The space of curves in a conformal 3-manifold

Ulrich Pinkall

Technische Universität Berlin

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Closed curves in $S^3$
Curves in a conformal 3-manifold

- $M$ a 3-manifold with conformal structure (equivalence class of Riemannian metrics)

- Main example: $M = S^3$

- $\mathcal{M} = \{\text{immersions } \gamma: S^1 \to M\}/\text{Diff}_0(S^1)$ (space of unparametrized oriented closed curves)

- More generally: Space of compact submanifolds of codimension $k$ in a conformal $n$-manifold
$\mathcal{M}$ is an infinite dimensional Frechet manifold ($C^{\infty}$-topology on closed curves in $M$).

What works as usual on Frechet manifolds?

- Defining tensors (like Riemannian metrics)
- Everything that involves only differentiation (like computing the Levi-Civita connection of a Riemannian metric)
Infinite dimensional manifolds

Where one has to be careful:

- No existence and uniqueness theorem for ODE’s on infinite-dimensional manifolds
  - Vector fields might not have integral curves
  - No geodesics with prescribed initial velocity

- Integration over $\mathcal{M}$ not easy
  - better not talk about volume of subsets of $\mathcal{M}$
Tangent bundle of $\mathcal{M}$

- $T_{\gamma}\mathcal{M} = \{\text{normal vector fields } Y \text{ along } \gamma\}$

- A compatible Riemannian metric $\langle , \rangle$ on $\mathcal{M}$ defines a Riemannian metric on $\mathcal{M}$:
  $$\langle Y, Z \rangle_{L^2} = \int \langle Y(s), Z(s) \rangle ds$$

- For a 1-parameter family $t \mapsto \gamma_t$, $t \in [0, 1]$ use Levi-Civita parallel translation along the orthogonal trajectories to transport normal vectors of $\gamma_0$ to normal vectors of $\gamma_1 \rightsquigarrow$ affine connection $\hat{\nabla}$ on $\mathcal{M}$
Levi-Civita connection of $\langle Y, Z \rangle_{L^2}$

- Vector field $\mathcal{H}$ on $\mathcal{M}$:
  \[ \mathcal{H}_\gamma = \text{Mean curvature vector field along } \gamma \]

- Tensor field $C$ on $\mathcal{M}$:
  \[ C : T_\gamma \mathcal{M} \times T_\gamma \mathcal{M} \to T_\gamma \mathcal{M} \]
  \[ C_X Y = \langle X, \mathcal{H} \rangle Y + \langle Y, \mathcal{H} \rangle X - \langle X, Y \rangle \mathcal{H} \]

- $\hat{\nabla} + \frac{1}{2} C$ is the Levi-Civita connection of $\langle Y, Z \rangle_{L^2}$
Canonical affine connection on $\mathcal{M}$

- $\nabla := \hat{\nabla} + C$ is a conformally invariant affine connection on $\mathcal{M}$

- $\nabla$ admits no parallel Riemannian metric, but the conformally invariant function

$$L : T\mathcal{M} \to \mathbb{R}_+$$

$$\frac{1}{L(Y)} = \int 1/|Y(s)| ds$$

is invariant under parallel translation:

$$\nabla L = 0$$

- $L$ is called the *harmonic mean Lagrangian*
Harmonic mean Lagrangian

- $L$ vanishes on normal vector fields $Y \in T_\gamma M$ that have zeroes
- $L$ is homogeneous of degree one, hence for curves $t \mapsto \gamma_t \in \mathcal{M}, \quad t \in [a, b]$

  the functional

  \[
  \mathcal{L} = \int_a^b L(\dot{\gamma}) \in \mathbb{R}_+
  \]

  is parametrization-independent

- $\mathcal{L}$ measures in a conformally invariant way the “length” of a curve in $\mathcal{M}$
Let \( f : S^1 \times [a, b] \to M \) be an immersed cylinder, viewed as a curve \( t \mapsto \gamma_t \) in \( M \). Then the following are equivalent:

- \( \gamma \) is a geodesic of \( \nabla \)
- \( \gamma \) is a critical point of \( \mathcal{L} \)
- \( f \) is isothermic and the curves \( \gamma_t \) make an angle of 45° with the curvature lines of \( f \)

\( \leadsto \) variational characterization of isothermic surfaces
The space of circles in $S^3$ is a 6-dimensional totally geodesic submanifold $Circ(S^3)$ of $M$.

Geodesics in $Circ(S^3)$ are special minimal surfaces (helicoids) with respect to some constant curvature metric on a subset of $S^3$. 
Rotation of normal vector fields $Y$ by $90^\circ$, $Y \mapsto J(Y)$ defines an almost complex structure on $\mathcal{M}$:

$$J : T\gamma \mathcal{M} \to T\gamma \mathcal{M}$$

The Nijenhuis-Tensor of $J$ vanishes.

For any compatible metric $\langle \cdot , \cdot \rangle$ on $\mathcal{M}$ the Levi-Civita connection of the $L^2$-metric $\langle \cdot , \cdot \rangle_{L^2}$ induced on $\mathcal{M}$ leaves $J$ parallel.

Hence $\langle \cdot , \cdot \rangle_{L^2}$ is a Kähler metric on $\mathcal{M}$.

$\nabla J = 0$ for the canonical connection.
Holomorphic curves in $\mathcal{M}$

- Locally a holomorphic curve
  
  $$f : U \to \mathcal{M}, \quad U \subset \mathbb{C}$$

  defines a fibration
  
  $$\phi : f^{-1}(U) \to U$$

- $\phi$ is a conformal submersion

- Classical topic in case $\mathcal{M} = S^3$ and $f(z)$ is a round circle for all $z \in U$ ("isotropic circle congruences")
The space of round circles in $S^3$ is a totally geodesic complex submanifold of $\mathcal{M}$.

So is the space of straight lines in a non-euclidean geometry embedded in $S^3$.

The Hopf fibration is a holomorphic 2-sphere in $\mathcal{M}$. 
The total torsion modulo $2\pi$ of any unit normal vector field $N$ along $\gamma$ is conformally invariant and independent of $N$:

$$\mathcal{T}(\gamma) \in S^1 = \mathbb{R}/2\pi$$

$\mathcal{T}$ is the monodromy in the normal bundle of $\gamma$.

In case $M$ is simply connected:

- $M$ is connected.
- $\mathcal{T}$ can be defined modulo $4\pi$ (use only normal vector fields with even linking number) $\leadsto \mathcal{T}/2 \in S^1$ is well-defined.
- An isomorphism of fundamental groups is induced by

$$\mathcal{T}/2 : M \to S^1$$
Critical points of the total torsion

- $\gamma$ is a critical point of $T \iff$

$$R(N, JN)\gamma' + H' = 0$$

where $N$ is any unit normal vector field along $\gamma$.

- In standard $S^3$: $\iff \gamma$ is a round circle.

- Define in general $\gamma$ to be a round circle in $M$ if it is a critical point of $T$.

- Question: Do there always exist closed round circles? How many?