The space of curves in a conformal 3-manifold

Ulrich Pinkall

Technische Universität Berlin

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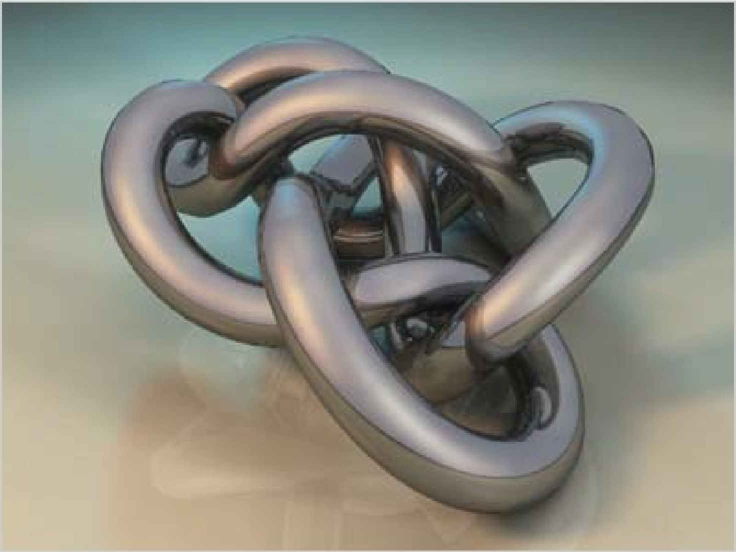
Joint work with F. Henrich

Closed curves in S^3



Curves in a conformal 3-manifold

- M a 3-manifold with conformal structure (equivalence class of Riemannian metrics)
- Main example: $M = S^3$
- $\mathcal{M} = \{\text{immersions } \gamma : S^1 \to M\}/\mathit{Diff}_0(S^1)$ (space of unparametrized oriented closed curves)
- More generally: Space of compact submanifolds of codimension k in a conformal n-manifold

Infinite dimensional manifolds

$\mathcal{M}$ is an infinite dimensional Frechet manifold (C^∞-topology on closed curves in M).

What works as usual on Frechet manifolds?

- Defining tensors (like Riemannian metrics)
- Everything that involves only differentiation (like computing the Levi-Civita connection of a Riemannian metric)

Infinite dimensional manifolds

Where one has to be careful:

- No existence and uniqueness theorem for ODE's on infinite-dimensional manifolds $\rightsquigarrow$
  - Vector fields might not have integral curves
  - No geodesics with prescribed initial velocity
- Integration over $\mathcal{M}$ not easy $\rightsquigarrow$ better not talk about volume of subsets of $\mathcal{M}$

Tangent bundle of $\mathcal{M}$

- $T_\gamma\mathcal{M} = \{$normal vector fields Y along $\gamma\}$
- A compatible Riemannian metric $\langle,\rangle$ on M defines a Riemannian metric on $\mathcal{M}$:

$$\langle Y, Z\rangle_{L^2} = \int \langle Y(s), Z(s)\rangle ds$$

- For a 1-parameter family $t \mapsto \gamma_t, t \in [0,1]$ use Levi-Civita parallel translation along the orthogonal trajectories to transport normal vectors of γ_0 to normal vectors of $\gamma_1 \rightsquigarrow$ affine Connection $\hat{\nabla}$ on $\mathcal{M}$

Levi-Civita connection of $\langle Y, Z\rangle_{L^2}$

- Vector field $\mathcal{H}$ on $\mathcal{M}$:

$$\mathcal{H}_\gamma = \text{Mean curvature vector field along } \gamma$$

- Tensor field C on $\mathcal{M}$:

$$C : T_\gamma\mathcal{M} \times T_\gamma\mathcal{M} \to T_\gamma\mathcal{M}$$

$$C_X Y = \langle X, \mathcal{H}\rangle Y + \langle Y, \mathcal{H}\rangle X - \langle X, Y\rangle\mathcal{H}$$

- $\hat{\nabla} + \frac{1}{2}C$ is the Levi-Civita connection of $\langle Y, Z\rangle_{L^2}$

Canonical affine connection on $\mathcal{M}$

- $\nabla := \hat{\nabla} + C$ is a conf. invariant affine connection on $\mathcal{M}$
- ∇ admits no parallel Riemannian metric, but the conformally invariant function
$$L : T\mathcal{M} \to \mathbb{R}_+$$
$$1/L(Y) = \int 1/|Y(s)|ds$$
is invariant under parallel translation:
$$\nabla L = 0$$
- L is called the *harmonic mean Lagrangian*

Harmonic mean Lagrangian

- L vanishes on normal vector fields $Y \in T_\gamma\mathcal{M}$ that have zeroes
- L is homogeneous of degree one, hence for curves

$$t \mapsto \gamma_t \in \mathcal{M}, \quad t \in [a, b]$$

the functional

$$\mathcal{L} = \int_a^b L(\dot\gamma) \in \mathbb{R}_+$$

is parametrization-independent

- $\mathcal{L}$ measures in a conformally invariant way the “length” of a curve in $\mathcal{M}$

Geodesics in $\mathcal{M}$

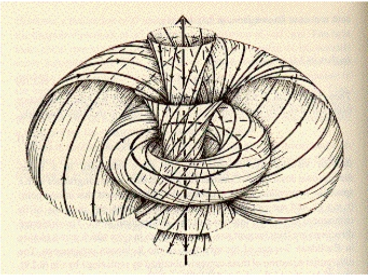
Let $f : S^1 \times [a, b] \to M$ be an immersed cylinder, viewed as a curve $t \mapsto \gamma_t$ in $\mathcal{M}$. Then the following are equivalent:

- γ is a geodesic of ∇
- γ is a critical point of $\mathcal{L}$
- f is isothermic and the curves γ_t make an angle of 45° with the curvature lines of f

$\rightsquigarrow$ variational characterization of isothermic surfaces

Circles in S^3

- The space of circles in S^3 is a 6-dimensional totally geodesic submanifold $Circ(S^3)$ of $\mathcal{M}$
- Geodesics in $Circ(S^3)$ are special minimal surfaces (helicoids) with respect to some constant curvature metric on a subset of S^3



Complex structure of $\mathcal{M}$

- Rotation of normal vector fields Y by 90°, $Y \mapsto J(Y)$ defines an almost complex structure on $\mathcal{M}$:

$$J : T_\gamma\mathcal{M} \to T_\gamma\mathcal{M}$$

- The Nijenhuis-Tensor of J vanishes
- For any compatible metric $\langle , \rangle$ on M the Levi-Civita connection of the L^2-metric $\langle , \rangle_{L^2}$ induced on $\mathcal{M}$ leaves J parallel
- Hence $\langle , \rangle_{L^2}$ is a Kähler metric on $\mathcal{M}$
- $\nabla J = 0$ for the canonical connection

Holomorphic curves in $\mathcal{M}$

- Locally a holomorphic curve

$$f : U \to \mathcal{M}, \quad U \subset \mathbb{C}$$

defines a fibration

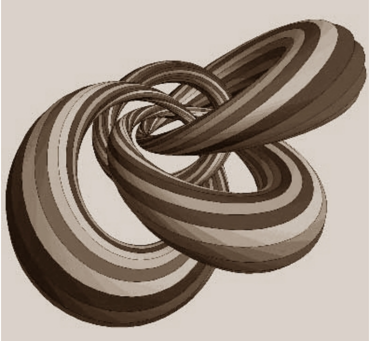
$$\phi : f^{-1}(U) \to U$$

- ϕ is a conformal submersion

- Classical topic in case $M = S^3$ and $f(z)$ is a round circle for all $z \in U$ ("isotropic circle congruences")

Hopf fibration

- The space of round circles in S^3 is a totally geodesic complex submanifold of $\mathcal{M}$
- So is the space of straight lines in a non-euclidean geometry embedded in S^3
- The Hopf fibration is a holomorphic 2-sphere in $\mathcal{M}$



Total torsion

- The total torsion modulo 2π of any unit normal vector field N along γ is conformally invariant and independent of N:

$$\mathcal{T}(\gamma) \in S^1 = \mathbb{R}/2\pi$$

- $\mathcal{T}$ is the monodromy in the normal bundle of γ.
- In case M is simply connected:
  - $\mathcal{M}$ is connected.
  - $\mathcal{T}$ can be defined modulo 4π (use only normal vector fields with even linking number) $\rightsquigarrow \mathcal{T}/2 \in S^1$ is well-defined.
  - An isomorphism of fundamental groups is induced by

$$\mathcal{T}/2 : \mathcal{M} \to S^1$$

Critical points of the total torsion

- γ is a critical point of $\mathcal{T} \Leftrightarrow$

$$R(N, JN)\gamma' + \mathcal{H}' = 0$$

  where N is any unit normal vector field along γ.

- In standard S^3: $\Leftrightarrow \gamma$ is a round circle.

- Define in general γ to be a *round circle* in M if it is a critical point of $\mathcal{T}$.

- Question: Do there always exist closed round circles? How many?