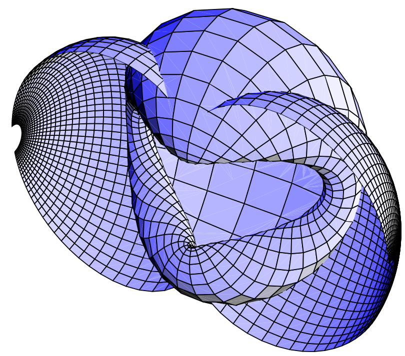
# Soliton Spheres

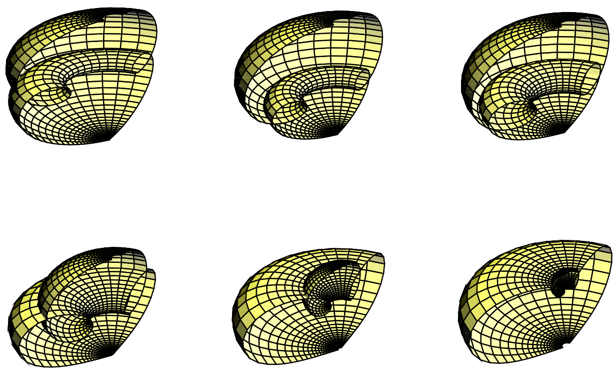
A conformal immersion $f\colon \mathbb{CP}^1 \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ is called a soliton sphere if equality holds in the quaternionic Plücker formula, i.e., if

$$W(L) = 4\pi(n^2 + \operatorname{ord} H)$$

for a linear system $H \subset H^0(L)$ of holomorphic sections in the spin bundle induced by $f$. Even though this definition involves Euclidean quantities, it can be shown to be conformally invariant. Examples of soliton spheres are Dirac spheres, Taimanov's soliton spheres and Willmore spheres.



Dirac sphere.



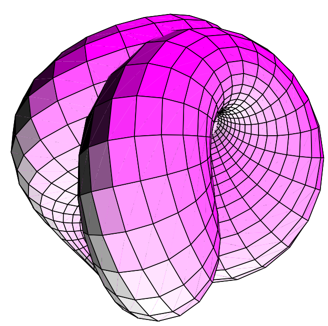
Reflectionless spheres.
The mKdV–flow of a rotational symmetric Dirac sphere.

A section of the spin bundle $L$ is holomorphic if — in coordinates — it is contained in the kernel of the two–dimensional Dirac operator

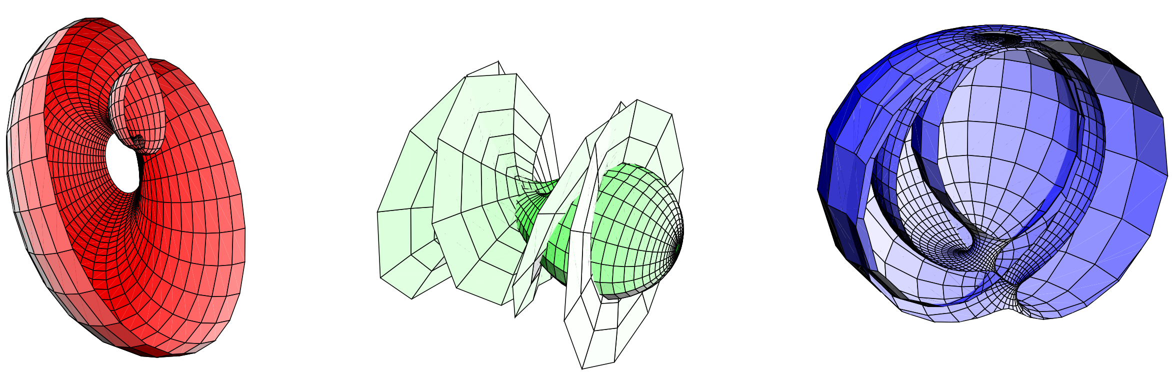
$$\mathcal{D} = \begin{pmatrix} U(x,y) & \partial_z \\ -\bar{\partial}_z & U(x,y) \end{pmatrix}$$

where $U$ is a real potential. Taimanov calls $f$ a reflectionless sphere, if in a cylindrical representation its potential $U$ is a one–dimensional reflexionless potential for the operator $L = \begin{pmatrix} 2U(x) & \partial_x \\ -\partial_x & 2U(x). \end{pmatrix}$ Those spheres are soliton spheres, since by the trace formula equality holds in the Plücker formula for $H = H^0(L)$.

Basepoint–free linear systems correspond to holomorphic curves in $\mathbb{HP}^n$ via Kodaira correspondence. The dual curve of that curve in $\mathbb{HP}^n$ has a holomorphic twistor lift if and only if equality holds in the Plücker formula for the corresponding linear system. Thus, spin bundles of soliton spheres correspond to certain holomorphic curves of genus 0 in $\mathbb{CP}^{2n+1}$which are given by $2n+2$ polynomials.
All Willmore spheres in $\mathbb{R}^3$ are soliton spheres and can be obtained from holomorphic curves in $\mathbb{CP}^3$ with hyperbolic minimal twistor projection by taking the (backward) Bäcklund transformation. This Bäcklund transformation can be calculated algebraically. Hence, this procedure gives rational conformal parametrizations of Willmore spheres.



Willmore sphere.



Three soliton spheres.

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