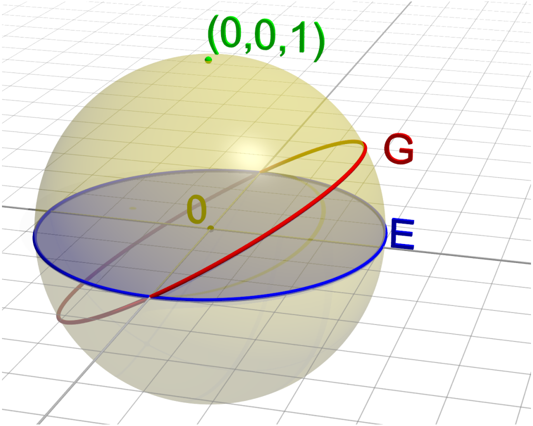
h maps great circles to small circles through $\zeta = (0,0,z)^\top$.

1) Stereographic projection π

- G ... Great circle of $\mathbb{S}^2$
- E ... Equator of $\mathbb{S}^2$

- G intersects E in two antipodal points (or is identical to E)

- $\pi(E) = E$
- $\pi(G)$ is a circle or line in $\mathbb{R}^2$ that intersects $\pi(E)$ in two antipodal points



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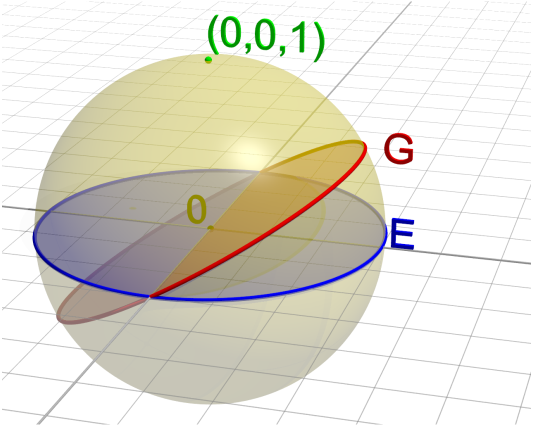

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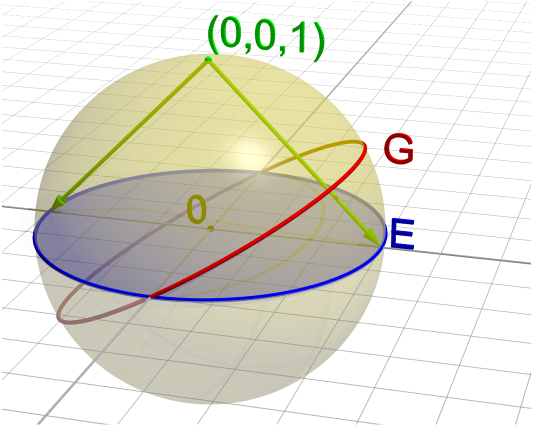

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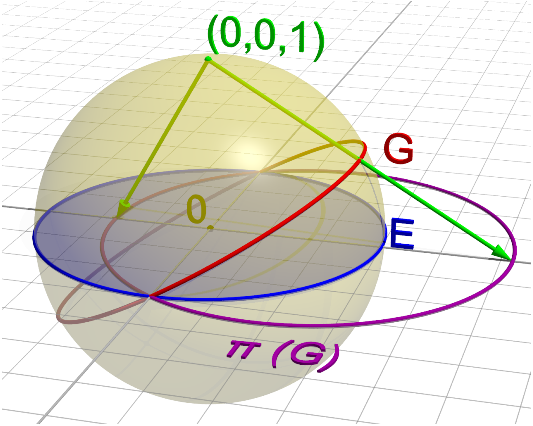
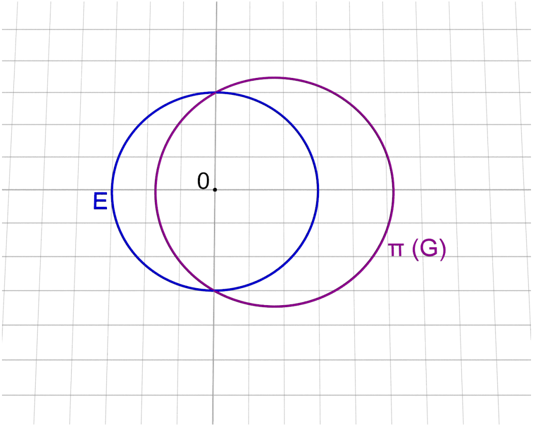


2) Scaling σ in the plane

- Uniform scaling with scale factor $s = \sqrt{\frac{1+z}{1-z}}$
- Unit circle E is mapped to the circle $\sigma(\pi(E))$ with radius s
- $\sigma(\pi(G))$ intersects $\sigma(\pi(E))$ in two antipodal points



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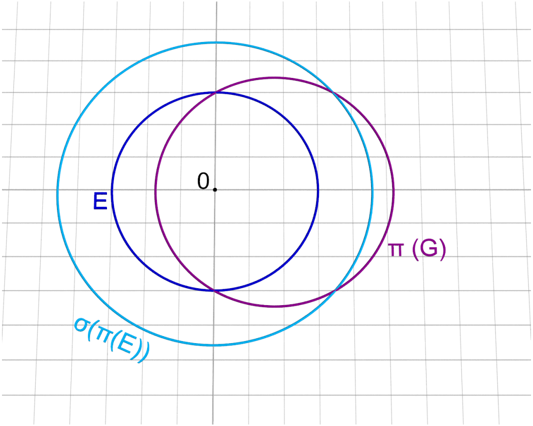

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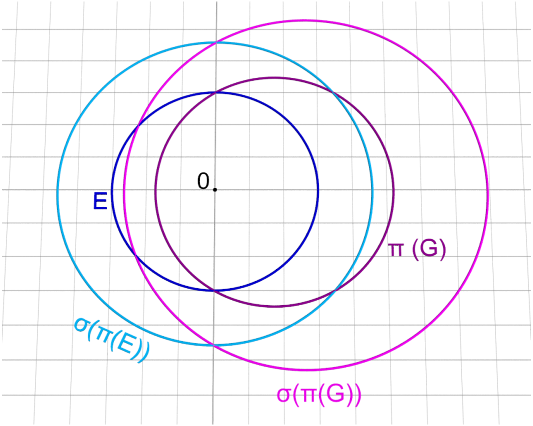
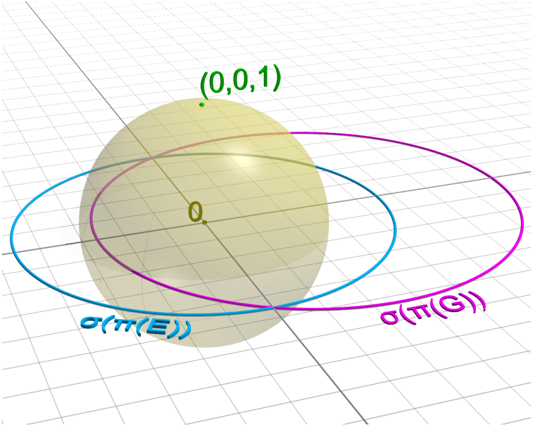


3) Inverse stereographic projection π^{-1}

- The circle with radius s is mapped to the circle of latitude z; $h(E)$
- $h(G) = \pi^{-1}(\sigma(\pi(G)))$ intersects $h(E)$ in two antipodal points
- $h(G)$ is a small circle through $\zeta = (0,0,z)^\top$



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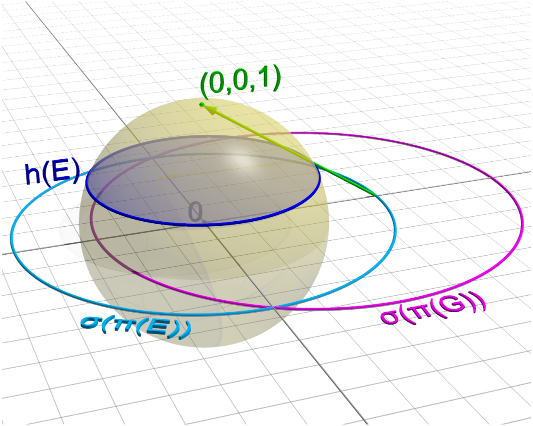

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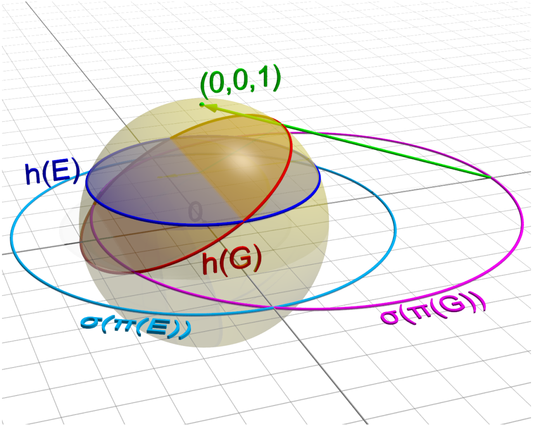

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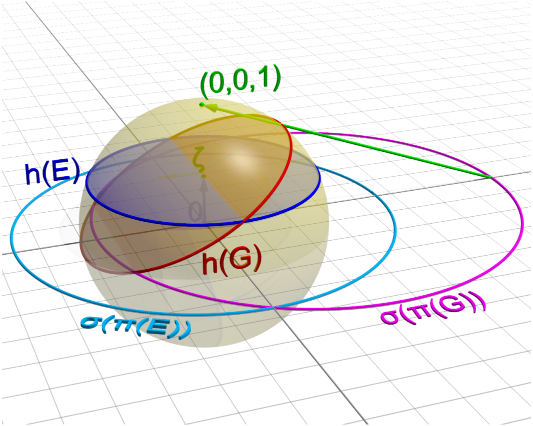
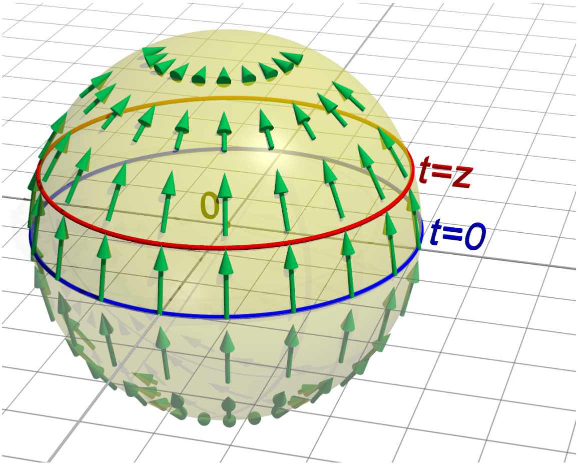


The resulting map h



$$h(\boldsymbol{\eta}) = \begin{pmatrix} \frac{\sqrt{1-z^2}}{1-z\eta_3}\eta_1 \\ \frac{\sqrt{1-z^2}}{1-z\eta_3}\eta_2 \\ \frac{z+\eta_3}{1-z\eta_3} \end{pmatrix}$$

$h\colon \mathbb{S}^2 \to \mathbb{S}^2$ is conformal

Theorem

Let $z \in [0,1)$. The generalized Radon transform $\mathcal{U}$ can be represented with the operators $\mathcal{M}, \mathcal{F}, \mathcal{N} \colon L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ via

$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M}.$$

These operators are defined for $f \in C(\mathbb{S}^2)$ by

- $\mathcal{M}f(\boldsymbol{\xi}) = \dfrac{\sqrt{1-z^2}}{1+z\xi_3}\,[f \circ h](\boldsymbol{\xi})$
- $\mathcal{F}$ ... Funk–Radon transform
- $\mathcal{N}f(\boldsymbol{\xi}) = f\left(\dfrac{1}{\sqrt{1-z^2\xi_3^2}}\left(\xi_1, \xi_2, \sqrt{1-z^2}\xi_3\right)\right)$

Nullspace of $\mathcal{U}$

Theorem

$\mathbf{R}$ ... Reflection of the sphere about the point $\zeta = (0, 0, z)^\top$

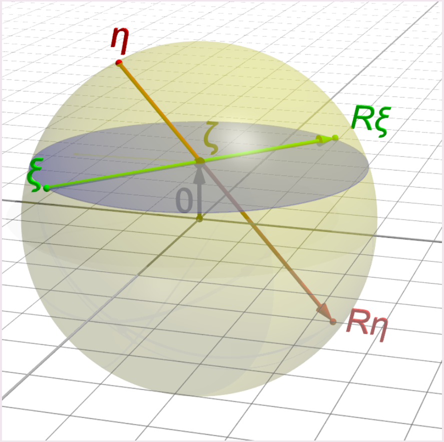
$f \in L^2(\mathbb{S}^2)$

We have

$$\mathcal{U}f = 0$$

if and only if for almost all $\eta \in \mathbb{S}^2$

$$f(\eta) = -f(\mathbf{R}\eta)\frac{1 - z^2}{1 + z^2 - 2z\eta_3}.$$



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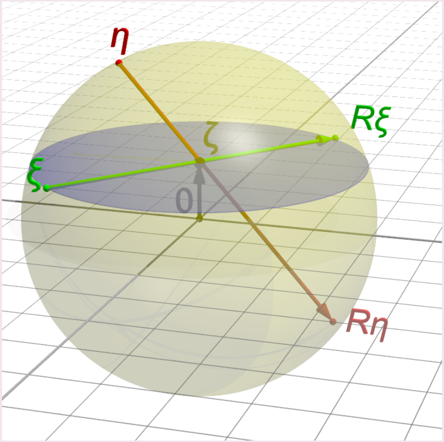

Range of $\mathcal{U}$

Theorem

The generalized Radon transform

$$\mathcal{U}\colon \tilde{L}^2_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is continuous and bijective.

- $\tilde{L}^2_{\mathrm{e}}(\mathbb{S}^2) = \left\{ f \in L^2(\mathbb{S}^2) \mid f(\boldsymbol{\eta}) = f(\mathbf{R}\boldsymbol{\eta}) \frac{1-z^2}{1+z^2-2z\eta_3} \right\}$
- $H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$ ... Sobolev space of smoothness $1/2$ that contains only even functions

Sketch of proof

We show

$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M} \colon \tilde{L}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$$

is continuous and bijective.

- $\mathcal{M} \colon \tilde{L}_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ is unitary
- $\mathcal{F} \colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective
- $\mathcal{N}, \mathcal{N}^{-1} \colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ are continuous and $\mathcal{N}, \mathcal{N}^{-1} \colon H_{\mathrm{e}}^1(\mathbb{S}^2) \to H_{\mathrm{e}}^1(\mathbb{S}^2)$ are continuous
- $\mathcal{N} \colon H_{\mathrm{e}}^{1/2}(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective (shown with interpolation theory [Triebel, 1995])

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Inversion formula

Based on the inversion of the Funk–Radon transform by [Helgason, 1980]

$$f(\boldsymbol{\eta}) = \frac{1}{2\pi}\,\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \int_{\langle \boldsymbol{\xi},\boldsymbol{\eta}\rangle^2 = 1-w^2} \mathcal{F}f(\boldsymbol{\xi})\,\mathrm{d}\ell(\boldsymbol{\xi})\,\frac{1}{\sqrt{u^2-w^2}}\,\mathrm{d}w\bigg|_{u=1}$$

Theorem

Let $z \in [0,1)$ and $f \in \tilde{L}^2_{\mathrm{e}}(\mathbb{S}^2)$. For $\boldsymbol{\eta} \in \mathbb{S}^2$,

$$f(\boldsymbol{\eta}) = \frac{1-z^2}{2\pi(1-zv)}\,\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \int_{\mathscr{S}_z(\boldsymbol{\eta},w)} \mathcal{U}f\left(\frac{(\sqrt{1-z^2}(\xi_1,\xi_2),\xi_3)}{\sqrt{1-z^2+z^2\xi_3^2}}\right)\mathrm{d}s(\boldsymbol{\xi})\,\frac{\mathrm{d}w}{\sqrt{u^2-w^2}}\bigg|_{u=1}$$

where $\mathrm{d}s$ is the arc-length on the circle

$$\mathscr{S}_z(\boldsymbol{\eta},w) = \left\{\boldsymbol{\xi} \in \mathbb{S}^2 \mid \left\langle \boldsymbol{\xi}, \left(\sqrt{1-z^2}(\eta_1,\eta_2),\eta_3 - z\right)\right\rangle = (1-z\eta_3)\sqrt{1-w^2}\right\}.$$

Inversion via Fourier expansion

- The last inversion formula is numerically unstable
- Use the factorization
$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M}$$
- $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$ can be computed explicitly
- For $\mathcal{F}^{-1}$: **Fourier expansion of the Funk–Radon transform** combined with the mollifier method (as regularization)
[Louis et al., 2011] [Hielscher & Q., 2015]
- Efficient implementation of the spherical Fourier transform available
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