The cone-beam transform and spherical convolution operators

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The cone-beam transform consists of integrating a function defined on the three-dimensional space along every ray that starts on a certain scanning set. Based on Grangeat's formula, Louis [2016, *Inverse Problems* **32** 115005] states reconstruction formulas based on a new generalized Funk–Radon transform on the sphere.

In this article, we give a singular value decomposition of this generalized Funk–Radon transform. We use this result to derive a singular value decomposition of the cone-beam transform with sources on the sphere thus generalizing a result of Kazantsev [2015, *J. Inverse Ill-Posed Probl.* 23(2):173–185].

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1 Introduction

The cone-beam transform integrates a function $f\colon \mathbb{R}^d \to \mathbb{R}$ along every ray that starts in a certain scanning set $\Gamma \subset \mathbb{R}^d$. We define the cone-beam transform, or divergent beam X-ray transform, by

$$\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}) = \int_0^\infty f(\boldsymbol{a} + t\boldsymbol{\omega})\,\mathrm{d}t, \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-1},\ \boldsymbol{a} \in \Gamma.$$

The cone-beam transform is widely used in medical imaging and nondestructive testing of three-dimensional objects. Uniqueness of the reconstruction was shown under rather weak assumptions, namely that Γ is an infinite set with positive distance to the convex

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hull of the support of f, see [13]. An explicit inversion formula [35] is known in case the Tuy–Kirillov completeness condition is satisfied, which states that the scanning set Γ intersects every hyperplane hitting $\operatorname{supp} f$ transversally, see [26, Chap. 2]. In 3D, if the Tuy–Kirillov condition is not satisfied, one can stably detect singularities of f only along planes that meet the scanning curve Γ, see [29].

The present article's main focus is the setting where the scanning set Γ covers the whole sphere, where, however, in most practical application, consider Γ to be only a curve, cf. [8, 33]. In 2015, Kazantsev [16] showed the singular value decomposition of the cone-beam transform $\mathcal{D}$ where the function f is supported inside the unit ball $\mathbb{B}^3$ and the scanning set Γ is the sphere $\mathbb{S}^2$. The singular value decomposition of the parallel beam X-ray transform is due to Maaß [20]. In 2016, Louis [19] gave new inversion formulas for the cone-beam transform for dimension $d = 3$. The proof of these formulas utilized Grangeat's formula and a generalized Funk–Radon transform $\mathcal{S}^{(j)}$ on the sphere, which is defined for a function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ by

$$\mathcal{S}_d^{(j)} f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \delta^{(j)}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

where $\delta^{(j)}$ denotes the j-th derivative of the Dirac delta function and $j \in \mathbb{N}_0$. This definition can be imagined by first taking the j-th derivative of f in direction of $\boldsymbol{\xi}$ and then computing the integral along the subsphere perpendicular to $\boldsymbol{\xi}$, which is a great circle for $d = 3$. For $j = 0$, we obtain the Funk or Funk–Radon transform $\mathcal{S}_d^{(0)}$, which assigns to f its integrals along all great circles. The generalized Funk–Radon transform $\mathcal{S}_d^{(j)}$ belongs to the class of convolution operators on the sphere, cf. [15].

The aim of the present paper is twofold. Firstly, we perform a comprehensive analysis on the properties of the generalized Funk–Radon transform $\mathcal{S}_d^{(j)}$. Secondly, we utilize Grangeat's formula and our previous findings in order to obtain the singular value decomposition of the cone-beam transform $\mathcal{D}$.

In Theorem 3.2, we derive the singular value decomposition of the generalized Funk–Radon transform $\mathcal{S}_d^{(j)}$. This allows us to provide a characterization of its nullspace and range in Section 3.2. In particular, we show that $\mathcal{S}_d^{(j)}$ is a continuous and open operator between the Sobolev spaces $H^s(\mathbb{S}^{d-1}) \to H^{s-j+\frac{d-2}{2}}(\mathbb{S}^{d-1})$ for $s \in \mathbb{R}$. This behavior is explained by the fact that $\mathcal{S}_d^{(j)}$ consists of j derivatives and an integration along a $d-2$ dimensional manifold. In Section 3.3, we consider special cases of $\mathcal{S}^{(j)}$, which equals the hemispherical transform for $j = -1$ and the spherical cosine transform for $j = -2$. The similarly defined integro-differential Radon transform from Makai et al. [21] and the Blaschke–Levy representation coincide with $\mathcal{S}_d^{(j)}$ for certain but not all parameters, see Section 3.4.

Grangeat's formula states a connection between the cone-beam transform $\mathcal{D}$, the generalized Funk–Radon transform $\mathcal{S}_d^{(d-2)}$ and the Radon transform, which computes the integrals along hyperplanes in $\mathbb{R}^d$. Based on Grangeat's formula and our results on $\mathcal{S}_d^{(j)}$, we derive a singular value decomposition of the cone-beam transform $\mathcal{D}$ in Section 4.2, where we assume that the function f is supported on the unit ball $\mathbb{B}^d$, the spatial dimension d is odd and the scanning set is $\Gamma = \mathbb{S}^{d-1}$. This contains an alternative proof of the result of Kazantsev [16] for $d = 3$. We analyze the asymptotic behavior of the singular values of the cone-beam transform in Section 4.3. The smallest singular values grow of the order $\mathcal{O}(m^{-1/2})$ independently of the spatial dimension d, which means that the inversion is ill-posed of degree 1/2, which is the same as for the Radon transform in 2D. We also obtain a constant upper bound on the singular values. However, it is open whether this bound is strict.

The outline of this paper is as follows. In Section 2, we summarize some basic facts about spherical harmonics. In Section 3, we show the singular value decomposition of the generalized Funk–Radon transform $\mathcal{S}_d^{(j)}$ and its bijectivity in certain Sobolev spaces. In Section 4, we prove the singular value decomposition of the cone-beam transform $\mathcal{D}$ and compute bounds on the singular values. Finally, we state our previous results for the practically most relevant case $\mathbb{B}^3$ in Section 5.

2 Harmonic analysis on the sphere

In this section, we are going to summarize some basic facts about harmonic analysis on the $(d-1)$-dimensional unit sphere $\mathbb{S}^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$ as it can be found in [3]. We denote the volume of $\mathbb{S}^{d-1}$ by

$$\left|\mathbb{S}^{d-1}\right| = \int_{\mathbb{S}^{d-1}} \mathrm{d}\mathbb{S}^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \begin{cases} \frac{2\pi^{d/2}}{(\frac{d}{2}-1)!}, & d \text{ even} \\ \frac{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}}{(d-2)!!}, & d \text{ odd}. \end{cases} \tag{2.1}$$

In the following, we give an orthonormal basis of the Lebesgue space $L^2(\mathbb{S}^{d-1})$, which is the space of square-integrable functions $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{S}^{d-1})} = \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi})\, g(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}.$$

The Legendre polynomial $P_{n,d}$ of degree $n \in \mathbb{N}_0$ in dimension $d \geq 3$ is given by the Rodrigues formula [3, (2.70)]

$$P_{n,d}(t) = (-1)^n \frac{(d-3)!!}{(2n+d-3)!!} (1-t^2)^{\frac{3-d}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (1-t^2)^{n+\frac{d-3}{2}}, \qquad t \in [-1,1]. \tag{2.2}$$

The classical Legendre polynomials are $P_n = P_{n,3}$. Here, we use this name also in the general situation $d > 3$ as in [24, 3]. The Legendre polynomials are orthogonal with respect to the weight function $w_d(t) = (1-t^2)^{\frac{d-3}{2}}$. They satisfy the orthogonality relation

$$\langle P_{n,d}, P_{m,d}\rangle_{w_d} = \int_{-1}^{1} P_{n,d}(t)\, P_{m,d}(t)\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t = \delta_{n,m} \frac{\left|\mathbb{S}^{d-1}\right|}{N_{n,d}\left|\mathbb{S}^{d-2}\right|}, \tag{2.3}$$

where

$$N_{n,d} = \frac{(2n+d-2)\,(n+d-3)!}{n!\,(d-2)!}. \tag{2.4}$$

Up to normalization, the Legendre polynomial $P_{n,d}$ is equal to the Gegenbauer or ultraspherical polynomial [3, (2.145)]

$$C_n^{(\frac{d-2}{2})} = \binom{n+d-3}{n} P_{n,d}. \tag{2.5}$$

The Gegenbauer polynomial $C_n^{(\alpha)}$ for $\alpha > -1/2$ satisfies the explicit expression [1, 22.3]

$$C_n^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m \, \Gamma(n-m+\alpha)}{m! \, (n-2m)!} (2t)^{n-2m}. \tag{2.6}$$

and the Rodrigues formula

$$C_n^{(\alpha)}(t) = \frac{(-1)^n \, \Gamma(\alpha + \frac{1}{2}) \, \Gamma(n+2\alpha)}{2^n \, n! \, \Gamma(2\alpha) \, \Gamma(\alpha + n + \frac{1}{2})} (1-t^2)^{\frac{1}{2}-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (1-t^2)^{n+\alpha-\frac{1}{2}}. \tag{2.7}$$

We define the space $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ as the range of the unnormalized projection operator

$$L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1}), \; f \mapsto \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi}) \, P_{n,d}(\boldsymbol{\xi}^\top(\cdot)) \, \mathrm{d}\boldsymbol{\xi}.$$

The space $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ consists of harmonic polynomials that are homogeneous of degree n, restricted to the sphere $\mathbb{S}^{d-1}$. Let $Y_{n,d}^k$ for $k = 1, \ldots, N_{n,d}$, be an orthonormal basis of $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ in $L^2(\mathbb{S}^{d-1})$. The addition theorem [3, (2.24)] states that for $n \in \mathbb{N}_0$

$$\sum_{k=1}^{N_{n,d}} Y_{n,d}^k(\boldsymbol{\xi}) \, \overline{Y_{n,d}^k(\boldsymbol{\eta})} = \frac{N_{n,d}}{\left|\mathbb{S}^{d-1}\right|} \, P_{n,d}(\boldsymbol{\xi}^\top \boldsymbol{\eta}), \qquad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^{d-1}. \tag{2.8}$$

We define the Gaunt coefficients

$$G_{n_1,k_1,n_2,k_2}^{n,k,d} = \int_{\mathbb{S}^{d-1}} Y_{n_1,d}^{k_1}(\boldsymbol{\xi}) \, Y_{n_2,d}^{k_2}(\boldsymbol{\xi}) \, \overline{Y_{n,d}^k(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi}.$$

Then the product of two spherical harmonics can be written as the sum

$$Y_{n_1,d}^{k_1}(\boldsymbol{\xi}) \, Y_{n_2,d}^{k_2}(\boldsymbol{\xi}) = \sum_{\substack{n=|n_1-n_2| \\ n-n_1-n_2 \text{ even}}}^{n_1+n_2} \sum_{k=1}^{N_{n,d}} G_{n_1,k_1,n_2,k_2}^{n,k_1+k_2,d} \, Y_{n,d}^k(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{2.9}$$

3 Spherical convolution

The spherical convolution of a function $\psi\colon [-1,1] \to \mathbb{C}$ with a function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ is defined by

$$[\psi \star f](\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta}) \, \psi(\boldsymbol{\xi}^\top \boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

The Funk–Hecke formula [3, Thm. 2.22] states that for a spherical harmonic $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ and $\psi \in L^1[-1,1]$ where $\int_{-1}^1 |\psi(t)| \, (1-t^2)^{\frac{d-3}{2}} \, \mathrm{d}t$ is finite, we have

$$[\psi \star Y_{n,d}](\boldsymbol{\xi}) = Y_{n,d}(\boldsymbol{\xi}) \, \left|\mathbb{S}^{d-2}\right| \int_{-1}^1 \psi(t) \, P_{n,d}(t) \, (1-t^2)^{\frac{d-3}{2}} \, \mathrm{d}t. \tag{3.1}$$

3.1 Generalized Funk–Radon transform

For $j \in \mathbb{N}_0$, we define the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ for $f \in C^\infty(\mathbb{S}^{d-1})$ by [19]

$$\mathcal{S}_d^{(j)} f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \delta^{(j)}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

Here, $\delta^{(j)}$ denotes the j-th derivative of the Dirac delta distribution, which is defined by its application to a test function $\psi \in C^\infty[-1,1]$

$$\int_{-1}^{1} \delta^{(j)}(t)\, \psi(t)\, \mathrm{d}t = (-1)^j \int_{-1}^{1} \delta(t)\, \psi^{(j)}(t)\, \mathrm{d}t = (-1)^j \psi^{(j)}(0).$$

For $j = 0$, the operator $\mathcal{S}^{(0)}$ is the Funk–Radon transform, cf. [30, L. 2.2]. As for now, we define $\mathcal{S}_d^{(j)}$ only for smooth functions. However, we will later extend it by density to appropriate Sobolev spaces in Section 3.2.

Remark 3.1. We explain the above definition of $\mathcal{S}_d^{(j)}$ and justify why we can apply the Funk–Hecke formula (3.1) for $\psi = \delta^{(j)}$. Let $f \in C^\infty(\mathbb{S}^{d-1})$. We observe that for any $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$

$$\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta} = t} f(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta})\, \mathrm{d}t,$$

where $\mathrm{d}\lambda$ is the standard surface measure on the sub-sphere $\{\boldsymbol{\eta} \in \mathbb{S}^{d-1} : \boldsymbol{\xi}^\top \boldsymbol{\eta} = t\}$. Then we have

$$\begin{aligned}
\mathcal{S}_d^{(j)} f(\boldsymbol{\xi}) &= \int_{\mathbb{S}^{d-1}} \delta^{(j)}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} \\
&= \int_{-1}^{1} \delta^{(j)}(t)\, (1-t^2)^{-\frac{1}{2}} \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta} = t} f(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta})\, \mathrm{d}t \\
&= (-1)^j \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j (1-t^2)^{-\frac{1}{2}} \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta} = t} f(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta}) \Bigg|_{t=0}.
\end{aligned}$$

We use the following generalized Funk–Hecke formula [5, (4.2.10)]

$$\int_{\boldsymbol{\xi}^\top \boldsymbol{\eta} = t} Y_{n,d}^k(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta}) = \left|\mathbb{S}^{d-2}\right| \, (1-t^2)^{\frac{d-2}{2}}\, P_{n,d}(t)\, Y_{n,d}^k(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in (-1,1).$$

Hence,

$$\mathcal{S}_d^{(j)} Y_{n,d}^k(\boldsymbol{\xi}) = \left|\mathbb{S}^{d-2}\right| \, (-1)^j \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j (1-t^2)^{\frac{d-3}{2}} P_{n,d}(t) \Bigg|_{t=0} Y_{n,d}^k(\boldsymbol{\xi}). \tag{3.2}$$

(3.2) can also be obtained by applying the Funk–Hecke formula (3.1) for $\psi = \delta^{(j)}$. □

In the following, we use double factorials defined by $n!! = n(n-2)\cdots 2$ for n even or $n!! = n(n-2)\cdots 1$ for n odd and $0!! = 1$. The Gamma function is defined for $x > 0$ by $\Gamma(x) = \int_0^\infty y^{x-1}\, \mathrm{e}^{-y}\, \mathrm{d}y$ and satisfies $\Gamma(x+1) = x\,\Gamma(x)$ as well as $\Gamma(n) = (n-1)!$ if n is a positive integer.

Theorem 3.2. *Let $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}_d^{(j)} \colon C(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ satisfies the eigenvalue decomposition*

$$\mathcal{S}_d^{(j)} Y_{n,d}^k = \hat{\mathcal{S}}_d^{(j)}(n)\, Y_{n,d}^k, \qquad n \in \mathbb{N}_0,\ k = 1, \ldots, N_{n,d},$$

with the eigenvalues for $n+j$ even and ($n \geq j-d+3$ or d even)

$$\hat{\mathcal{S}}_d^{(j)}(n) = \left|\mathbb{S}^{d-2}\right| (-1)^{\frac{n+j}{2}} \frac{(n+j-1)!!\,(d-3)!!}{(n-j+d-3)!!} \tag{3.3}$$

$$= \pi^{\frac{d-2}{2}} (-1)^{\frac{n+j}{2}}\, 2^{j+1} \frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{n-j+d-1}{2}\right)} \tag{3.4}$$

and otherwise

$$\hat{\mathcal{S}}_d^{(j)}(n) = 0.$$

Proof. Let $n \in \mathbb{N}_0$ and $k \in \{1, \ldots, N_{n,d}\}$. By (3.2), we have

$$\mathcal{S}_d^{(j)} Y_{n,d}(\boldsymbol{\xi}) = \left|\mathbb{S}^{d-2}\right| (-1)^j\, Y_{n,d}^k(\boldsymbol{\xi}) \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\bigg|_{t=0}. \tag{3.5}$$

By Rodrigues' formula (2.2), we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\bigg|_{t=0} = (-1)^n \frac{(d-3)!!}{(2n+d-3)!!} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j} (1-t^2)^{n+\frac{d-3}{2}}\bigg|_{t=0}. \tag{3.6}$$

We want to apply the generalized binomial theorem, which states for $a, b, z \in \mathbb{C}$

$$(a+b)^z = \sum_{k=0}^{\infty} \binom{z}{k} a^{z-k}\, b^k, \tag{3.7}$$

where the binomial coefficient of $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$ is defined by

$$\binom{z}{k} = \frac{z\,(z-1)\cdots(z-k+1)}{k!}. \tag{3.8}$$

The binomial theorem implies

$$(1-t^2)^{n+\frac{d-3}{2}} = \sum_{k=0}^{\infty} \binom{n+\frac{d-3}{2}}{k} (-1)^k\, t^{2k}. \tag{3.9}$$

Evaluating the $(n+j)$-th derivative of (3.9) at $t=0$ and taking into account that $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\ell t^{2k}\Big|_{t=0} = (2k)!\,\delta_{\ell,2k}$, we obtain if $n+j$ is even

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j} (1-t^2)^{n+\frac{d-3}{2}}\bigg|_{t=0} &= \sum_{k=0}^{\infty} \binom{n+\frac{d-3}{2}}{k} (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j} t^{2k}\bigg|_{t=0} \\ &= \binom{n+\frac{d-3}{2}}{\frac{n+j}{2}} (-1)^{\frac{n+j}{2}}\, (n+j)!\end{aligned} \tag{3.10}$$

and zero otherwise. By its definition in (3.8), the binomial coefficient $\binom{z}{k}$ is zero if and only if both z is a nonnegative integer and $z < k$. Hence, the binomial coefficient $\binom{n+\frac{d-3}{2}}{\frac{n+j}{2}}$ from (3.10) is nonzero if and only if $\frac{d-3}{2}$ is not an integer or $n+\frac{d-3}{2} \geq \frac{n+j}{2}$. This condition can be simplified to that d is even or $n \geq j-d+3$. Then we have

$$\begin{aligned}\binom{n+\frac{d-3}{2}}{\frac{n+j}{2}} &= \frac{\left(\frac{2n+d-3}{2}\right)\left(\frac{2n+d-3}{2}-1\right)\cdots\left(\frac{2n+d-3}{2}-\frac{n+j}{2}+1\right)}{\left(\frac{n+j}{2}\right)!} \\ &= \frac{(2n+d-3)!!}{(n-j+d-3)!!\,(n+j)!!}.\end{aligned} \tag{3.11}$$

Combining (3.6), (3.10) and (3.11), we obtain

$$\begin{aligned}&\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\Bigg|_{t=0} \\ &= (-1)^n \frac{(d-3)!!}{(2n+d-3)!!}\frac{(2n+d-3)!!}{(n+j)!!(n-j+d-3)!!}(-1)^{\frac{n+j}{2}}(n+j)! \\ &= (-1)^{\frac{n-j}{2}}\frac{(n+j-1)!!\,(d-3)!!}{(n-j+d-3)!!}\end{aligned}$$

if $n+j$ is even and ($n \geq j-d+3$ or d even), and zero otherwise. Plugging into (3.5) shows (3.3). Inserting the volume (2.1) of $\mathbb{S}^{d-2}$ into (3.3), we have for $n+j$ even and ($n \geq j-d+3$ or d even)

$$\begin{aligned}\hat{\mathcal{S}}_d^{(j)}(n) &= \left|\mathbb{S}^{d-2}\right|\,(-1)^{\frac{n-j}{2}}\frac{(n+j-1)!!\,(d-3)!!}{(n-j+d-3)!!} \\ &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}(-1)^{\frac{n-j}{2}}\frac{(n+j-1)(n+j-3)\cdots 1}{(n-j+d-3)(n-j+d-5)\cdots(d-1)} \\ &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}(-1)^{\frac{n-j}{2}}\frac{2^{\frac{n+j}{2}}\left(\frac{n+j-1}{2}\right)\left(\frac{n+j-1}{2}-1\right)\cdots\left(\frac{1}{2}\right)}{2^{\frac{n-j}{2}}\left(\frac{n-j+d-1}{2}-1\right)\left(\frac{n-j+d-1}{2}-2\right)\cdots\left(\frac{d-1}{2}\right)} \\ &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}(-1)^{\frac{n-j}{2}}2^j\frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{n-j+d-1}{2}\right)} \\ &= \pi^{\frac{d-2}{2}}(-1)^{\frac{n-j}{2}}2^{j+1}\frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{n-j+d-1}{2}\right)},\end{aligned}$$

which shows (3.4). ■

Theorem 3.2 traces back to [23] for $j=0$ on $\mathbb{S}^2$. The case $j=0$ and d arbitrary was shown in [5].

3.2 $\mathcal{S}^{(j)}$ in Sobolev spaces

In this section, we extend $\mathcal{S}^{(j)}$ to a continuous operator between Sobolev spaces. The spherical Sobolev space $H^s(\mathbb{S}^{d-1})$ of order $s \in \mathbb{R}$ is defined as completion of $C^\infty(\mathbb{S}^{d-1})$

with respect to the Sobolev norm

$$\|f\|_{H^s(\mathbb{S}^{d-1})}^2 = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left(n + \tfrac{d-2}{2}\right)^{2s} \left| \left\langle f, Y_{n,d}^k \right\rangle_{L^2(\mathbb{S}^{d-1})} \right|^2, \tag{3.12}$$

cf. [3, (3.98)]. The spherical harmonics $Y_{n,d}^k$ are dense in $H^s(\mathbb{S}^{d-1})$. The Sobolev spaces are nested: we have $H^s(\mathbb{S}^{d-1}) \hookrightarrow H^t(\mathbb{S}^{d-1})$ whenever $s > t$. The space $H^0(\mathbb{S}^{d-1})$ can be identified with $L^2(\mathbb{S}^{d-1})$. If s is a positive integer, $H^s(\mathbb{S}^{d-1})$ can be imagined as the space of functions defined on $\mathbb{S}^{d-1}$ whose (distributional) derivatives up to order s are in $L^2(\mathbb{S}^{d-1})$.

In the following lemma, we derive an asymptotic approximation of the eigenvalues $\hat{\mathcal{S}}_d^{(j)}(n)$ from Theorem 3.2. We use the notation of asymptotic equivalence $a(n) \simeq b(n)$ for $n \to \infty$ if $\lim_{n\to\infty} a(n)/b(n) = 1$.

Lemma 3.3. *Let $j \in \mathbb{N}_0$. We have for $n \to \infty$ with $n + j$ even and $n \geq j$*

$$\left|\hat{\mathcal{S}}_d^{(j)}(n)\right| \simeq n^{j-\frac{d-2}{2}}\, \pi^{\frac{d-1}{2}}\, 2^{\frac{d}{2}},$$

Proof. Let $n + j$ be even and $n \geq j$. We apply Stirling's approximation of the Gamma function

$$\Gamma(x) \simeq \sqrt{2\pi}\, x^{x-\frac{1}{2}}\, \mathrm{e}^{-x}, \qquad x \to \infty$$

to the eigenvalues (3.4) and obtain for $n \to \infty$

$$\begin{aligned}
\left|\hat{\mathcal{S}}_d^{(j)}(n)\right| &= \pi^{\frac{d-2}{2}}\, 2^{j+1}\, \frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{n-j+d-1}{2}\right)} \\
&\simeq \pi^{\frac{d-2}{2}}\, 2^{j+1}\, \frac{\left(\frac{n+j+1}{2}\right)^{\frac{n+j}{2}} \mathrm{e}^{-\frac{n+j+1}{2}}}{\left(\frac{n-j+d-1}{2}\right)^{\frac{n-j+d-2}{2}} \mathrm{e}^{-\frac{n-j+d-1}{2}}} \\
&= \pi^{\frac{d-2}{2}}\, 2^{\frac{d}{2}}\, \mathrm{e}^{\frac{d-2}{2}-j}\, \frac{(n+j+1)^{\frac{n+j}{2}}}{(n-j+d-1)^{\frac{n-j+d-2}{2}}} \\
&= 2^{\frac{d}{2}}\, \pi^{\frac{d-2}{2}}\, \mathrm{e}^{\frac{d-2}{2}-j} \left(1 + \frac{2j-d+2}{n-j+d-1}\right)^{\frac{n}{2}} (n+j+1)^{\frac{j}{2}}\, (n-j+d-1)^{\frac{j+2-d}{2}}.
\end{aligned}$$

Considering that $\mathrm{e}^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$, we obtain

$$\left|\hat{\mathcal{S}}_d^{(j)}(n)\right| \simeq 2^{\frac{d}{2}}\, \pi^{\frac{d-2}{2}}\, \mathrm{e}^{\frac{d-2}{2}-j}\, \mathrm{e}^{\frac{2j-d+2}{2}}\, n^{j+\frac{2-d}{2}} = 2^{\frac{d}{2}}\, \pi^{\frac{d-2}{2}}\, n^{j-\frac{d-2}{2}}. \qquad \blacksquare$$

The following mapping property of $\mathcal{S}_d^{(j)}$ between Sobolev spaces was shown for $j = 0$ in [34, § 4].

Theorem 3.4. *Let $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}_d^{(j)}$ extends to a continuous operator*

$$\mathcal{S}_d^{(j)} \colon H^s(\mathbb{S}^{d-1}) \to H^{s-j+\frac{d-2}{2}}(\mathbb{S}^{d-1}). \tag{3.13}$$

If $j > \frac{d-2}{2}$, then $\mathcal{S}_d^{(j)} \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ is compact. The nullspace of $\mathcal{S}_d^{(j)}$ is the closed linear span

$$\overline{\operatorname{span}} \left\{ \mathscr{Y}_{n,d}(\mathbb{S}^{d-1}) : n + j \text{ odd or } (n \le j - d + 1 \text{ and } d \text{ odd}) \right\}.$$

If d is odd and $j \ge d-1$, the nullspace of $\mathcal{S}_d^{(j)}$ comprises the sum of all polynomials of degree up to $j-d+1$ and all odd (even) functions whenever j is even (odd). Otherwise, the null-space of $\mathcal{S}_d^{(j)}$ comprises all odd (even) functions whenever j is even (odd).

Proof. By the definition (3.12) of the Sobolev space, $\mathcal{S}_d^{(j)}$ is continuous if and only if the sequence

$$n \mapsto \left| \hat{\mathcal{S}}_d^{(j)}(n) \right| \left(n + \frac{d-2}{2} \right)^{-j + \frac{d-2}{2}}$$

has an upper bound, which follows from Lemma 3.3. The compactness for $j > \frac{d-2}{2}$ follows because then the eigenvalues $\hat{\mathcal{S}}_d^{(j)}(n)$ converge to 0 for $n \to \infty$. The nullspace of $\mathcal{S}_d^{(j)}$ consists of the closed span of all spherical harmonics $Y_{n,d}^k$ where $n \in \mathbb{N}_0$ satisfies $\hat{\mathcal{S}}_d^{(j)}(n) = 0$. ∎

The order of smoothness $s - j + \frac{d-2}{2}$ of the Sobolev space in (3.13) is not unexpected, because $\mathcal{S}_d^{(j)}$ consists of j differentiations, which lower the order of smoothness by j, and the integration along a $(d-2)$-dimensional submanifold, which raises the order of smoothness by $\frac{d-2}{2}$.

3.3 Special cases of j

In this section, we take a look at $\mathcal{S}_d^{(j)}$ for certain special choices of j, some of which are already well-known operators from literature. Even though $\mathcal{S}_d^{(j)}$ was initially defined only for $j \in \mathbb{N}_0$, we can extend it to negative j by the singular value decomposition (3.3). Inserting $j = -1$ in equation (3.4) of the eigenvalues yields for odd n

$$\hat{\mathcal{S}}_d^{(-1)}(n) = \left| \mathbb{S}^{d-2} \right| (-1)^{\frac{n-1}{2}} \frac{(n-2)!!\,(d-3)!!}{(n+d-2)!!} = 2\,(-1)^{\frac{n-1}{2}} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+d}{2}\right)}.$$

Hence, $\mathcal{S}_d^{(-1)}$ is the modified hemispherical transform [31]

$$\mathcal{S}_d^{(-1)} f(\boldsymbol{\xi}) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\; f(\boldsymbol{\eta}) \,\mathrm{d}\boldsymbol{\eta}. \tag{3.14}$$

Inserting $j = -2$ gives the eigenvalues

$$\hat{\mathcal{S}}_d^{(-2)}(n) = \begin{cases} 2 \left| \mathbb{S}^{d-2} \right| (-1)^{\frac{n-2}{2}} \frac{(n-3)!!\,(d-2)!!}{(n+d-1)!!}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

of the spherical cosine transform, cf. [12, L. 3.4.5],

$$\mathcal{S}_d^{(-2)} f(\boldsymbol{\xi}) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \left|\boldsymbol{\xi}^\top \boldsymbol{\eta}\right| \, f(\boldsymbol{\eta}) \,\mathrm{d}\boldsymbol{\eta}.$$

In the case $d = 2j+2$, which in particular implies that d is even, we have the eigenvalues

$$\hat{\mathcal{S}}_{2j+2}^{(j)}(n) = \begin{cases} |\mathbb{S}^{2j}| \, (-1)^{\frac{n+j}{2}} \, (2j-1)!! = 2\,(2\pi)^j \, (-1)^{\frac{n+j}{2}}, & n+j \text{ even} \\ 0, & n+j \text{ odd}, \end{cases}$$

which are, except for their sign, independent of n. Hence, if j is even (odd), the operator $\mathcal{S}_{2j+2}^{(j)} \colon L^2(\mathbb{S}^{2j+1}) \to L^2(\mathbb{S}^{2j+1})$ restricted to the even (odd) functions is an isometry.

The following theorem shows an inversion formula for the Funk–Radon transform in even dimensions.

Theorem 3.5. *Let $d \geq 2$ be even. Then any even function $f \colon \mathbb{S}^{d-1} \to \mathbb{C}$ can be reconstructed from its Funk–Radon transform $g = \mathcal{S}_d^{(0)} f$ by*

$$f = \frac{1}{|\mathbb{S}^{d-2}|^2 \, ((d-3)!!)^2} \, \mathcal{S}_d^{(d-2)} g.$$

Proof. Let $n \in \mathbb{N}_0$ be even. On the one hand, we have the eigenvalues

$$\hat{\mathcal{S}}_d^{(d-2)}(n) = \left|\mathbb{S}^{d-2}\right| \, (-1)^{\frac{n+d-2}{2}} \frac{(n+d-3)!! \, (d-3)!!}{(n-1)!!}.$$

On the other hand, the Funk–Radon transform $\mathcal{S}_d^{(0)}$ has the eigenvalues

$$\hat{\mathcal{S}}_d^{(0)}(n) = \left|\mathbb{S}^{d-2}\right| \, (-1)^{\frac{n}{2}} \frac{(n-1)!! \, (d-3)!!}{(n+d-3)!!}.$$

Hence, the product of the two operators has the constant eigenvalues

$$\widehat{\mathcal{S}_d^{(d-2)} \mathcal{S}_d^{(0)}}(n) = \left|\mathbb{S}^{d-2}\right|^2 \, (-1)^{\frac{d-2}{2}} \, (d-3)!!^2.$$

■

3.4 Similar transforms

In this section, we consider two integral transforms, which are equal to $\mathcal{S}_d^{(j)}$ for certain but not all parameters j.

Integro-differential transform For $j \in \mathbb{N}_0$ and $\vartheta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we define the integro-differential transform $\mathcal{R}_{d,\vartheta}^{(j)} \colon C^j(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ by

$$\mathcal{R}_{d,\vartheta}^{(j)} f(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}^\top \boldsymbol{\omega} = 0} \left(\frac{\partial}{\partial \vartheta}\right)^j f(\boldsymbol{\xi} \sin\vartheta + \boldsymbol{\omega} \cos\vartheta) \,\mathrm{d}\boldsymbol{\omega}, \qquad \boldsymbol{\xi} \in \mathbb{S}^d,$$

which was introduced in [21]. The operator $\mathcal{R}_{d,\vartheta}^{(0)}$ has been investigated in [32]. For $j > 0$, we first take the j-th derivative of f perpendicular to the circle of integration. It was shown in [22] that the nullspace of the operator $\mathcal{R}_{d,\vartheta}^{(j)}$ is

$$\overline{\operatorname{span}}\left\{ \mathscr{Y}_{n,d}(\mathbb{S}^{d-1}) : \left(\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right)^{j} P_{n,d}(\sin\vartheta) = 0 \right\}.$$

If $\vartheta = 0$, we write $\mathcal{R}_d^{(j)} = \mathcal{R}_{d,0}^{(j)}$. If $j \geq 1$, the null–space of $\mathcal{R}_d^{(j)} \colon C^j(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ equals for j odd (even) the set $\{f \in C^m(\mathbb{S}^d) : f$ is even (f is the sum of an odd function and a constant)$\}$.

Theorem 3.6. *We have* $\mathcal{S}_d^{(0)} = \mathcal{R}_d^{(0)}$ *and* $\mathcal{S}_d^{(1)} = -\mathcal{R}_d^{(1)}$.

Proof. For $j = 0$, we see that $\mathcal{S}_d^{(0)} = \mathcal{R}_d^{(0)}$ is the Funk–Radon transform. By [22], the operator $\mathcal{R}_d^{(j)}$ has as eigenfunctions the spherical harmonics $Y_{n,d}^k$ and the eigenvalues

$$\hat{\mathcal{R}}_d^{(j)}(n) = \left|\mathbb{S}^{d-2}\right| \frac{\mathrm{d}}{\mathrm{d}\vartheta} P_{n,d}(\sin\vartheta)\Big|_{\vartheta=0}.$$

Theorem 3.2 shows that $\mathcal{S}_d^{(j)}$ has the same eigenfunctions. Hence, the two operators coincide if their respective eigenvalues do. We have for $j = 1$ on the one hand

$$\hat{\mathcal{R}}_d^{(1)}(n) = \left|\mathbb{S}^{d-2}\right| P_{n,d}'(\sin\vartheta)\, \cos\vartheta\big|_{\vartheta=0} = \left|\mathbb{S}^{d-2}\right| P_{n,d}'(0)$$

and on the other hand

$$\hat{\mathcal{S}}_d^{(1)}(n) = -\left|\mathbb{S}^{d-2}\right| \frac{\mathrm{d}}{\mathrm{d}t} P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\Big|_{t=0} = -\left|\mathbb{S}^{d-2}\right| P_{n,d}'(0).$$ ■

Remark 3.7. Theorem 3.6 does not hold for all j. We have for $d = 3$ and $j = 2$

$$\hat{\mathcal{R}}_3^{(2)}(n) = 2\pi\big(P_n''(\sin\vartheta)\, \cos^2\vartheta - P_n'(\sin\vartheta)\, \sin\vartheta\big)\Big|_{\vartheta=0} = 2\pi\, P_n''(0) = \hat{\mathcal{S}}_3^{(2)}(n).$$

However, for $j = 3$

$$\begin{aligned}\hat{\mathcal{R}}_3^{(3)}(n) &= 2\pi\big(P_n'''(\sin\vartheta)\, \cos^3\vartheta - 3P_n''(\sin\vartheta)\, \cos\vartheta\, \sin\vartheta - P_n'(\sin\vartheta)\, \cos\vartheta\big)\Big|_{\vartheta=0} \\ &= 2\pi\left(P_n'''(0) - P_n'(0)\right)\end{aligned}$$

does not coincide with $-\hat{\mathcal{S}}_3^{(3)}(n) = 2\pi\, P_n'''(0)$. □

Blaschke–Levy reprsentation Another related transform is the α-cosine transform or Blaschke–Levy representation [17, 31]

$$\mathcal{H}^{(\alpha)} f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \left|\boldsymbol{\xi}^\top \boldsymbol{\eta}\right|^{\alpha} f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}$$

with singular values

$$\hat{\mathcal{H}}^{(\alpha)}(n) = \begin{cases} (-1)^{n/2} \dfrac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+d+\alpha}{2}\right)}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Hence, for j even and $\alpha = -j-1$, the α-cosine transform $\mathcal{H}^\alpha$ is, up to a constant factor, equal to the generalized Funk–Radon transform $\mathcal{S}_d^{(j)}$.

4 Cone-beam transform

4.1 Connection of Radon and cone-beam transform

Radon transform We define the Radon transform $\mathcal{R}$ on the d-dimensional unit ball $\mathbb{B}^d = \{\boldsymbol{x} \in \mathbb{R}^d : \|x\| \leq 1\}$ by [25, Sec. II.1]

$$\begin{aligned} &\mathcal{R} \colon L^2(\mathbb{B}^d) \to L^2(\mathbb{S}^{d-1} \times [-1,1], w_{d/2}^{-1}) \\ &\mathcal{R}f(\boldsymbol{\omega}, s) = \int_{\boldsymbol{x}^\top \boldsymbol{\omega} = s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \end{aligned} \tag{4.1}$$

with the weight function

$$w_\nu(s) = (1-s^2)^{\nu - 1/2}, \qquad s \in [-1,1].$$

The Radon transform on the unit ball has the following singular value decomposition [18]. For $m \in \mathbb{N}_0$, $l = 0, \ldots, m$ with $m + l$ even and $k = 1, \ldots, N_{l,d}$, we have

$$\mathcal{R}\widetilde{V}_{m,l,k}(\boldsymbol{\omega}, s) = \frac{\sqrt{2m+d}\,\Gamma(\frac{d}{2})\, m!}{2^{1-d}\, \pi^{1-\frac{d}{2}}\, (m+d-1)!} \,(1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s)\, Y_{l,d}^k(\boldsymbol{\omega}), \tag{4.2}$$

where

$$\widetilde{V}_{m,l,k}(s\boldsymbol{\omega}) = \sqrt{2m+d}\, s^l P_{\frac{m-l}{2}}^{\left(0, l+\frac{d-2}{2}\right)}(2s^2-1)\, Y_{l,d}^k(\boldsymbol{\omega}), \qquad s \in [0,1],\ \boldsymbol{\omega} \in \mathbb{S}^{d-1} \tag{4.3}$$

and $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n and orders $\alpha, \beta > -1$. The set

$$\left\{\widetilde{V}_{m,l,k} : l \in \mathbb{N}_0,\ m \in \{l, l+2, l+4, \ldots\},\ k \in \{1, \ldots, N_{l,d}\}\right\}$$

is an orthonormal basis of $L^2(\mathbb{B}^d)$ consisting of polynomials of degree $m \in \mathbb{N}_0$.

Remark 4.1. We compute the norm of $\widetilde{V}_{m,l,k}$ in $L^2(\mathbb{B}^d)$. The Jacobi polynomials satisfy

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1) n!}.$$

We have

$$\|\widetilde{V}_{m,l,k}\|^2_{L^2(\mathbb{B}^d)} = (2m+d)\int_0^1 s^{2l}\left|P^{\left(0,l+\frac{d-2}{2}\right)}_{\frac{m-l}{2}}(2s^2-1)\right|^2 s^{d-1}\,\mathrm{d}s\int_{\mathbb{S}^{d-1}}\left|Y^k_{l,d}(\boldsymbol{\omega})\right|^2\,\mathrm{d}\boldsymbol{\omega}.$$

By the normalization of the spherical harmonics and the substitution $t = 2s^2-1$ with $\mathrm{d}t = 4s\,\mathrm{d}s$, we obtain

$$\begin{aligned}\|\widetilde{V}_{m,l,k}\|^2_{L^2(\mathbb{B}^d)} &= \frac{2m+d}{4}\int_{-1}^1\left(\frac{t+1}{2}\right)^{l+\frac{d-2}{2}}\left|P^{\left(0,l+\frac{d-2}{2}\right)}_{\frac{m-l}{2}}(t)\right|^2\,\mathrm{d}t\\ &= \frac{2m+d}{4}2^{-l-\frac{d-2}{2}}\frac{2^{l+\frac{d}{2}}}{m+\frac{d}{2}}\frac{\Gamma(\frac{m-l}{2}+1)\,\Gamma(\frac{m+l+d}{2})}{(\frac{m-l}{2})!\,\Gamma(\frac{m+l+d}{2})} = 1.\end{aligned}$$

□

Cone-beam transform The cone-beam transform, which is also known as divergent beam X-ray transform, with scanning set $\Gamma \subset \mathbb{R}^d$ is defined by

$$\mathcal{D}f(\boldsymbol{a},\boldsymbol{\omega}) = \int_0^\infty f(\boldsymbol{a}+t\boldsymbol{\omega})\,\mathrm{d}t, \qquad \boldsymbol{\omega}\in\mathbb{S}^{d-1},\ \boldsymbol{a}\in\Gamma.$$

Grangeat's formula There is a relation between the Radon transform and the cone-beam transform. Let $h\colon \mathbb{R}\to\mathbb{R}$ be a function that is homogeneous of degree $1-d$. It was essentially shown in [13] (see also [26, Sec. 2.3] and [28, Sec. 2.2.1]) that

$$\int_{-\infty}^\infty \mathcal{R}f(\boldsymbol{\omega},s)\,h(s-\boldsymbol{a}^\top\boldsymbol{\omega})\,\mathrm{d}s = \int_{\mathbb{S}^{d-1}}\mathcal{D}f(\boldsymbol{a},\boldsymbol{\xi})\,h(\boldsymbol{\omega}^\top\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}. \tag{4.4}$$

Inserting $h = \delta^{(d-2)}$, we obtain Grangeat's formula, which was originally proved for $d=3$ in [11], stating that

$$(-1)^d\left(\frac{\partial}{\partial s}\right)^{d-2}\mathcal{R}f(\boldsymbol{\omega},\boldsymbol{a}^\top\boldsymbol{\omega}) = \mathcal{S}^{(d-2)}_d\mathcal{D}f(\boldsymbol{a},\boldsymbol{\omega}), \tag{4.5}$$

where $\mathcal{S}^{(d-2)}_d$ is applied with respect to $\boldsymbol{\omega}$.

The following theorem gives an alternative version of Grangeat's formula for $d=3$. However, it is not a special case of (4.4), because the function h is not homogeneous of degree -2.

Theorem 4.2. *Let $\boldsymbol{\omega}\in\mathbb{S}^2$ and $\boldsymbol{a}\in\mathbb{R}^3$. We have*

$$-\mathcal{S}^{(-1)}_3\frac{\partial}{\partial s}\mathcal{R}f(\boldsymbol{\omega},\boldsymbol{a}^\top\boldsymbol{\omega}) = \int_{\mathbb{S}^2}h(\boldsymbol{\xi}^\top\boldsymbol{\omega})\,\mathcal{D}f(\boldsymbol{a},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}, \tag{4.6}$$

where

$$h(x) = \frac{2x}{\sqrt{1-x^2}}, \qquad x\in(-1,1).$$

and $\mathcal{S}^{(-1)}_d$ is the modified hemispherical transform (3.14) *applied with respect to $\boldsymbol{\omega}$.*

Proof. Multiplying Grangeat's formula (4.5) for $d = 3$ with $\mathcal{S}^{(-1)}$, we obtain

$$-\mathcal{S}_3^{(-1)} \frac{\partial}{\partial s} \mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top \boldsymbol{\omega}) = \mathcal{S}_3^{(-1)} \mathcal{S}_3^{(1)} \mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}). \tag{4.7}$$

The left-hand side of the previous equation (4.7) is the same as that of equation (4.6). We are going to show the equality of the right-hand side of (4.6) and (4.7) by evaluation for all spherical harmonics $\mathcal{D}f = Y_n^k$, the assertion for general function $\mathcal{D}f$ follows by the density of the spherical harmonics Y_n^k. By [10, 8.922.4], we have

$$h(x) = \frac{2x}{\sqrt{1-x^2}} = \pi \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (2n+1) \frac{(n-2)!!\, n!!}{(n-1)!!\, (n+1)!!} P_n(x), \qquad x \in (-1, 1).$$

Then we have by the Funk–Hecke formula (3.1) for all $n \in \mathbb{N}_0$, $|k| \le n$ and $\boldsymbol{\xi} \in \mathbb{S}^2$

$$\begin{aligned} \int_{\mathbb{S}^2} h(\boldsymbol{\xi}^\top \boldsymbol{\omega})\, Y_n^k(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} &= 2\pi \int_{-1}^{1} h(t)\, P_n(t)\, \mathrm{d}t\, Y_n^k(\boldsymbol{\omega}) \\ &= \begin{cases} 4\pi^2 \frac{(n-2)!!\, n!!}{(n-1)!!\, (n+1)!!}\, Y_n^k(\boldsymbol{\omega}), & n \text{ odd} \\ 0, & n \text{ even}. \end{cases} \end{aligned}$$

On the other hand, the right-hand side of (4.7) evaluates for odd n by (3.3)

$$\mathcal{S}_3^{(-1)}\, \mathcal{S}_3^{(1)}\, Y_n^k = \hat{\mathcal{S}}_3^{(-1)}(n)\, \hat{\mathcal{S}}_3^{(1)}(n)\, Y_n^k = 4\pi^2 \frac{(n-2)!!\, n!!}{(n-1)!!\, (n+1)!!}\, Y_n^k,$$

which shows the assertion. ∎

4.2 Singular value decomposition

In the following, we consider the cone-beam transform $\mathcal{D}$ with scanning set $\Gamma = \mathbb{S}^{d-1}$ and we assume that the function f is supported in the unit ball $\mathbb{B}^d$. We see that $\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}) = 0$ for all $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ with $\boldsymbol{a}^\top \boldsymbol{\omega} \ge 0$ since the ray of integration is outside $\mathbb{B}^d$. We denote the odd part of the cone-beam transform $\mathcal{D}(\boldsymbol{a}, \cdot)$ by

$$\mathcal{D}^{(\text{odd})} f(\boldsymbol{a}, \boldsymbol{\omega}) = \frac{Df(\boldsymbol{a}, \boldsymbol{\omega}) - Df(\boldsymbol{a}, -\boldsymbol{\omega})}{2}.$$

Then

$$\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}) = 2\mathcal{D}^{(\text{odd})} f(\boldsymbol{a}, \boldsymbol{\omega})$$

for all $\boldsymbol{\omega} \in \mathbb{S}^2$ with $\boldsymbol{a}^\top \boldsymbol{\omega} < 0$ and $\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}) = 0$ otherwise.

Lemma 4.3. *Let $m \in \mathbb{N}_0$ and $d \ge 3$ be odd. Then*

$$C_{m+1}^{(\frac{d-2}{2})}(s) = (-1)^{\frac{d-1}{2}} \frac{(d-2)\, m!}{(m+d-1)!} \left(\frac{\partial}{\partial s}\right)^{d-2} (1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s). \tag{4.8}$$

Proof. We denote the right-hand side of (4.8) by $B_{m+1}^{(\frac{d-2}{2})}(s)$, i.e., we set

$$B_{m+1}^{(\frac{d-2}{2})}(s) = (-1)^{\frac{d-1}{2}} \frac{(d-2)\, m!}{(m+d-1)!} \left(\frac{\partial}{\partial s}\right)^{d-2} (1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s).$$

We obtain with Rodrigues' formula (2.7) for the Gegenbauer polynomials

$$\begin{aligned} B_{m+1}^{(\frac{d-2}{2})}(s) &= (-1)^{\frac{d-1}{2}} \frac{(d-2)\, m!}{(m+d-1)!} \frac{(-1)^m\, (\frac{d-1}{2})!\, (m+d-1)!}{2^m\, m!\, (d-1)!\, (m+\frac{d-1}{2})!} \left(\frac{\partial}{\partial s}\right)^{m+d-2} (1-s^2)^{m+\frac{d-1}{2}} \\ &= \frac{(-1)^{m+\frac{d-1}{2}}\, (d-2)\, (\frac{d-1}{2})!}{2^m\, (d-1)!\, (m+\frac{d-1}{2})!} \left(\frac{\partial}{\partial s}\right)^{m+d-2} (1-s^2)^{m+\frac{d-1}{2}}. \end{aligned}$$

We compute with the binomial theorem (3.7)

$$B_{m+1}^{(\frac{d-2}{2})}(s) = \frac{(-1)^{m+\frac{d-1}{2}}\, (d-2)\, (\frac{d-1}{2})!}{2^m\, (d-1)!\, (m+\frac{d-1}{2})!} \sum_{i=0}^{m+\frac{d-1}{2}} (-1)^i \binom{m+\frac{d-1}{2}}{i} \left(\frac{\partial}{\partial s}\right)^{m+d-2} s^{2i}$$

Considering

$$\left(\frac{\partial}{\partial s}\right)^{k} s^{2i} = \frac{(2i)!}{(2i-k)!} s^{2i-k},$$

we have

$$\begin{aligned} &B_{m+1}^{(\frac{d-2}{2})}(s) \\ &\quad= \frac{(-1)^{m+\frac{d-1}{2}}\, (d-2)\, (\frac{d-1}{2})!}{2^m\, (d-1)!\, (m+\frac{d-1}{2})!} \sum_{i=\lceil \frac{m+d-2}{2} \rceil}^{m+\frac{d-1}{2}} \binom{m+\frac{d-1}{2}}{i} \frac{(-1)^i\, (2i)!}{(2i-m-d+2)!}\, s^{2i-m-d+2}. \end{aligned}$$

Shifting the index $i \mapsto l$ with $i = m - l + \frac{d-1}{2}$, we obtain

$$\begin{aligned} &B_{m+1}^{(\frac{d-2}{2})}(s) \\ &= \frac{(-1)^{m+\frac{d-1}{2}}\, (d-2)\, (\frac{d-1}{2})!}{2^m\, (d-1)!\, (m+\frac{d-1}{2})!} \sum_{l=0}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(-1)^{m-l+\frac{d-1}{2}}\, (m+\frac{d-1}{2})!}{(m-l+\frac{d-1}{2})!\, l!} \frac{(2m-2l+d-1)!}{(m+1-2l)!}\, s^{m+1-2l} \\ &= \frac{(d-2)\, (\frac{d-1}{2})!}{2^m\, (d-1)!} \sum_{l=0}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(-1)^l}{(m-l+\frac{d-1}{2})!\, l!} \frac{(2m-2l+d-1)!}{(m+1-2l)!}\, s^{m+1-2l}. \end{aligned}$$

Because

$$\frac{(2m)!}{m!} = \frac{2^m\, (2m)!}{(2m)!!} = 2^m\, (2n-1)!!,$$

we have

$$\begin{aligned}
B_{m+1}^{(\frac{d-2}{2})}(s) &= \frac{(d-2)}{2^m\,(d-2)!!\,2^{\frac{d-1}{2}}} \sum_{l=0}^{\lfloor\frac{m+1}{2}\rfloor} \frac{(-1)^l\,2^{m-l+\frac{d-1}{2}}\,(2m-2l+d-2)!!}{l!\,(m+1-2l)!}\,s^{m+1-2l} \\
&= \frac{1}{(d-4)!!} \sum_{l=0}^{\lfloor\frac{m+1}{2}\rfloor} \frac{(-1)^l\,2^{-l}\,(2m-2l+d-2)!!}{l!\,(m+1-2l)!}\,s^{m+1-2l}.
\end{aligned}$$

We rewrite the quotient of double factorials with the Gamma function

$$\frac{(m+2k)!!}{m!!} = 2^k\left(\frac{m+2k}{2}\right)\left(\frac{m+2k-2}{2}\right)\cdots\left(\frac{m+2}{2}\right) = 2^k\frac{\Gamma\left(\frac{m+2k+2}{2}\right)}{\Gamma\left(\frac{2k+2}{2}\right)}$$

and obtain

$$B_{m+1}^{(\frac{d-2}{2})}(s) = \sum_{l=0}^{\lfloor\frac{m+1}{2}\rfloor} \frac{(-1)^l\,\Gamma(m-l+\frac{d}{2})}{\Gamma(\frac{d-2}{2})\,l!\,(m+1-2l)!}\,(2s)^{m+1-2l},$$

which is exactly the formula (2.6) for the Gegenbauer polynomial $C_{m+1}^{(\frac{d-2}{2})}(s)$. ■

Theorem 4.4. *Let $m \in \mathbb{N}_0$, $l = 0, \ldots, m$ with $l+m$ even, $k \in \{1, \ldots, N_{l,d}\}$ and $d \geq 3$ odd. The odd cone-beam transform $\mathcal{D}^{(odd)}\colon \mathbb{B}^d \to \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ satisfies for $\boldsymbol{a}, \boldsymbol{\omega} \in \mathbb{S}^{d-1}$*

$$\mathcal{D}^{(odd)}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = \mu_{m,d} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^{j}(\boldsymbol{a})} \sum_{n=m+1-l}^{l+m+1}{}' \nu_{n,d} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d}\, Y_{n,d}^{i}(\boldsymbol{\omega}),$$

where $\sum'$ denotes the summation over odd indices, $\widetilde{V}_{m,l,k}$ is given in (4.3) *and*

$$\mu_{m,d} = \sqrt{\frac{2^{d+1}\,\pi^{d-1}}{2m+d}}, \tag{4.9}$$

$$\nu_{n,d} = \frac{(-1)^{\frac{n+1}{2}}\,(n-1)!!}{(n+d-3)!!}. \tag{4.10}$$

Proof. Let $m \in \mathbb{N}_0$, $l \in \{0, \ldots, m\}$ with $m+l$ even, $k \in \{1, \ldots, N_{l,d}\}$ and d be odd. We have by the singular value decomposition (4.2) of the Radon transform and Lemma 4.3

$$\begin{aligned}
&\left(\frac{\partial}{\partial s}\right)^{d-2} \mathcal{R}\widetilde{V}_{m,l,k}(\boldsymbol{\omega}, s) \\
&= \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\,\sqrt{2m+d}\,\Gamma(\frac{d}{2})\,m!}{(m+d-1)!} \left(\frac{\partial}{\partial s}\right)^{d-2} (1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s)\, Y_{l,d}^k(\boldsymbol{\omega}) \\
&= \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\,\sqrt{2m+d}\,\Gamma(\frac{d}{2})\,m!}{(m+d-1)!} (-1)^{\frac{d-1}{2}}\,\frac{(m+d-1)!}{(d-2)\,m!}\, C_{m+1}^{(\frac{d-2}{2})}(s)\, Y_{l,d}^k(\boldsymbol{\omega}) \\
&= \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\,\sqrt{2m+d}\,\Gamma(\frac{d}{2})}{d-2} (-1)^{\frac{d-1}{2}}\, C_{m+1}^{(\frac{d-2}{2})}(s)\, Y_{l,d}^k(\boldsymbol{\omega}).
\end{aligned}$$

By Grangeat's formula (4.5) and the relation (2.5) between the Gegenbauer and the Legendre polynomials, we obtain

$$\mathcal{S}_d^{(d-2)}\mathcal{D}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = (-1)^{\frac{d+1}{2}} \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\,\sqrt{2m+d}\,\Gamma(\frac{d}{2})\,(m+d-2)!}{(m+1)!\,(d-2)!} P_{m+1,d}(\boldsymbol{a}^\top\boldsymbol{\omega})\,Y_{l,d}^k(\boldsymbol{\omega}).$$

By the addition formula (2.8) for spherical harmonics, we have

$$\mathcal{S}_d^{(d-2)}\mathcal{D}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = (-1)^{\frac{d+1}{2}} \left|\mathbb{S}^{d-1}\right| \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}{\sqrt{2m+d}} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^j(\boldsymbol{a})}\,Y_{m+1,d}^j(\boldsymbol{\omega})\,Y_{l,d}^k(\boldsymbol{\omega}).$$

By the multiplication formula (2.9) for spherical harmonics, we see that

$$\mathcal{S}_d^{(d-2)}\mathcal{D}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = (-1)^{\frac{d-1}{2}} \left|\mathbb{S}^{d-1}\right| \frac{2^{d+1}\,\pi^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}{\sqrt{2m+d}} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^j(\boldsymbol{a})} \sideset{}{'}\sum_{n=m+1-l}^{m+1+l} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d} Y_{n,d}^i(\boldsymbol{\omega}).$$

Since the generalized Funk–Radon transform $\mathcal{S}^{(d-2)}$ acts only on odd functions, we have $\mathcal{S}_d^{(d-2)}\mathcal{D} = \mathcal{S}_d^{(d-2)}\mathcal{D}^{(\mathrm{odd})}$. Then the eigenvalue decomposition (3.3) of $\mathcal{S}_d^{(d-2)}$ yields

$$\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = (-1)^{\frac{d+1}{2}} \frac{\left|\mathbb{S}^{d-1}\right|}{\left|\mathbb{S}^{d-2}\right|} \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}{\sqrt{2m+d}} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^j(\boldsymbol{a})}$$
$$\sideset{}{'}\sum_{n=m+1-l}^{l+m+1} (-1)^{\frac{n+d-2}{2}} \frac{(n-1)!!}{(n+d-3)!!\,(d-3)!!} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d} Y_{n,d}^i(\boldsymbol{\omega}).$$

Since, by (2.1),

$$\frac{\left|\mathbb{S}^{d-1}\right|}{\left|\mathbb{S}^{d-2}\right|} = \frac{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} = \frac{\sqrt{\pi}\,(\frac{d-3}{2})!}{\Gamma(\frac{d}{2})} = \frac{\sqrt{\pi}\,2^{-\frac{d-3}{2}}\,(d-3)!!}{\Gamma(\frac{d}{2})},$$

we obtain

$$\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = \frac{2^{\frac{d+1}{2}}\,\pi^{\frac{d-1}{2}}}{\sqrt{2m+d}} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^j(\boldsymbol{a})} \sideset{}{'}\sum_{n=m+1-l}^{l+m+1} (-1)^{\frac{n+1}{2}} \frac{(n-1)!!}{(n+d-3)!!} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d} Y_{n,d}^i(\boldsymbol{\omega}). \quad \blacksquare$$

Theorem 4.5. *The functions*

$$\frac{1}{\lambda_{m,l,d}}\,\mathcal{D}^{(odd)}\widetilde{V}_{m,l,k}, \qquad m\in\mathbb{N}_0,\ l=0,\dots,m,\ k=1,\dots,N_{l,d},\ l+m \text{ even}$$

are orthonormal in $L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$*, where*

$$\lambda_{m,l,d} = \sqrt{\frac{N_{m+1,d}\,\mu_{m,d}^2\,|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|^2} \sideset{}{'}\sum_{n=m+1-l}^{m+1+l} \nu_{n,d}^2\, N_{n,d}\,\langle P_{m+1,d}\,P_{l,d}, P_{n,d}\rangle_{w_d}} \tag{4.11}$$

and

$$\langle P_{m+1,d}\,P_{l,d}, P_{n,d}\rangle_{w_d} = \int_{-1}^{1} P_{m+1,d}(t)\,P_{l,d}(t)\,P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t.$$

Proof. Let $m,m'\in\mathbb{N}_0$, $l=0,\dots,m$, $l'=0,\dots,m'$, $k=1,\dots,N_{l,d}$, $k'=1,\dots,N_{l',d}$ such that $m+l$ and $m'+l'$ are even. We have

$$\begin{aligned}
&\left\langle \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}, \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m',l',k'}\right\rangle_{L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})} \\
&= \mu_{m,d}\,\mu_{m',d} \sum_{j=1}^{N_{m+1,d}} \sum_{j'=1}^{N_{m'+1,d}} \int_{\mathbb{S}^{d-1}} \overline{Y_{m+1,d}^{j}(\boldsymbol{a})}\, Y_{m'+1,d}^{j'}(\boldsymbol{a})\,\mathrm{d}\boldsymbol{a} \\
&\quad \sideset{}{'}\sum_{n=m+1-l}^{m+1+l} \sideset{}{'}\sum_{n'=m'+1-l'}^{m'+1+l'} \nu_{n,d}\,\nu_{n',d} \sum_{i=1}^{N_{n,d}} \sum_{i'=1}^{N_{n',d}} G_{m+1,j,l,k}^{n,i,d}\, \overline{G_{m'+1,j',l',k'}^{n',i',d}} \int_{\mathbb{S}^{d-1}} Y_{n,d}^{i}(\boldsymbol{\omega})\,\overline{Y_{n',d}^{i'}(\boldsymbol{\omega})}\,\mathrm{d}\boldsymbol{\omega}.
\end{aligned}$$

By the orthonormality of the spherical harmonics, we obtain

$$\begin{aligned}
&\left\langle \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}, \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m',l',k'}\right\rangle_{L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})} \\
&\qquad = \delta_{m,m'}\,\mu_{m,d}^2 \sum_{j=1}^{N_{m+1,d}} \sideset{}{'}\sum_{n=m+1-l}^{m+1+l} \nu_{n,d}^2 \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d}\, \overline{G_{m+1,j,l',k'}^{n,i,d}}.
\end{aligned}$$

We have by the definition of the Gaunt coefficients in (5.1)

$$\begin{aligned}
&\sum_{j=1}^{N_{m+1,d}} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d}\,\overline{G_{m+1,j,l',k'}^{n,i,d}} \\
&= \sum_{j=1}^{N_{m+1,d}} \sum_{i=1}^{N_{n,d}} \int_{\mathbb{S}^{d-1}} Y_{m+1,d}^{j}(\boldsymbol{\xi})\,Y_{l,d}^{k}(\boldsymbol{\xi})\,\overline{Y_{n,d}^{i}(\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\xi} \int_{\mathbb{S}^{d-1}} \overline{Y_{m+1,d}^{j}(\boldsymbol{\eta})\,Y_{l',d}^{k'}(\boldsymbol{\eta})}\,Y_{n,d}^{i}(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} \\
&= \frac{N_{m+1,d}\,N_{n,d}}{|\mathbb{S}^{d-1}|^2} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} P_{m+1,d}(\boldsymbol{\xi}^\top\boldsymbol{\eta})\,P_{n,d}(\boldsymbol{\xi}^\top\boldsymbol{\eta})\,Y_{l,d}^{k}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\,\overline{Y_{l',d}^{k'}(\boldsymbol{\eta})}\,\mathrm{d}\boldsymbol{\eta},
\end{aligned}$$

where the last equality follows from the addition formula (2.8) for spherical harmonics.

Applying the Funk–Hecke formula (3.1) to the inner integral, we obtain

$$\sum_{j=1}^{N_{m+1,d}} \sum_{i=1}^{N_{n,d}} G^{n,i,d}_{m+1,j,l,k} \overline{G^{n,i,d}_{m+1,j,l',k'}}$$
$$= \frac{N_{m+1,d}\, N_{n,d} \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2} \int_{-1}^{1} P_{m+1,d}(t)\, P_{n,d}(t)\; P_{l,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t \int_{\mathbb{S}^{d-1}} Y^k_{l,d}(\boldsymbol{\eta})\, Y^k_{l',d}(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}$$
$$= \delta_{l,l'}\,\delta_{k,k'}\,\frac{N_{m+1,d}\, N_{n,d}}{\left|\mathbb{S}^{d-1}\right|^2} \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} P_{m+1,d}(t)\, P_{n,d}(t)\; P_{l,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t,$$

where we used again the orthonormality of the spherical harmonics. By [6], the value of the integral $\langle P_{m+1,d}\, P_{n,d}, P_{l,d}\rangle_{w_d}$ is nonzero if and only if

$$n \in \{|m+1-l|\,,\, |m+1-l|+2,\, \ldots,\, m+1+l\}.$$

Hence, we have

$$\left\langle \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}, \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m',l',k'} \right\rangle_{L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})}$$
$$= \delta_{m,m'}\,\delta_{l,l'}\,\delta_{k,k'}\,\frac{N_{m+1,d}\,\mu^2_{m,d}\,\left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2} \sideset{}{'}\sum_{n=m+1-l}^{m+1+l} \nu^2_{n,d}\, N_{n,d}\, \langle P_{m+1,d}\, P_{n,d}, P_{l,d}\rangle_{w_d}.$$

■

4.3 Bounds on the singular values

4.3.1 Upper bound

In this section, we show that the singular values $\lambda_{m,l,d}$ of the cone-beam transform $\mathcal{D}^{(odd)}$, which are given in (4.11), are bounded independently of m and l, which implies that the cone-beam transform as operator $\mathcal{D}^{(\mathrm{odd})}\colon L^2(\mathbb{B}^d) \to L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$ is bounded.

Lemma 4.6. *Let $n \geq 1$ and $d \geq 3$ be odd integers. Then*

$$\nu^2_{n,d} N_{n,d} \leq \begin{cases} \pi, & d = 3 \\ \frac{d}{((d-2)!!)^2}, & d \geq 5. \end{cases}$$

Proof. We have by (4.10) and (2.4)

$$\nu^2_{n,d} N_{n,d} = \frac{(n-1)!!^2}{(n+d-3)!!^2}\,\frac{(2n+d-2)\,(n+d-3)!}{n!\,(d-2)!}$$
$$= \frac{(n-1)!!}{n!!}\,\frac{(n+d-4)!!}{(n+d-3)!!}\,\frac{2n+d-2}{(d-2)!}.$$

We note that, by [4], for $n \to \infty$

$$(2n-1)!! \simeq \frac{(2n)!!}{\sqrt{\pi(n+\frac{1}{2})}}.$$

Hence, we obtain

$$\nu_{n,d}^2 N_{n,d} \simeq \frac{\sqrt{\pi(\frac{n}{2}+1)}}{n+1} \frac{\sqrt{\pi(\frac{n+d-1}{2})}}{n+d-4} \frac{2n+d-2}{(d-2)!} \simeq \frac{\pi}{(d-2)!}. \tag{4.12}$$

Because

$$\begin{aligned} \frac{\nu_{n+2,d}^2 N_{n+2,d}}{\nu_{n,d}^2 N_{n,d}} &= \frac{n+1}{n+2} \frac{n+d-2}{n+d-1} \frac{2n+d+2}{2n+d-2} \\ &= \frac{2n^3+3dn^2+(d^2+3d-6)n+d^2-4}{2n^3+3dn^2+(d^2+3d-6)n+2d^2-6d+4}, \end{aligned}$$

the sequence $n \mapsto \nu_{n,d}^2 N_{n,d}$ is increasing for $d=3$ and decreasing for $d \geq 5$. The fact that

$$\nu_{1,d}^2 N_{1,d} = \frac{d}{((d-2)!!)^2}$$

completes the proof. ■

Theorem 4.7. *Let $d \geq 3$ be an odd integer and $m, l \in \mathbb{N}_0$ such that $l \leq m$ and $m+l$ is even. Then the singular values $\lambda_{m,l,d}$ satisfy*

$$\lambda_{m,l,d} \leq 2^{\frac{d+1}{4}} \pi^{\frac{d-1}{4}} \sqrt{C_d\,(d-2)!!} \sqrt{\frac{l+1}{2m+d}} \leq (2\pi)^{\frac{d-1}{4}} \sqrt{C_d\,(d-2)!!},$$

where

$$C_d = \begin{cases} \pi, & d=3 \\ \frac{d}{((d-2)!!)^2}, & d \geq 5. \end{cases}$$

In particular, we have $\lim_{m\to\infty} \lambda_{m,l,d} = 0$ for all $l \in \mathbb{N}_0$.

Proof. Because of the orthogonality (2.3) of the Legendre polynomials, we have

$$P_{l,d}\, P_{n,d} = \sum_{m=|l-n|-1}^{l+n-1} \frac{N_{m+1,d} \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|} \left\langle P_{m+1,d} P_{l,d}, P_{n,d} \right\rangle_{w_d} P_{m+1,d}.$$

Utilizing the fact that $P_{i,d}(1) = 1$ for all $i \in \mathbb{N}_0$, we obtain

$$1 = \sum_{m=|l-n|-1}^{l+n-1} \frac{N_{m+1,d} \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|} \left\langle P_{m+1,d} P_{l,d}, P_{n,d} \right\rangle_{w_d}. \tag{4.13}$$

Since all summands in the above sum (4.13) are non-negative, they are bounded by

$$\frac{N_{m+1,d}\left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|}\left\langle P_{m+1}P_l, P_n\right\rangle_{w_d} \le 1. \tag{4.14}$$

Inserting the bound from Lemma 4.6 into the definition of the singular values (4.11), we have

$$\begin{aligned}
\lambda_{m,l,d}^2 &= \frac{N_{m+1,d}\,\mu_{m,d}^2\left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2}\sum_{n=m+1-l}^{m+1+l}{}' \nu_{n,d}^2\,N_{n,d}\left\langle P_{m+1,d}\,P_{l,d}, P_{n,d}\right\rangle_{w_d}\\
&\le \frac{N_{m+1,d}\,\mu_{m,d}^2\left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2}\,C_d\sum_{n=m+1-l}^{m+1+l}{}' \left\langle P_{m+1,d}\,P_{l,d}, P_{n,d}\right\rangle_{w_d}.
\end{aligned}$$

With (4.14), we obtain

$$\begin{aligned}
\lambda_{m,l,d}^2 &\le C_d\,\frac{\mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|}\sum_{n=m+1-l}^{m+1+l}{}' 1\\
&= C_d\,\frac{\mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|}\,(l+1)\\
&= C_d\,2^{\frac{d+1}{2}}\,\pi^{\frac{d-1}{2}}\,(d-2)!!\,\frac{l+1}{2m+d},
\end{aligned}$$

where we inserted the formulas of $\mu_{m,d}$ from (4.9) and $\left|\mathbb{S}^{d-1}\right|$ from (2.1). ■

4.3.2 Lower bound

Theorem 4.8. *Let $d \ge 3$ be an odd integer. There exists a constant $c_d > 0$, which depends only on the dimension d, such that for all $m \in \mathbb{N}_0$ and $l \in \{0, \dots, m\}$ with $m+l$ even, the singular values admit the lower bound*

$$\left|\lambda_{m,l,d}\right| \ge c_d\,m^{-1/2}. \tag{4.15}$$

This bound is asymptotically tight, in the sense that the exponent $-1/2$ in (4.15) cannot be replaced by a greater one.

Proof. We extract the smallest value of $\nu_{n,d}^2$ in the following sum

$$\begin{aligned}
\lambda_{m,l,d}^2 &= \frac{N_{m+1,d}\,\mu_{m,d}^2\left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2}\sum_{n=m+1-l}^{m+1+l}{}' \nu_{n,d}^2\,N_{n,d}\left\langle P_{m+1,d}\,P_{l,d}, P_{n,d}\right\rangle_{w_d}\\
&\ge \frac{N_{m+1,d}\,\mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|}\left(\min_{n=m+1-l}^{m+1+l}\nu_{n,d}^2\right)\sum_{n=m+1-l}^{m+1+l}{}' \frac{N_{n,d}\left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|}\left\langle P_{m+1,d}\,P_{l,d}, P_{n,d}\right\rangle_{w_d}.
\end{aligned}$$

Utilizing (4.13) with the roles of $m+1$ and n interchanged, we obtain

$$\lambda_{m,l,d}^2 \geq \frac{N_{m+1,d}\,\mu_{m,d}^2}{|\mathbb{S}^{d-1}|} \min_{n=m+1-l}^{m+1+l} \nu_{n,d}^2.$$

Since the map

$$n \mapsto \nu_{n,d}^2 = \frac{(n-1)!!^2}{(n+d-3)!!^2}$$

is decreasing, we have

$$\min_{n=m+1-l}^{m+1+l} \nu_{n,d}^2 = \nu_{m+1+l,d}^2.$$

Because $0 \leq l \leq m$ and again $\nu_{m+1+l,d}^2$ decreases with respect to l, we further see that

$$\min_{n=m+1-l}^{m+1+l} \nu_{n,d}^2 \geq \nu_{2m+1,d}^2 = \frac{(2m)!!^2}{(2m+d-2)!!^2}.$$

Hence, we have

$$\begin{aligned}
\lambda_{m,l,d}^2 &\geq \frac{N_{m+1,d}\,\mu_{m,d}^2}{|\mathbb{S}^{d-1}|} \frac{(2m)!!^2}{(2m+d-2)!!^2} \\
&= \frac{2^{d+1}\,\pi^{d-1}}{(2m+d)} \frac{(d-2)!!}{2^{\frac{d+1}{2}}\,\pi^{\frac{d-1}{2}}} \frac{(2m+d)\,(m+d-2)!}{(m+1)!\,(d-2)!} \frac{(2m)!!^2}{(2m+d-2)!!^2} \\
&= \frac{2^{\frac{d+1}{2}}\,\pi^{\frac{d-1}{2}}}{(d-3)!!} \frac{(m+d-2)!}{(m+1)!} \frac{(2m)!!^2}{(2m+d-2)!!^2},
\end{aligned}$$

where we inserted (2.4), (4.9) and (2.1). We are going to apply Stirling's approximation of the factorial

$$n! \simeq \sqrt{2\pi}\, n^{n+1/2}\, \mathrm{e}^{-n}$$

and the double factorials, cf. [4],

$$(2n)!! \simeq \sqrt{\pi}\,(2n)^{n+1/2}\,\mathrm{e}^{-n}, \qquad (2n-1)!! \simeq \sqrt{2}\,(2n)^n\,\mathrm{e}^{-n}.$$

We obtain for $m \to \infty$

$$\begin{aligned}
\lambda_{m,l,d}^2 &\geq \frac{2^{\frac{d+1}{2}}\,\pi^{\frac{d-1}{2}}}{(d-3)!!} \frac{(m+d-2)!}{(m+1)!} \frac{(2m)!!^2}{(2m+d-2)!!^2} \\
&\simeq \frac{2^{\frac{d+1}{2}}\,\pi^{\frac{d-1}{2}}}{(d-3)!!} \frac{(m+d-2)^{m+d-3/2}\,\mathrm{e}^{-m-d+2}}{(m+1)^{m+3/2}\,\mathrm{e}^{-m-1}} \frac{(2m)^{2m+1}\,\mathrm{e}^{-2m}\pi}{2(2m+d-1)^{2m+d-1}\,\mathrm{e}^{-2m-d-1}} \\
&\simeq \frac{2^{\frac{3-d}{2}}\,\pi^{\frac{d+1}{2}}}{(d-3)!!}\,\mathrm{e}^4 \frac{(m+d-2)^{m+d-3/2}}{(m+1)^{m+3/2}} \frac{m^{2m+1}}{(m+\frac{d-1}{2})^{2m+d-1}}.
\end{aligned}$$

Hence, there exists a constant $c_d > 0$ such that

$$\lambda_{m,l,d} \geq \sqrt{\frac{2^{\frac{d+1}{2}}\,\pi^{\frac{d-1}{2}}}{(d-3)!!} \frac{(m+d-2)!}{(m+1)!} \frac{(2m)!!^2}{(2m+d-2)!!^2}} \geq c_d\, m^{-1/2}.$$

In order to show that this bound is tight, we consider the case m even and $l = 0$. We have by (2.3)

$$\lambda_{m,0,d}^2 = \frac{N_{m+1,d}\, \mu_{m,d}^2\, \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2}\, \nu_{m+1,d}^2\, N_{m+1,d}\, \langle P_{m+1,d}, P_{m+1,d}\rangle_{w_d}$$
$$= \frac{N_{m+1,d}\, \mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|}\, \nu_{m+1,d}^2.$$

By (4.12), we have for $m \to \infty$

$$\lambda_{m,0,d}^2 \simeq \frac{\pi}{(d-2)!}\, \frac{\mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|}. \qquad \blacksquare$$

Remark 4.9. While the lower bound $\mathcal{O}(m^{-1/2})$ on the singular values $\lambda_{m,l,d}$ is asymptotically strict, we have only shown that they can be bounded from above by a constant in Theorem 4.7. However, the degree of ill-posedness of the reconstruction problem depends on the behavior of the smallest singular values, which is here $\mathcal{O}(m^{-1/2})$ and so the same as for the Radon transform in 2D. $\square$

5 Cone-beam transform in $\mathbb{R}^3$

In this section, we state the singular value decomposition of the cone-beam transform $\mathcal{D}^{(odd)}$ from Section 4 for the dimension $d = 3$. This case was already shown in [16]. Before we state the result, we give some formulas of spherical harmonics and Gaunt coefficients in this case. We write a point $\boldsymbol{\xi} \in \mathbb{S}^2$ in cylindrical coordinates

$$\boldsymbol{\xi}(\varphi, t) = (\cos\varphi\sqrt{1-t^2}, \sin\varphi\sqrt{1-t^2}, t)^\top, \quad \varphi \in [0, 2\pi),\ t \in [-1, 1].$$

We define the normalized associated Legendre functions of degree $n \in \mathbb{N}_0$ and order $k = -n, \dots, n$ by

$$\widetilde{P}_n^k = \sqrt{\frac{2n+1}{4\pi}\, \frac{(n-k)!}{(n+k)!}}\, \frac{(-1)^k}{2^n n!}\, \left(1-t^2\right)^{k/2} \frac{\mathrm{d}^{n+k}}{\mathrm{d}t^{n+k}} \left(t^2-1\right)^n.$$

The spherical harmonics

$$Y_n^k(\boldsymbol{\xi}(\varphi, t)) = \widetilde{P}_n^k(t)\, \mathrm{e}^{\mathrm{i}k\varphi}, \qquad \boldsymbol{\xi}(\varphi, t) \in \mathbb{S}^2,$$

of degree $n \in \mathbb{N}_0$ and order $k \in \{-n, \dots, n\}$ form an orthonormal basis of $L^2(\mathbb{S}^2)$, see [36, Chapter 5] and also [7]. The Gaunt coefficients

$$G_{n_1,k_1,n_2,k_2}^{n,k} = \int_{\mathbb{S}^2} Y_{n_1}^{k_1}(\boldsymbol{\xi})\, Y_{n_2}^{k_2}(\boldsymbol{\xi})\, \overline{Y_n^k(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} \tag{5.1}$$

are zero unless all the conditions

$$k = k_1 + k_2,\ |k_1| \le n_1,\ |k_2| \le n_2,\ |k| \le n$$

and

$$n = |n_1 - n_2|\,, |n_1 - n_2| + 2, \dots, n_1 + n_2$$

hold. An explicit representation of Gaunt coefficients can be found in [9]. The Gaunt coefficients are closely related to the Clebsch–Gordan coefficients, cf. [36].

Theorem 5.1. *The odd cone-beam transform $\mathcal{D}^{(odd)} \colon L^2(\mathbb{B}^3) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ has the singular value decomposition*

$$\mathcal{D}^{(\mathrm{odd})} \widetilde{V}_{m,l,k} = \lambda_{m,l}\, W_{m,l,k}, \qquad m \in \mathbb{N}_0,\ 0 \le l \le m,\ m+l \text{ even},\ k \in \{-l, \dots, l\}.$$

The polynomials

$$\widetilde{V}_{m,l,k}(s\boldsymbol{\omega}) = \sqrt{2m+3}\, s^l P_{\frac{m-l}{2}}^{(0,l+\frac{1}{2})}(2s^2-1)\, Y_l^k(\boldsymbol{\omega}), \qquad s \in [0,1],\ \boldsymbol{\omega} \in \mathbb{S}^2,$$

form an orthonormal basis of $L^2(\mathbb{B}^3)$. The singular values are given by

$$\lambda_{m,l} = \sqrt{2\pi \sum_{n=m+1-l}^{m+1+l}{}' \frac{(2n+1)(n-1)!!^2}{n!!^2} \langle P_{m+1}\, P_n, P_l \rangle},$$

where $\sum'$ denotes the summation over odd indices and

$$\langle P_{m+1}\, P_n, P_l \rangle = \frac{2(l+m-n)!!\,(l-m+n-2)!!\,(-l+m+n)!!\,(l+m+n+1)!!}{(l+m-n+1)!!\,(l-m+n-1)!!\,(-l+m+n+1)!!\,(l+m+n+2)!!}$$

for $n \in \{|m+1-l|\,,\ |m+1-l|+2,\ \dots,\ m+1+l\}$ and zero otherwise. The singular values satisfy

$$c_1 m^{-1/2} \le \lambda_{m,l} \le c_2 m^{-1/8} \tag{5.2}$$

for some constants c_1, c_2 independent of m and l. Furthermore, the functions

$$W_{m,l,k}(\boldsymbol{a}, \boldsymbol{\omega}) = \frac{4\pi}{\lambda_{m,l}\sqrt{2m+3}} \sum_{j=-m-1}^{m+1} \overline{Y_{m+1}^j(\boldsymbol{a})} \sum_{n=m+1-l}^{m+1+l}{}' \frac{(-1)^{\frac{n+1}{2}}\,(n-1)!!}{n!!} G_{m+1,j,l,k}^{n,j+k} Y_n^{j+k}(\boldsymbol{\omega})$$

for $\boldsymbol{a}, \boldsymbol{\omega} \in \mathbb{S}^2$ are orthonormal in $L^2(\mathbb{S}^2 \times \mathbb{S}^2)$.

Proof. The singular value decomposition is a special case of Theorems 4.4 and 4.5. The triple product $\langle P_{m+1}\, P_n, P_l \rangle$ is computed in [27] (see also [2]). The lower bound of the singular values $\lambda_{m,l}$ in (5.2) is due to Theorem 4.8. It is left to show the upper bound in (5.2), which we do as in [16]. Changing roles of m and n in (4.13), we have

$$1 = \sum_{n=m+1-l}^{m+1+l} \frac{2n+1}{2} \langle P_{m+1} P_l, P_n \rangle\,. \tag{5.3}$$

Furthermore, since $|P_n(t)| \le 1$ for $|t| \le 1$, we obtain the inequality

$$\langle P_{m+1}P_l, P_n\rangle \le \int_{-1}^{1} |P_{m+1,d}(t)\, P_{n,d}(t)\;\, P_{l,d}(t)|\, \mathrm{d}t \le \int_{-1}^{1} |P_{m+1,d}(t)\, P_{n,d}(t)|\, \mathrm{d}t.$$

By the Cauchy-Schwarz inequality,

$$\langle P_{m+1}P_l, P_n\rangle \le \|P_{m+1}\|_{L^2(-1,1)}\, \|P_n\|_{L^2(-1,1)} = \frac{2}{\sqrt{(2m+3)\,(2n+1)}}. \tag{5.4}$$

Based on Wallis' product, it was shown in [14] that for $m \in \mathbb{N}_0$

$$\frac{(2m)!!^2}{(2m+1)!!^2} \le \frac{\pi}{2(2m+1)}.$$

Hence, we have

$$\begin{aligned}\lambda_{m,l}^2 &= 2\pi \sum_{n=1}^{m+1+l}{}' \frac{(2n+1)(n-1)!!^2}{n!!^2} \langle P_{m+1}\, P_n, P_l\rangle \\ &\le 2\pi^2 \sum_{n=1}^{2m+1}{}' \frac{2n+1}{2n} \langle P_{m+1}\, P_n, P_l\rangle\end{aligned}$$

because $0 \le l \le m$. By the Cauchy-Schwarz inequality, we have

$$\lambda_{m,l}^2 \le 2\pi^2 \sqrt{\sum_{n=1}^{2m+1}{}' \frac{2n+1}{2} \langle P_{m+1}\, P_n, P_l\rangle} \sqrt{\sum_{n=1}^{2m+1}{}' \frac{2n+1}{2n^2} \langle P_{m+1}\, P_n, P_l\rangle}$$

Inserting (5.3) and (5.4), we obtain

$$\begin{aligned}\lambda_{m,l}^2 &= 2\pi^2 \sqrt{\sum_{n=1}^{2m+1}{}' \frac{2n+1}{2n^2} \langle P_{m+1}\, P_n, P_l\rangle} \\ &\le \frac{2\pi^2}{(2m+3)^{\frac{1}{4}}} \sqrt{\sum_{n=1}^{2m+1}{}' \frac{\sqrt{2n+1}}{n^2}}.\end{aligned}$$

The last sum converges for $m \to \infty$ and thus can be bounded from above by a constant intependent of m, which implies that $\lambda_{m,l}^2 \in \mathcal{O}(m^{-1/4})$. ■

Remark 5.2. The upper bound on the singular values $\lambda_{m,l} \in \mathcal{O}(m^{-1/8})$ may not be optimal. The authors think that the upper bound can be improved to $\mathcal{O}(m^{-1/2})$ even in general dimension d. This conjecture is backed by numerical computations as well as the following observation, which is not a proof though. We consider Grangeat's formula (4.5)

$$(-1)^d \left(\frac{\partial}{\partial s}\right)^{d-2} \mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top \boldsymbol{\omega}) = \mathcal{S}_d^{(d-2)} \mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}).$$

We know that the singular values of the Radon transform $\mathcal{R}$ are $\mathcal{O}(m^{(1-d)/2})$ and those of the $d-2$ differentiations are $\mathcal{O}(m^{d-2})$, so the left side should behave like $\mathcal{O}(m^{(d-3)/2})$. On the right side, $\mathcal{S}_d^{(d-2)}$ has the singular values $\mathcal{O}(m^{(d-2)/2})$ by Lemma 3.3, so the singular values of the cone-beam transform $\mathcal{D}$ should behave like $\mathcal{O}(m^{-1/2})$. □

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, DC, USA, 1972.

[2] W. A. Al-Salam. On the product of two Legendre polynomials. *Math. Scand.*, 4:239–242, 1956.

[3] K. Atkinson and W. Han. *Spherical Harmonics and Approximations on the Unit Sphere: An Introduction*, volume 2044 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[4] F. L. Bauer. Remark on Stirling's formula and on approximations for the double factorial. *Math. Intelligencer*, 29(2):10–14, 2007.

[5] H. Berens, P. L. Butzer, and S. Pawelke. Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationsverhalten. *Publ. Res. Inst. Math. Sci.*, 4:201–268, 1968.

[6] H. Chaggara and W. Koepf. On linearization coefficients of Jacobi polynomials. *Appl. Math. Lett.*, 23(5):609–614, 2010.

[7] F. Dai and Y. Xu. *Approximation Theory and Harmonic Analysis on Spheres and Balls*. Springer Monographs in Mathematics. Springer, New York, 2013.

[8] D. V. Finch. Cone beam reconstruction with sources on a curve. *SIAM J. Appl. Math.*, 45(4):665–673, 1985.

[9] J. A. Gaunt. The triplets of helium. *Phil. Trans. R. Soc. Lond. A*, 228(659-669):151–196, 1929.

[10] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Academic Press New York, seventh edition, 2007.

[11] P. Grangeat. Mathematical framework of cone beam 3D reconstruction via the first derivative of the Radon transform. In G. T. Herman, A. K. Louis, and F. Natterer, editors, *Mathematical Methods in Tomography: Proceedings of a Conference held in Oberwolfach, Germany, 5–11 June, 1990*, pages 66–97. Springer, Berlin, Heidelberg, 1991.

[12] H. Groemer. *Geometric applications of Fourier series and spherical harmonics*, volume 61 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1996.

[13] C. Hamaker, K. T. Smith, D. C. Solmon, and S. L. Wagner. The divergent beam X-ray transform. *Rocky Mountain J. Math.*, 10(1):253–283, 1980.

[14] R. Hielscher, D. Potts, and M. Quellmalz. An SVD in spherical surface wave tomography. In B. Hofmann, A. Leitao, and J. P. Zubelli, editors, *New Trends in Parameter Identification for Mathematical Models*, Trends in Mathematics, pages 121–144. Birkhäuser, Basel, 2018.

[15] R. Hielscher and M. Quellmalz. Optimal mollifiers for spherical deconvolution. *Inverse Problems*, 31(8):085001, 2015.

[16] S. G. Kazantsev. Singular value decomposition for the cone-beam transform in the ball. *J. Inverse Ill-Posed Probl.*, 23(2):173–185, 2015.

[17] A. Koldobsky. Inverse formula for the Blaschke-Levy representation. *Houston J. Math.*, 23(1):95–108, 1997.

[18] A. K. Louis. Orthogonal function series expansions and the null space of the Radon transform. *SIAM J. Math. Anal.*, 15(3):621–633, 1984.

[19] A. K. Louis. Exact cone beam reconstruction formulae for functions and their gradients for spherical and flat detectors. *Inverse Problems*, 32(11):115005, 2016.

[20] P. Maass. The x-ray transform: singular value decomposition and resolution. *Inverse Problems*, 3(4):729–741, 1987.

[21] E. Makai, H. Martini, and T. Ódor. Maximal sections and centrally symmetric bodies. *Mathematika*, 47(1-2):19–30, 2000.

[22] E. Makai, H. Martini, and T. Ódor. On an integro-differential transform on the sphere. *Studia Sci. Math. Hungar.*, 38(1-4):299–312, 2001.

[23] H. Minkowski. Über die Körper konstanter Breite. In D. Hilbert, editor, *Gesammelte Abhandlungen von Hermann Minkowski*, volume 2, pages 277–279. B. G. Teubner, Leipzig, 1911.

[24] C. Müller. *Analysis of spherical symmetries in Euclidean spaces*, volume 129 of *Applied Mathematical Sciences*. Springer, New York, 1998.

[25] F. Natterer. *The Mathematics of Computerized Tomography*. John Wiley & Sons Ltd, Chichester, 1986.

[26] F. Natterer and F. Wübbeling. *Mathematical Methods in Image Reconstruction*. SIAM, Philadelphia, PA, USA, 2000.

[27] F. E. Neumann. *Beiträge zur Theorie der Kugelfunctionen: erste und zweite Abtheilung.* B.G. Teubner, Leipzig, 1878.

[28] V. P. Palamodov. *Reconstruction from Integral Data.* Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, 2016.

[29] E. T. Quinto. Singularities of the X-ray transform and limited data tomography in $\mathbb{R}^2$ and $\mathbb{R}^3$. *SIAM J. Math. Anal.*, 24(5):1215–1225, 1993.

[30] M. Riplinger. *Numerische Inversion der sphärischen Radontransformation und der Kosinustransformation.* PhD thesis, Universität des Saarlandes, Saarbrücken, 2011.

[31] B. Rubin. Inversion and characterization of the hemispherical transform. *J. Anal. Math.*, 77(1):105–128, 1999.

[32] R. Schneider. Functions on a sphere with vanishing integrals over certain subspheres. *J. Math. Anal. Appl.*, 26:381–384, 1969.

[33] B. D. Smith. Cone-beam tomography: recent advances and a tutorial review. *Optical Engineering*, 29(5):524–534, 1990.

[34] R. S. Strichartz. L^p estimates for Radon transforms in Euclidean and non-Euclidean spaces. *Duke Math. J.*, 48(4):699–727, 1981.

[35] H. K. Tuy. An inversion formula for cone-beam reconstruction. *SIAM J. Appl. Math.*, 43(3):546–552, 1983.

[36] D. Varshalovich, A. Moskalev, and V. Khersonskii. *Quantum Theory of Angular Momentum.* World Scientific Publishing, Singapore, 1988.