A generalization of the spherical Radon transform to circles passing through a fixed point

Michael Quellmalz

Technische Universität Chemnitz, Faculty of Mathematics

8th International Conference
“Inverse Problems: Modeling & Simulation”

26 May 2016

- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f \colon \mathbb{S}^2 \to \mathbb{C}$
- Circles on the sphere are intersections of the sphere with planes:

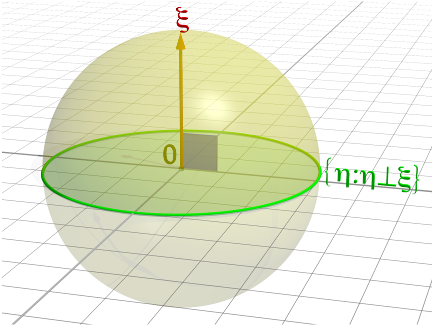
$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x\},$$

for $\boldsymbol{\xi} \in \mathbb{S}^2$, $x \in [-1, 1]$

- **Spherical Radon transform** (a.k.a. **Funk–Radon transform**)

$$\mathcal{F} \colon C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$

$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0} f(\boldsymbol{\eta}) \, \mathrm{d}\lambda(\boldsymbol{\eta})$$

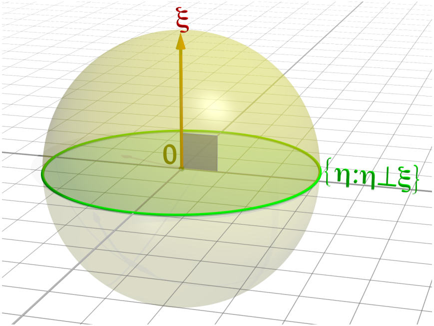


[Funk, 1913]

- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^2 \to \mathbb{C}$
- Circles on the sphere are intersections of the sphere with planes:

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x\},$$

for $\boldsymbol{\xi} \in \mathbb{S}^2,\ x \in [-1,1]$



- **Spherical Radon transform** (a.k.a. **Funk–Radon transform**)

$$\mathcal{F}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$

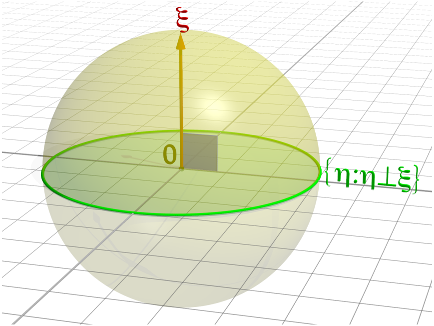
$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0} f(\boldsymbol{\eta}) \,\mathrm{d}\lambda(\boldsymbol{\eta})$$

[Funk, 1913]

- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f : \mathbb{S}^2 \to \mathbb{C}$
- Circles on the sphere are intersections of the sphere with planes:

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x\},$$

  for $\boldsymbol{\xi} \in \mathbb{S}^2,\ x \in [-1, 1]$



- **Spherical Radon transform** (a.k.a. **Funk–Radon transform**)

$$\mathcal{F} : C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$

$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0} f(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta})$$

[Funk, 1913]

Generalized Radon transform

[Salman, 2015]

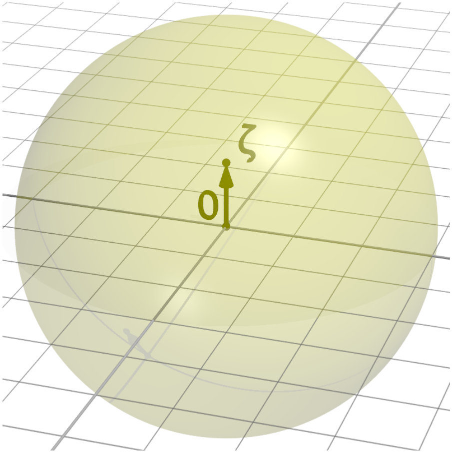
Replace 0 by an arbitrary point

$$\zeta = (0,0,z)^\top, \qquad 0 \le z < 1$$

inside the sphere.

Circle through ζ is

$$\{\eta \in \mathbb{S}^2 : \langle \xi, \eta \rangle = \underbrace{\langle \xi, \zeta \rangle}_{= z\xi_3}\}.$$



Definition

$$\mathcal{U} : C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$
$$\mathcal{U}f(\xi) = \int_{\langle \xi,\eta \rangle = z\xi_3} f(\eta)\,\mathrm{d}\lambda(\eta)$$

Generalized Radon transform

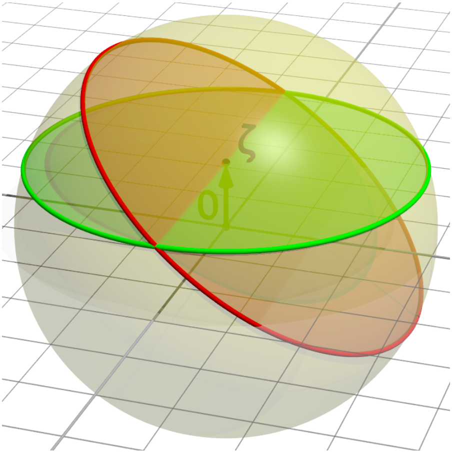
[Salman, 2015]

Replace 0 by an arbitrary point

$$\boldsymbol{\zeta} = (0, 0, z)^\top, \qquad 0 \le z < 1$$

inside the sphere.
Circle through $\boldsymbol{\zeta}$ is

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \underbrace{\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle}_{= z\xi_3}\}.$$



Definition

$$\mathcal{U}: C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$
$$\mathcal{U}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z\xi_3} f(\boldsymbol{\eta}) \, \mathrm{d}\lambda(\boldsymbol{\eta})$$

Generalized Radon transform

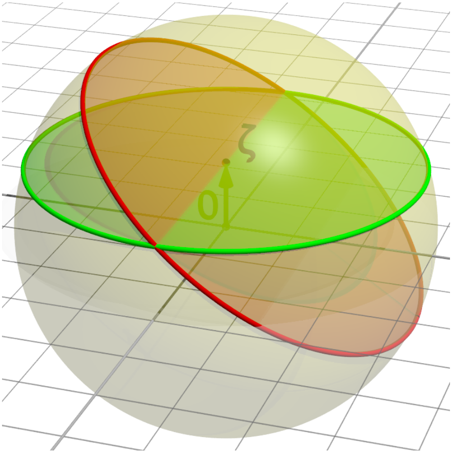
[Salman, 2015]

Replace 0 by an arbitrary point

$$\zeta = (0,0,z)^\top, \qquad 0 \le z < 1$$

inside the sphere.
Circle through ζ is

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \underbrace{\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle}_{= z\xi_3}\}.$$



Definition

$$\mathcal{U} \colon C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$
$$\mathcal{U} f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z\xi_3} f(\boldsymbol{\eta}) \, \mathrm{d}\lambda(\boldsymbol{\eta})$$

What is known about the spherical Radon transform

Theorem [Strichartz, 1981]

The spherical Radon transform

$$\mathcal{F}\colon L^2_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is continuous and bijective.
Its nullspace consists of all odd functions $f(\boldsymbol{\xi}) = -f(-\boldsymbol{\xi})$.

Question

What about the generalized Radon transform?

What is known about the spherical Radon transform

Theorem [Strichartz, 1981]

The spherical Radon transform

$$\mathcal{F}\colon L^2_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is continuous and bijective.
Its nullspace consists of all odd functions $f(\boldsymbol{\xi}) = -f(-\boldsymbol{\xi})$.

Question

What about the generalized Radon transform?

From great circles to small circles

Definition

Define the map

$$h\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad h = \pi^{-1} \circ \sigma \circ \pi$$

consisting of

1. Stereographic projection $\pi\colon \mathbb{S}^2 \to \mathbb{R}^2$
2. Scaling in the plane $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
3. Inverse stereographic projection $\pi^{-1}\colon \mathbb{R}^2 \to \mathbb{S}^2$

We show that

h maps great circles to small circles through $\zeta = (0,0,z)^\top$.

From great circles to small circles

Definition

Define the map

$$h \colon \mathbb{S}^2 \to \mathbb{S}^2, \quad h = \pi^{-1} \circ \sigma \circ \pi$$

consisting of

1. Stereographic projection $\pi \colon \mathbb{S}^2 \to \mathbb{R}^2$
2. Scaling in the plane $\sigma \colon \mathbb{R}^2 \to \mathbb{R}^2, \ \boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
3. Inverse stereographic projection $\pi^{-1} \colon \mathbb{R}^2 \to \mathbb{S}^2$

We show that

h maps great circles to small circles through $\zeta = (0, 0, z)^\top$.

From great circles to small circles

Definition

Define the map

$$h\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad h = \pi^{-1} \circ \sigma \circ \pi$$

consisting of

1. Stereographic projection $\pi\colon \mathbb{S}^2 \to \mathbb{R}^2$
2. Scaling in the plane $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
3. Inverse stereographic projection $\pi^{-1}\colon \mathbb{R}^2 \to \mathbb{S}^2$

We show that

h maps great circles to small circles through $\zeta = (0, 0, z)^\top$.

From great circles to small circles

Definition

Define the map

$$h\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad h = \pi^{-1} \circ \sigma \circ \pi$$

consisting of

1. Stereographic projection $\pi\colon \mathbb{S}^2 \to \mathbb{R}^2$
2. Scaling in the plane $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
3. Inverse stereographic projection $\pi^{-1}\colon \mathbb{R}^2 \to \mathbb{S}^2$

We show that

h maps great circles to small circles through $\zeta = (0, 0, z)^\top$.

From great circles to small circles

Definition

Define the map

$$h\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad h = \pi^{-1} \circ \sigma \circ \pi$$

consisting of

1. Stereographic projection $\pi\colon \mathbb{S}^2 \to \mathbb{R}^2$
2. Scaling in the plane $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
3. Inverse stereographic projection $\pi^{-1}\colon \mathbb{R}^2 \to \mathbb{S}^2$

We show that

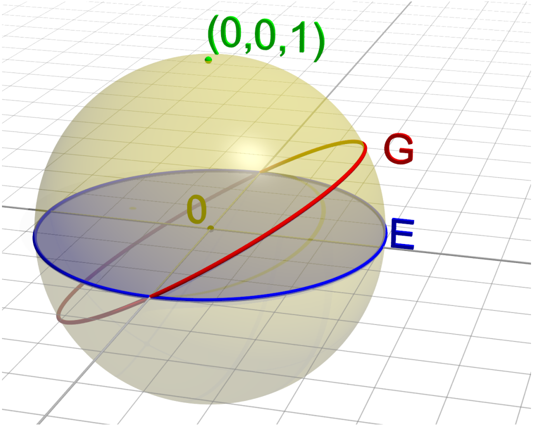
h maps great circles to small circles through $\zeta = (0,0,z)^\top$.

1) Stereographic projection π

- G ... Great circle of $\mathbb{S}^2$
- E ... Equator of $\mathbb{S}^2$

- G intersects E in two antipodal points (or is identical to E)

- $\pi(E) = E$
- $\pi(G)$ is a circle or line in $\mathbb{R}^2$ that intersects $\pi(E)$ in two antipodal points

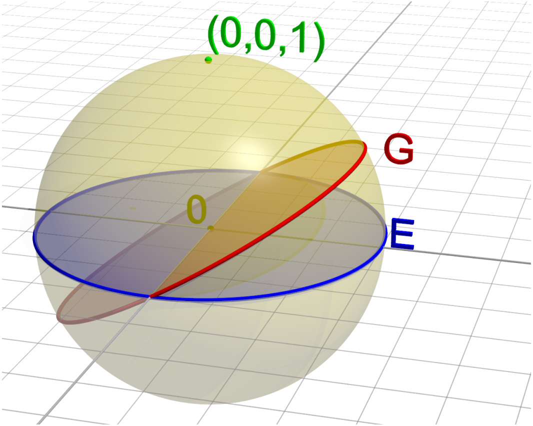


1) Stereographic projection π

- G ... Great circle of $\mathbb{S}^2$
- E ... Equator of $\mathbb{S}^2$

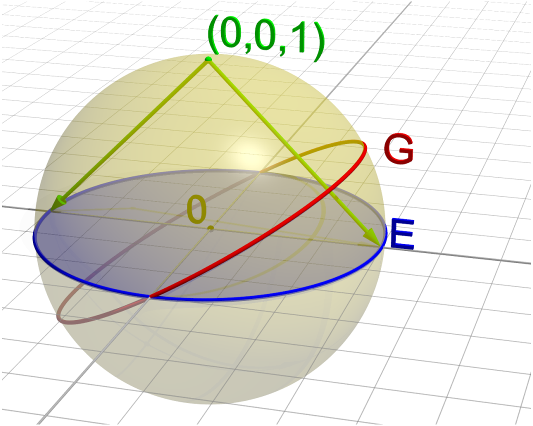
- G intersects E in two antipodal points (or is identical to E)

- $\pi(E) = E$
- $\pi(G)$ is a circle or line in $\mathbb{R}^2$ that intersects $\pi(E)$ in two antipodal points



1) Stereographic projection π

- G ... Great circle of $\mathbb{S}^2$
- E ... Equator of $\mathbb{S}^2$
- G intersects E in two antipodal points (or is identical to E)
- $\pi(E) = E$
- $\pi(G)$ is a circle or line in $\mathbb{R}^2$ that intersects $\pi(E)$ in two antipodal points

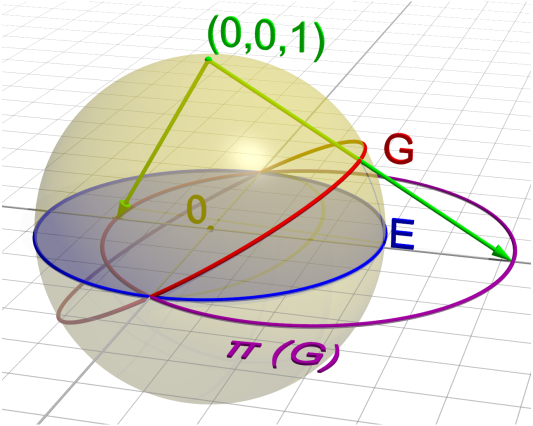


1) Stereographic projection π

- G ... Great circle of $\mathbb{S}^2$
- E ... Equator of $\mathbb{S}^2$

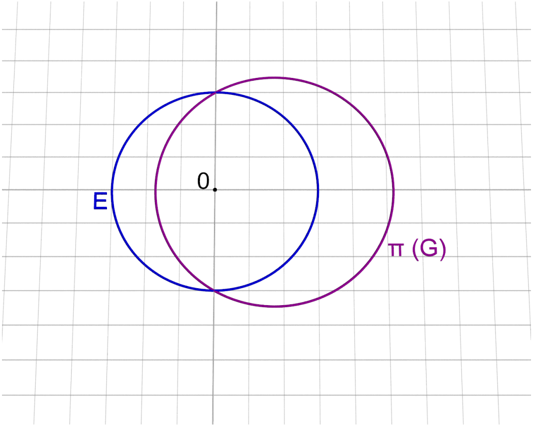
- G intersects E in two antipodal points (or is identical to E)

- $\pi(E) = E$
- $\pi(G)$ is a circle or line in $\mathbb{R}^2$ that intersects $\pi(E)$ in two antipodal points



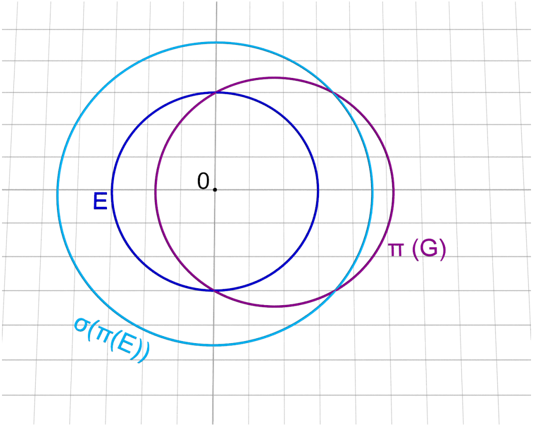
2) Scaling σ in the plane

- Uniform scaling with scale factor $s = \sqrt{\frac{1+z}{1-z}}$
- Unit circle E is mapped to the circle $\sigma(\pi(E))$ with radius s
- $\sigma(\pi(G))$ intersects $\sigma(\pi(E))$ in two antipodal points



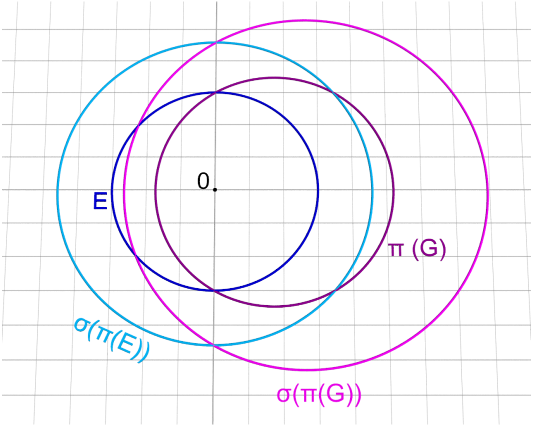
2) Scaling σ in the plane

- Uniform scaling with scale factor $s = \sqrt{\frac{1+z}{1-z}}$
- Unit circle E is mapped to the circle $\sigma(\pi(E))$ with radius s
- $\sigma(\pi(G))$ intersects $\sigma(\pi(E))$ in two antipodal points



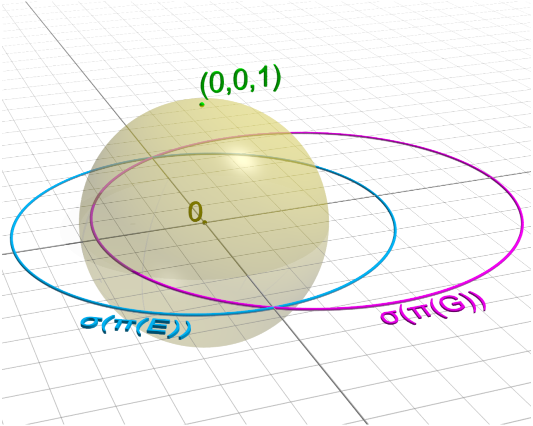
2) Scaling σ in the plane

- Uniform scaling with scale factor $s = \sqrt{\frac{1+z}{1-z}}$
- Unit circle E is mapped to the circle $\sigma(\pi(E))$ with radius s
- $\sigma(\pi(G))$ intersects $\sigma(\pi(E))$ in two antipodal points



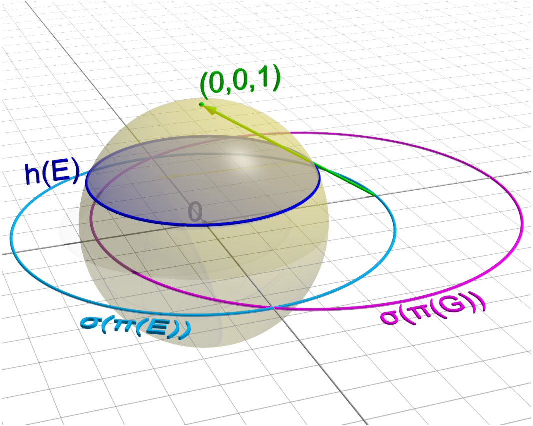
3) Inverse stereographic projection π^{-1}

- The circle with radius s is mapped to the circle of latitude z; $h(E)$
- $h(G) = \pi^{-1}(\sigma(\pi(G)))$ intersects $h(E)$ in two antipodal points
- $h(G)$ is a small circle through $\zeta = (0, 0, z)^\top$



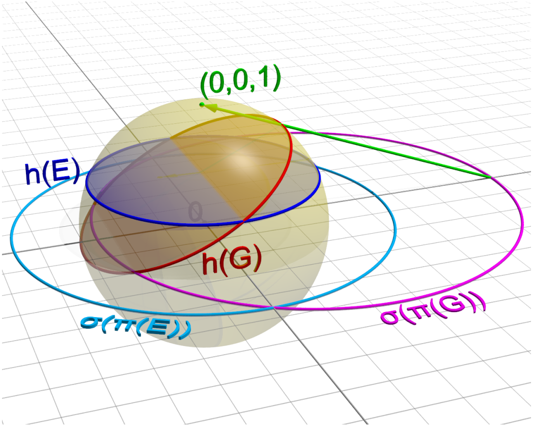
3) Inverse stereographic projection π^{-1}

- The circle with radius s is mapped to the circle of latitude z; $h(E)$
- $h(G) = \pi^{-1}(\sigma(\pi(G)))$ intersects $h(E)$ in two antipodal points
- $h(G)$ is a small circle through $\zeta = (0, 0, z)^\top$



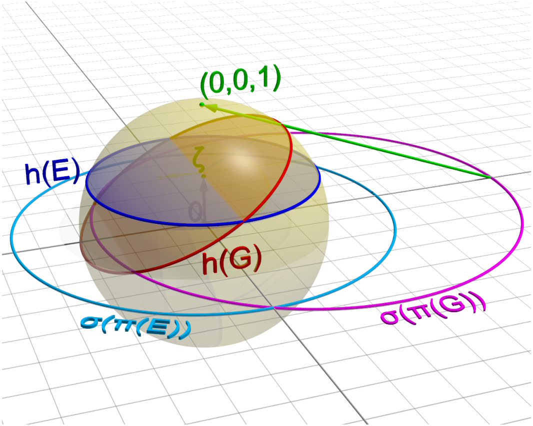
3) Inverse stereographic projection π^{-1}

- The circle with radius s is mapped to the circle of latitude z; $h(E)$
- $h(G) = \pi^{-1}(\sigma(\pi(G)))$ intersects $h(E)$ in two antipodal points
- $h(G)$ is a small circle through $\zeta = (0, 0, z)^\top$

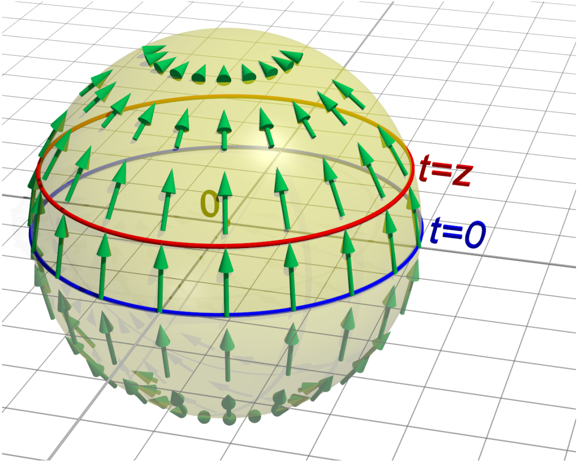


3) Inverse stereographic projection π^{-1}

- The circle with radius s is mapped to the circle of latitude z; $h(E)$
- $h(G) = \pi^{-1}(\sigma(\pi(G)))$ intersects $h(E)$ in two antipodal points
- $h(G)$ is a small circle through $\zeta = (0, 0, z)^\top$



The resulting map $h\colon \mathbb{S}^2 \to \mathbb{S}^2$



$$h(\boldsymbol{\eta}) = \begin{pmatrix} \dfrac{\sqrt{1-z^2}}{1-z\eta_3}\eta_1 \\ \dfrac{\sqrt{1-z^2}}{1-z\eta_3}\eta_2 \\ \dfrac{z+\eta_3}{1-z\eta_3} \end{pmatrix}$$

Theorem

Let $z \in [0,1)$. *The generalized Radon transform* $\mathcal{U}$ *can be represented with the operators* $\mathcal{M}, \mathcal{F}, \mathcal{N} \colon L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ *via*

$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M}.$$

These operators are defined for $f \in C(\mathbb{S}^2)$ by

- $\mathcal{M}f(\boldsymbol{\xi}) = \dfrac{\sqrt{1-z^2}}{1+z\xi_3}\,[f \circ h](\boldsymbol{\xi})$
- $\mathcal{F}$ ... spherical Radon transform
- $\mathcal{N}f(\boldsymbol{\xi}) = f\left(\dfrac{1}{\sqrt{1-z^2\xi_3^2}}\left(\xi_1, \xi_2, \sqrt{1-z^2}\xi_3\right)\right)$

Nullspace of $\mathcal{U}$

Theorem

$\mathbf{R}$ ... Reflection of the sphere about the point $\zeta = (0, 0, z)^\top$

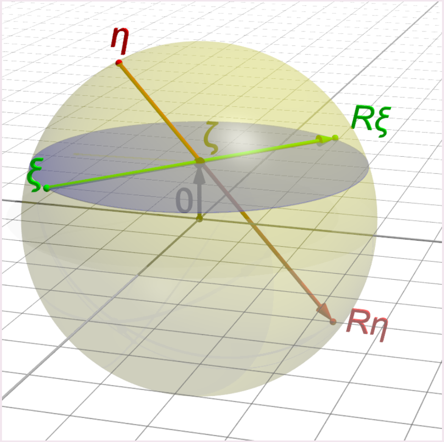
$f \in L^2(\mathbb{S}^2)$

We have

$$\mathcal{U}f = 0$$

if and only if for almost all $\eta \in \mathbb{S}^2$

$$f(\eta) = -f(\mathbf{R}\eta)\frac{1 - z^2}{1 + z^2 - 2z\eta_3}.$$



Nullspace of $\mathcal{U}$

Theorem

$\mathbf{R}$ ... Reflection of the sphere about the point $\zeta = (0, 0, z)^\top$

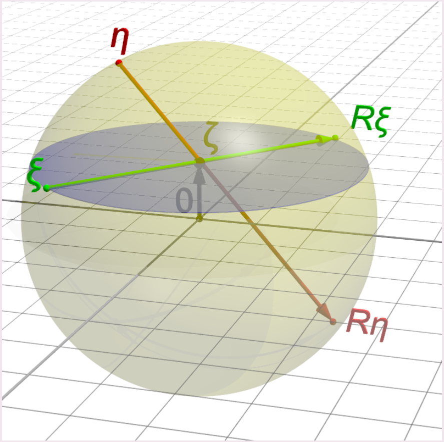
$f \in L^2(\mathbb{S}^2)$

We have

$$\mathcal{U} f = 0$$

if and only if for almost all $\boldsymbol{\eta} \in \mathbb{S}^2$

$$f(\boldsymbol{\eta}) = -f(\mathbf{R}\boldsymbol{\eta}) \frac{1 - z^2}{1 + z^2 - 2z\eta_3}.$$



Range of $\mathcal{U}$

Theorem

The generalized Radon transform

$$\mathcal{U}\colon \tilde{L}^2_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is continuous and bijective.

- $\tilde{L}^2_{\mathrm{e}}(\mathbb{S}^2) = \left\{ f \in L^2(\mathbb{S}^2) \mid f(\boldsymbol{\eta}) = f(\mathbf{R}\boldsymbol{\eta}) \dfrac{1-z^2}{1+z^2-2z\eta_3} \right\}$
- $H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$ ... Sobolev space of smoothness $1/2$ that contains only even functions

Sketch of proof

We show

$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M} \colon \tilde{L}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$$

is continuous and bijective.

- $\mathcal{M} \colon \tilde{L}_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ is unitary
- $\mathcal{F} \colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective
- $\mathcal{N}, \mathcal{N}^{-1} \colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ are continuous and $\mathcal{N}, \mathcal{N}^{-1} \colon H_{\mathrm{e}}^1(\mathbb{S}^2) \to H_{\mathrm{e}}^1(\mathbb{S}^2)$ are continuous
- $\mathcal{N} \colon H_{\mathrm{e}}^{1/2}(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective (shown with interpolation theory [Triebel, 1995])

Sketch of proof

We show

$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M}\colon \tilde{L}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$$

is continuous and bijective.

- $\mathcal{M}\colon \tilde{L}_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ is unitary
- $\mathcal{F}\colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective
- $\mathcal{N}, \mathcal{N}^{-1}\colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ are continuous and $\mathcal{N}, \mathcal{N}^{-1}\colon H_{\mathrm{e}}^1(\mathbb{S}^2) \to H_{\mathrm{e}}^1(\mathbb{S}^2)$ are continuous
- $\mathcal{N}\colon H_{\mathrm{e}}^{1/2}(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective (shown with interpolation theory [Triebel, 1995])

Sketch of proof

We show

$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M} \colon \tilde{L}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$$

is continuous and bijective.

- $\mathcal{M} \colon \tilde{L}_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ is unitary
- $\mathcal{F} \colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective
- $\mathcal{N}, \mathcal{N}^{-1} \colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ are continuous and $\mathcal{N}, \mathcal{N}^{-1} \colon H_{\mathrm{e}}^1(\mathbb{S}^2) \to H_{\mathrm{e}}^1(\mathbb{S}^2)$ are continuous
- $\mathcal{N} \colon H_{\mathrm{e}}^{1/2}(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective (shown with interpolation theory [Triebel, 1995])

Sketch of proof

We show

$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M}\colon \tilde{L}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$$

is continuous and bijective.

- $\mathcal{M}\colon \tilde{L}_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ is unitary
- $\mathcal{F}\colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective
- $\mathcal{N}, \mathcal{N}^{-1}\colon L_{\mathrm{e}}^2(\mathbb{S}^2) \to L_{\mathrm{e}}^2(\mathbb{S}^2)$ are continuous and $\mathcal{N}, \mathcal{N}^{-1}\colon H_{\mathrm{e}}^1(\mathbb{S}^2) \to H_{\mathrm{e}}^1(\mathbb{S}^2)$ are continuous
- $\mathcal{N}\colon H_{\mathrm{e}}^{1/2}(\mathbb{S}^2) \to H_{\mathrm{e}}^{1/2}(\mathbb{S}^2)$ is continuous and bijective (shown with interpolation theory [Triebel, 1995])

Inversion formula

Theorem

Let $z \in [0,1)$ and $f \in \tilde{L}^2_{\mathrm{e}}(\mathbb{S}^2)$. For $\boldsymbol{\eta} \in \mathbb{S}^2$,

$$f(\boldsymbol{\eta}) = \frac{1-z^2}{2\pi(1-zv)} \frac{\mathrm{d}}{\mathrm{d}u} \int_0^u \int_{\mathscr{S}_z(\boldsymbol{\eta},w)} \mathcal{U}f \left(\frac{(\sqrt{1-z^2}(\xi_1,\xi_2),\xi_3)}{\sqrt{1-z^2+z^2\xi_3^2}} \right) \mathrm{d}s(\boldsymbol{\xi}) \frac{\mathrm{d}w}{\sqrt{u^2-w^2}} \bigg|_{u=1}$$

where $\mathrm{d}s$ is the arc-length on the circle

$$\mathscr{S}_z(\boldsymbol{\eta},w) = \left\{ \boldsymbol{\xi} \in \mathbb{S}^2 \mid \left\langle \boldsymbol{\xi}, \left(\sqrt{1-z^2}(\eta_1,\eta_2), \eta_3 - z \right) \right\rangle = (1-z\eta_3)\sqrt{1-w^2} \right\}.$$

- based on an inversion formula of the spherical Radon transform by [Helgason, 1980]

Inversion formula

Theorem

Let $z \in [0,1)$ and $f \in \tilde{L}^2_{\mathrm{e}}(\mathbb{S}^2)$. For $\boldsymbol{\eta} \in \mathbb{S}^2$,

$$f(\boldsymbol{\eta}) = \frac{1-z^2}{2\pi(1-zv)} \frac{\mathrm{d}}{\mathrm{d}u} \int_0^u \int_{\mathscr{S}_z(\boldsymbol{\eta},w)} \mathcal{U}f\left(\frac{(\sqrt{1-z^2}(\xi_1,\xi_2),\xi_3)}{\sqrt{1-z^2+z^2\xi_3^2}}\right) \mathrm{d}s(\boldsymbol{\xi}) \frac{\mathrm{d}w}{\sqrt{u^2-w^2}}\Bigg|_{u=1}$$

where $\mathrm{d}s$ is the arc-length on the circle

$$\mathscr{S}_z(\boldsymbol{\eta},w) = \left\{\boldsymbol{\xi} \in \mathbb{S}^2 \mid \left\langle \boldsymbol{\xi}, \left(\sqrt{1-z^2}(\eta_1,\eta_2),\eta_3-z\right)\right\rangle = (1-z\eta_3)\sqrt{1-w^2}\right\}.$$

- based on an inversion formula of the spherical Radon transform by [Helgason, 1980]

Numerical inversion (via Fourier expansion)

- The last inversion formula is numerically unstable
- Utilize factorization

$$\mathcal{U}^{-1} = \mathcal{M}^{-1}\mathcal{F}^{-1}\mathcal{N}^{-1}$$

- $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$ can be computed explicitly
- For $\mathcal{F}^{-1}$: **Fourier expansion of the spherical Radon transform** combined with the mollifier method (as regularization)

  [Louis et al., 2011] [Hielscher & Q., 2015]
- Efficient implementation of the spherical Fourier transform available

  [Keiner & Potts, 2008]

Numerical inversion (via Fourier expansion)

- The last inversion formula is numerically unstable
- Utilize factorization

$$\mathcal{U}^{-1} = \mathcal{M}^{-1}\mathcal{F}^{-1}\mathcal{N}^{-1}$$

- $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$ can be computed explicitly
- For $\mathcal{F}^{-1}$: **Fourier expansion of the spherical Radon transform** combined with the mollifier method (as regularization)
  [Louis et al., 2011] [Hielscher & Q., 2015]
- Efficient implementation of the spherical Fourier transform available
  [Keiner & Potts, 2008]

Numerical inversion (via Fourier expansion)

- The last inversion formula is numerically unstable
- Utilize factorization

$$\mathcal{U}^{-1} = \mathcal{M}^{-1}\mathcal{F}^{-1}\mathcal{N}^{-1}$$

- $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$ can be computed explicitly
- For $\mathcal{F}^{-1}$: **Fourier expansion of the spherical Radon transform** combined with the mollifier method (as regularization)

  [Louis et al., 2011] [Hielscher & Q., 2015]
- Efficient implementation of the spherical Fourier transform available

  [Keiner & Potts, 2008]

Numerical inversion (via Fourier expansion)

- The last inversion formula is numerically unstable
- Utilize factorization

$$\mathcal{U}^{-1} = \mathcal{M}^{-1}\mathcal{F}^{-1}\mathcal{N}^{-1}$$

- $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$ can be computed explicitly
- For $\mathcal{F}^{-1}$: **Fourier expansion of the spherical Radon transform** combined with the mollifier method (as regularization)

  [Louis et al., 2011] [Hielscher & Q., 2015]
- Efficient implementation of the spherical Fourier transform available

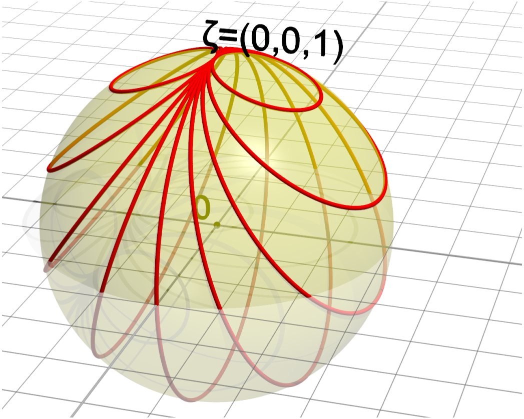
  [Keiner & Potts, 2008]

Numerical inversion (via Fourier expansion)

- The last inversion formula is numerically unstable
- Utilize factorization
$$\mathcal{U}^{-1} = \mathcal{M}^{-1}\mathcal{F}^{-1}\mathcal{N}^{-1}$$
- $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$ can be computed explicitly
- For $\mathcal{F}^{-1}$: **Fourier expansion of the spherical Radon transform** combined with the mollifier method (as regularization)
[Louis et al., 2011] [Hielscher & Q., 2015]
- Efficient implementation of the spherical Fourier transform available
[Keiner & Potts, 2008]

The case $z = 1$ (spherical slice transform) [Abouelaz & Daher, 1993]

$$\mathcal{V}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi},\boldsymbol{\eta}\rangle=\xi_3} f(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$

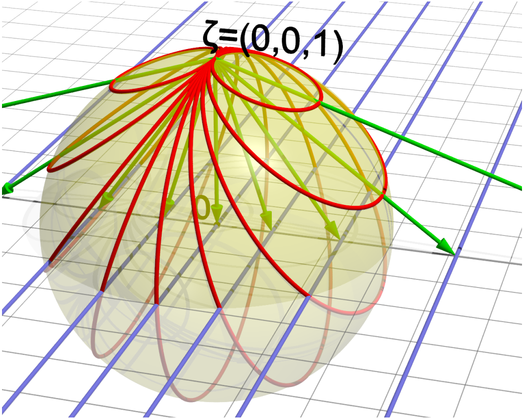


- "Circles through the North pole"
- Stereographic projection turns circles into lines in the plane
- $\mathcal{V}$ is injective for all bounded functions

[Rubin, 2015]

The case $z = 1$ (spherical slice transform) [Abouelaz & Daher, 1993]

$$\mathcal{V}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi},\boldsymbol{\eta}\rangle = \xi_3} f(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$

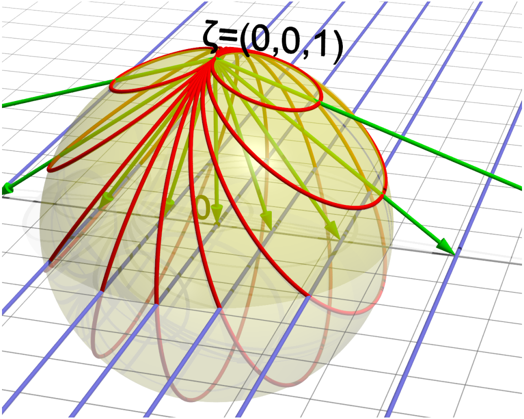


- "Circles through the North pole"
- Stereographic projection turns circles into lines in the plane
- $\mathcal{V}$ is injective for all bounded functions

[Rubin, 2015]

The case $z = 1$ (spherical slice transform) [Abouelaz & Daher, 1993]

$$\mathcal{V}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \xi_3} f(\boldsymbol{\eta})\, \mathrm{d}s(\boldsymbol{\eta})$$



- “Circles through the North pole”
- Stereographic projection turns circles into lines in the plane
- $\mathcal{V}$ is injective for all bounded functions

[Rubin, 2015]

Selected bibliography

P. Funk.
Über Flächen mit lauter geschlossenen geodätischen Linien.
Math. Ann., 74(2): 278 – 300, 1913.

Y. Salman.
An inversion formula for the spherical transform in S^2 for a special family of circles of integration.
Anal. Math. Phys., 6(1): 43 – 58, 2016.

M. Quellmalz.
A generalization of the Funk–Radon transform to circles passing through a fixed point.
Preprint, Faculty of Mathematics, TU Chemnitz, 2015.

\endinput