Navigating Shape Space

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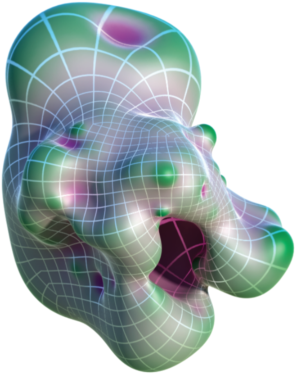
- **Shape space**:
  Space of *all* possible surface shapes
- No special geometry
- No special parameter mesh
- **Modelling** or **animation**:
  "Navigation in shape space"



- Space $\tilde{\mathcal{M}}$ of all *textured surfaces*:

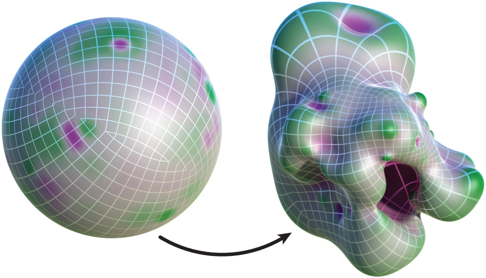
  $\tilde{\mathcal{M}}$ = space of immersions

  $$S^2 \to \mathbb{R}^3$$

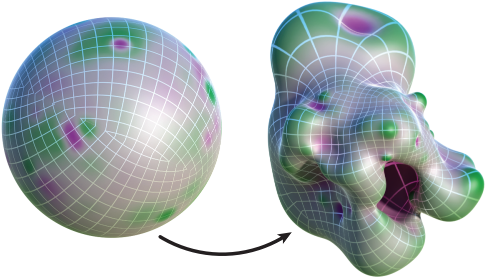
  modulo translation, rotation and scale.

- Shape space $\hat{\mathcal{M}}$ of (not parametrized) surfaces:

  $\hat{\mathcal{M}} = \tilde{\mathcal{M}}/\mathrm{Diff}(S^2)$



- $\hat{\mathcal{M}} = \tilde{\mathcal{M}}/\mathrm{Diff}(S^2)$ is difficult to work with.
- $\hat{\mathcal{M}}$ is **hard** to discretize (because the mesh itself is a kind of "texture").
- Cannot be avoided for some applications (e.g. medical).

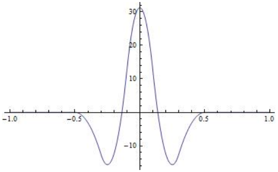


- Simpler case: Shape space of curves in $\mathbb{R}^2$ (not necessarily closed)
- Parametrize by

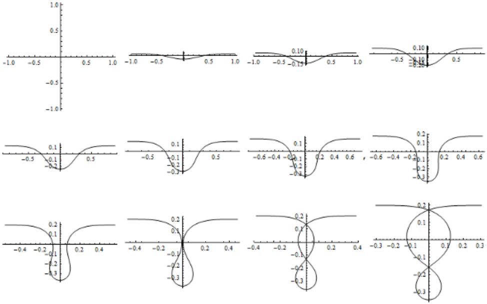
$$\gamma : [0,1] \to \mathbb{R}^2$$

with $|\gamma'| = \text{const}$

- Shape uniquely defined by the curvature density $\kappa\, ds$ on $[0,1]$
- Shape space is a vector space!



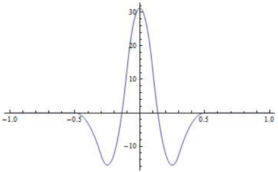
curvature $\kappa = t\rho$



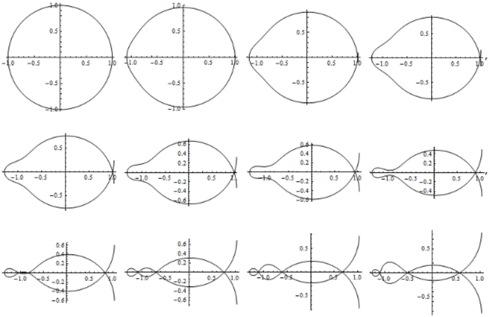
- For closed curves κ has to satisfy

$$\oint \kappa \, ds \in 2\pi\mathbb{N}$$

- Shape space is a collection of parallel hyperplanes!
- More conditions to kill the remaining translational period (let us ignore these for now)



curvature $\kappa = \kappa_0 + t\rho$

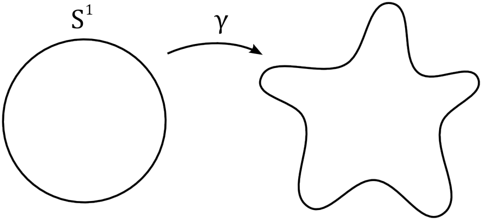


- Every closed plane curve admits a parametrization

  $$\gamma : S^1 \to \mathbb{R}^2$$

  with constant speed.

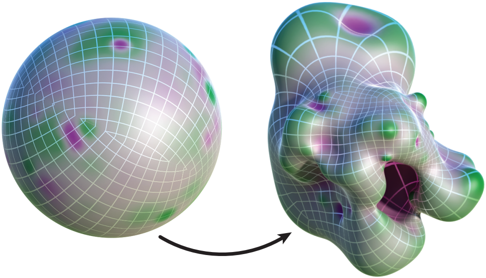
- γ is unique up to precomposition with a rotation of S^1.



If we ignore this ambiguity (and the problem with periods) the shape space of closed plane curves is

$$\mathcal{M} = \{ \text{densities } \ \kappa\, ds \ \text{ on } S^1 \,|\, \oint \kappa\, ds \in 2\pi\mathbb{N} \}$$

- Every topological sphere in $\mathbb{R}^3$ admits a conformal parametrization $f : S^2 \to \mathbb{R}^3$.
- f is unique up to precomposition with a Möbius transformation of S^2.
- Similar situation as with constant speed parametrizations of plane curves



- Let
$$f : S^2 \to \mathbb{R}^3$$
be a conformal immersion.

- The analogue of the curvature density $\kappa\, ds$ is the mean curvature half-density
$$u = H\, ds$$

- ds is a function on the tangent bundle:
$$ds(X) = |df(X)|$$

- We often write
$$ds = |df|$$

Mean Curvature Half-Density

- ds is the square root of the induced Riemannian metric

$$ds^2 = |df|^2$$

- In the conformal setting a compatible metric is the same as a volume form S^2.

- $u^2 = H^2 |df|^2$ is a 2-form and can be integrated.

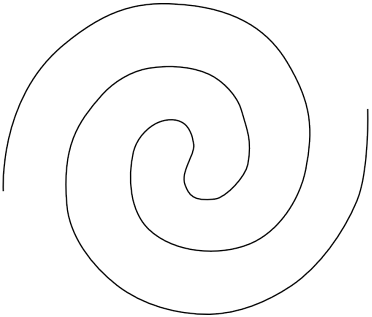
- The product of half-densities is a 2-form and can be integrated.

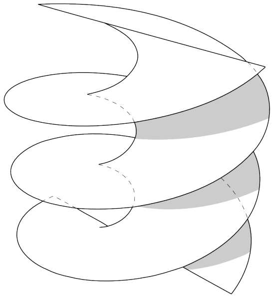
- The space $\mathcal{H}$ of all half-densities is a euclidean vector space.

The set $\mathcal{M}$ of all half-densities on S^2 for which there is a conformal immersion $f : S^2 \to \mathbb{R}^3$ with mean curvature half-density

$$H|df| = u$$

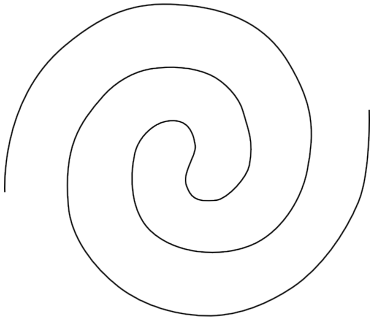
is a hypersurface in $\mathcal{H}$.

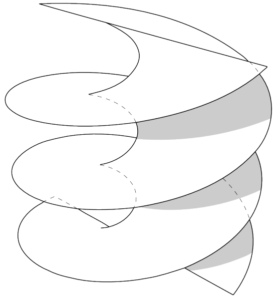




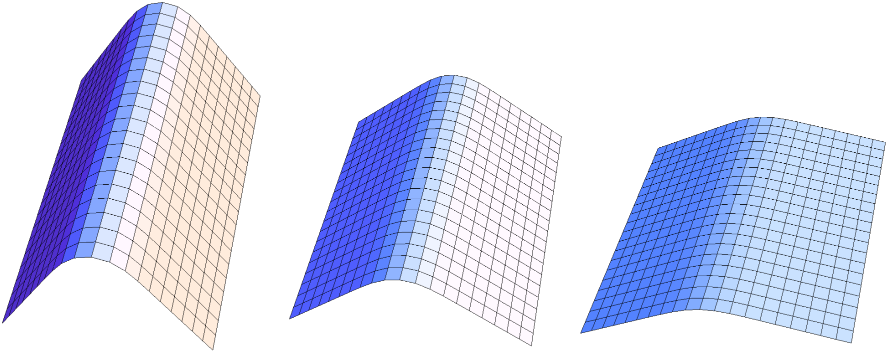
- For most $u \in \mathcal{M}$ the corresponding surface f is unique up to translation, rotation and scale.

- $\mathcal{M}$ is a decent model for shape space!





Scaling the mean curvature half-density



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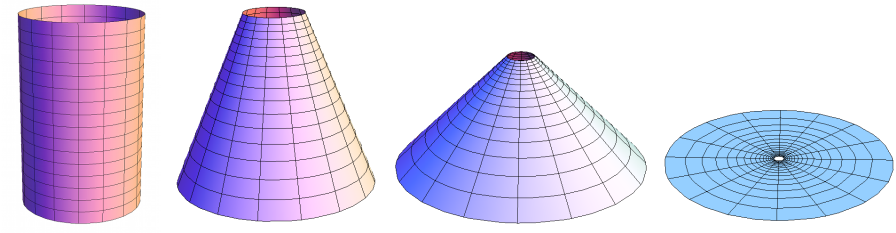

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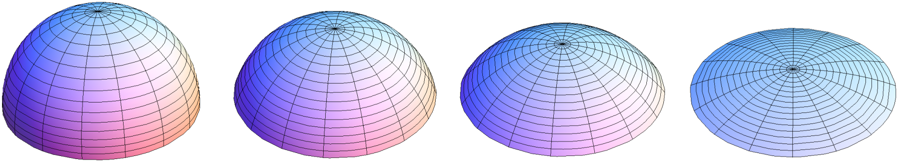

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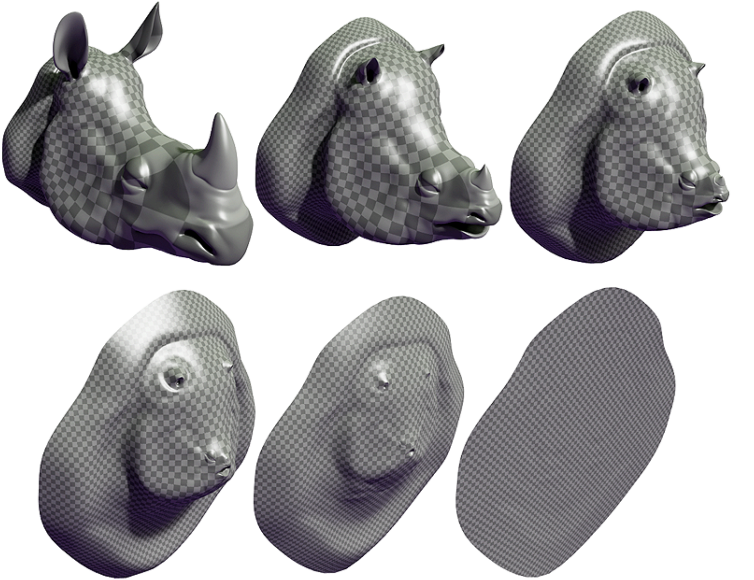
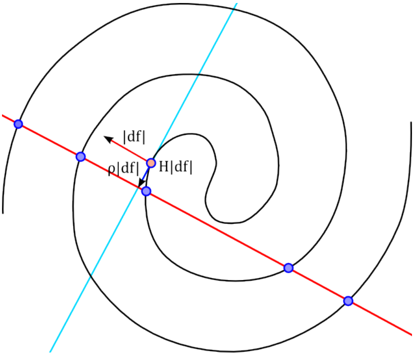


Scaling the mean curvature half-density

- Normal to $\mathcal{M}$ at a half-density u is $N = |df|$.
- Move out tangentially and project back to $\mathcal{M}$ along N
- Eigenvalue problem (numerical linear algebra)
- After moving out far enough the projection jumps to a different sheet of $\mathcal{M}$



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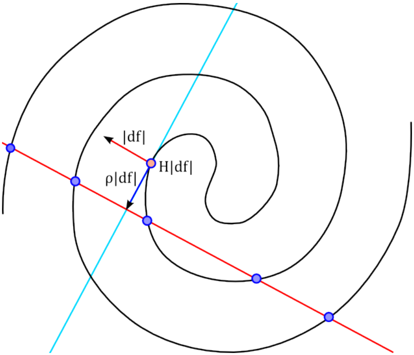
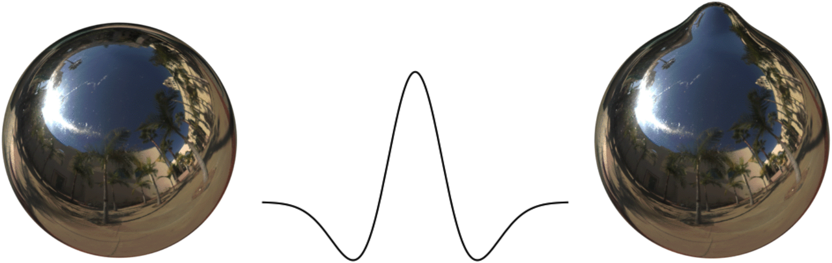


Bumps

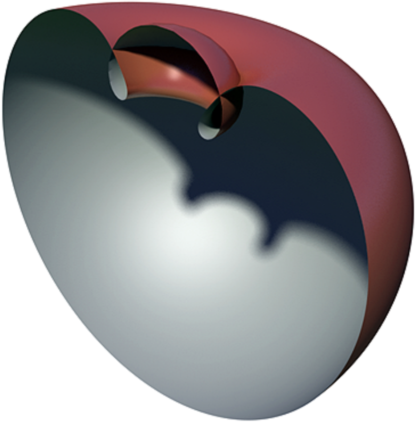
$$g = \text{Gaussian bump}$$

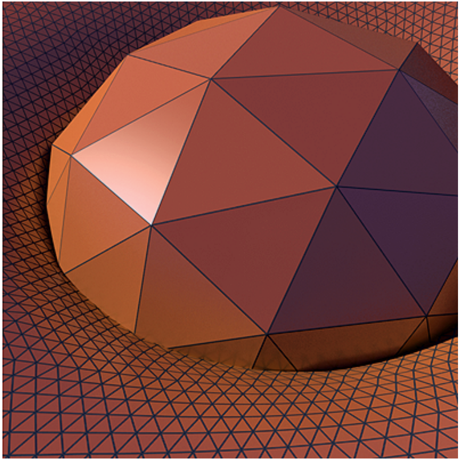
$$\rho = -\Delta g$$



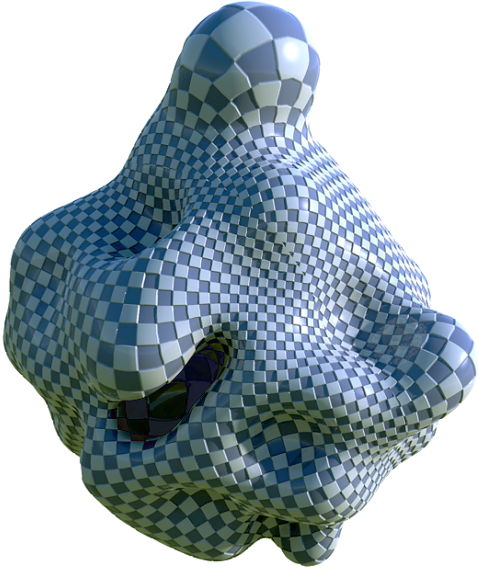
Bigger bumps

Huge bumps





Many moderate bumps



Dirac Bunnies