Polygonal smoke

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jointly with Steffen Weißmann

DFG Research Center MATHEON
Mathematics for key technologies

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Polygonal smoke
Claim: The whole smoke can be modelled as a collection of entangled smoke rings.

Smoke rings move on their own, but they also interact.

Interaction can even imply a topology change (reconnection)
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Vortex filaments
Colliding vortex rings
A velocity vector field $v$ is uniquely determined by its vorticity $\omega = \text{curl} \, v$.

$v$ is given by the Biot-Savart formula:

$$v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \omega(y) \times \frac{x - y}{|x - y|^3} \, dy$$

In an ideal fluid $\omega$ flows with the velocity $v$ it generates:

$$\dot{\omega} = \text{curl} \, (v \times \omega) = [\omega, v]$$
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Away from obstacles vorticity is neither created nor destroyed

Just swept along with the flow

All vorticity originates at the boundaries of obstacles

*Kaffeeöffelexperiment* by Felix Klein
Origin of vorticity

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Vorticity originates as 2-dimensional vortex sheets

Vortex sheets roll up into 1-dimensional structures ("smoke rings")
Vortex sheet roll up

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Airplane rides on a giant vortex ring

Extends back to where it took off

Vorticity concentrated on a filament
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Vorticity concentrated on a filament
Suppose all vorticity is concentrated in a small tube of radius $R$ around a space curve $\gamma$ (like water flowing through the tube).

Then away from $\gamma$ the velocity field is given by

$$v(x) = K \oint \frac{\gamma' \times (x - \gamma)}{|x - \gamma|^3}$$
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Evolution of $\gamma$: Evaluate velocity $\nu$ on $\gamma$$ \rightsquigarrow$

$$\dot{\gamma} \approx C_f K \log(R) \gamma' \times \gamma''$$

- Scale down $K$ as $R \rightarrow 0$$ \rightsquigarrow$ smoke ring flow
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Integrable system equivalent to the non-linear Schroedinger equation (Hashimoto 1972)
\[ \dot{\gamma} = \gamma' \times \gamma'' \]

- Curve moves orthogonal to its osculating plane.
- Speed is proportional to the curvature.
- Length is constant.
- Area vector is constant.
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For a space curve:

\[ A = \frac{1}{2} \oint \gamma \times \gamma' \]

For a closed polygon:

\[ A = \frac{1}{2} \sum_{i=1}^{n} \gamma_i \times \gamma_{i+1} \]

\( b \) a unit vector \( \sim \) \( \langle A, b \rangle \) is the algebraic area of the orthogonal projection of \( \gamma \) onto a plane with normal vector \( b \)
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▷ \( b \) a unit vector \( \sim \langle A, b \rangle \) is the algebraic area of the orthogonal projection of \( \gamma \) onto a plane with normal vector \( b \)
A quadrilateral $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in $\mathbb{R}^3$ is called a *skew parallelogram* of twist $\tau$ if the difference vector

$$V = \frac{\gamma_3 + \gamma_1}{2} - \frac{\gamma_2 + \gamma_0}{2}$$

between the centers of its diagonals is a multiple of its area vector:

$$V = \tau A$$

Opposite sides of a skew parallelogram have the same length.
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A polygon $\eta$ is called a Darboux transform with rod-length $\rho$ and twist $\tau$ if all quadrilaterals

$$\eta_i \quad \eta_{i+1}$$

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are skew parallelograms with

$$|\eta_i - \gamma_i| = \rho$$

and twist $\tau$.

For generic $\rho, \tau$ every closed polygon has exactly two closed Darboux transforms.

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Closed Darboux transforms of a closed polygon have the same:

- length
- Area vector

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Iterating Darboux transforms with the same $\rho$ and $\tau$ gives “discrete Lund-Regge surfaces” (Schieß 2007)
Iterated Darboux transforms

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Using twists $\tau$ and $-\tau$ in an alternating fashion preserves reflectional symmetries.

Forget the odd iterations.

Excellent discrete version of the smoke-ring flow.
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To make this practical, several features have to be included:

- Interaction between thick vortex rings
- Obstacles
- Vorticity generation at obstacle boundaries ("vortex shedding")
- Topology changes ("vortex reconnection")
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What is missing?

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See Steffen’s talk on Thursday!