Darboux Transformations: From Elastic Curves to Elastic Surfaces and Beyond

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Abstract

The aim of this report is to explain a recent result about the generalized Darboux transformation and its relation to integrable systems: the generalized Darboux transformation preserves equality in the quaternionic Plücker estimate. A special case of this is the theorem that the generalized Darboux transforms of Willmore (or elastic) spheres are twistor holomorphic curves. As a toy model of the general case we investigate a 1–dimensional reduction: surfaces of revolution. In this case the Darboux transformation induces a transformation of the meridian curve, which one might view as a curve in the hyperbolic half plane. The mentioned result then generalizes the following fact: The asymptotically geodesic planar elastic curves are Darboux transformations of the meridian curve of the round sphere, which is the only twistor holomorphic surface of revolution.

1 Weierstrass Representation

Let $U \subset \mathbb{C}$ be a simply connected open domain. It is convenient to identify the imaginary quaternions $\operatorname{Im}\mathbb{H} = \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$ with $\mathbb{R}^3$. The stretch rotations of $\mathbb{R}^3$ may then for example be written as $x \mapsto \bar{\lambda}x\lambda$ for a quaternion $\lambda \in \mathbb{H}$ and its quaternionic conjugate $\bar{\lambda}$. The (generalized) *Weierstrass representation* of a conformal immersions from U to $\mathbb{R}^3$ then reads: a smooth map $f\colon U \to \operatorname{Im}\mathbb{H}$ is a conformal immersion if and only if there exists $q\colon U \to \mathbb{R}$ and a nowhere vanishing $\lambda\colon U \to \mathbb{C}^2 \cong \mathbb{C} \oplus \mathbf{k}\mathbb{C} = \mathbb{H}$, unique up to sign, that satisfies the following Dirac equation

$$\begin{pmatrix} q & \partial \\ -\bar{\partial} & q \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{D}$$

and

$$df = \bar{\lambda}\mathbf{j}dz\lambda.$$

The Dirac potential q is determined by $2q|dz| = H|df|$ (where H is the mean curvature of f), the induced metric satisfies $|df| = (|\lambda_1|^2 + |\lambda_2|^2)|dz|$, and the Willmore or elastic energy $W(f) = \int_{f(U)} H^2 = 4\int_U q^2$. A global version of this representation in terms of a uniformizing coordinate was given by Taimanov [17, 20]. Another global version in terms of quaternionic holomorphic line bundles on Riemann surfaces that does not refer to a special coordinate was introduced by Pedit and Pinkall [15].

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2 mNV Flow

The integrable hierarchy of the *modified Novikov–Veselov flow*

$$\begin{aligned} q_t &= q_{zzz} + q_{\bar{z}\bar{z}\bar{z}} + 3(q_z p + q_{\bar{z}}\bar{p}) + \tfrac{3}{2}(q p_z + q\bar{p}_{\bar{z}}), \\ p_{\bar{z}} &= (q^2)_z, \end{aligned} \tag{mNV}$$

which was introduced by Bogdanov[1], preserves the dimension of the space of solutions to (D). The indices in this equation indicate partial derivatives.
The time evolution of the solutions to (D) is provided by the "A" operator in its "L, A, B" triple representation. Konopelchenko [11] and Taimanov [17] suggested to study the induced flow on the conformal immersions $f\colon U \to \mathbb{R}^3$. The theorem of Taimanov that the mNV flow deforms tori into tori and preserves their conformal type and Willmore energy showed a way to attack the Willmore conjecture [16] by applying powerful methods of the theory of integrable systems.

3 Möbius Invariant Representation

The Weierstrass representation of Section 1 is invariant under stretch rotations, since the space of solutions to (D) is invariant under quaternionic right multiplication. There is also a Möbius invariant representation [15] of conformal immersions into $\mathbb{R}^4$ in terms of solutions to a Dirac equation. It is again convenient to identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. The orientation preserving Möbius transformations are then represented by fractional linear transformations (verbatim as in the case of Möbius transformations of $\mathbb{C} = \mathbb{R}^2$). The local version of the Möbius invariant representation is then as follows (for the global version see Section 8): a smooth map $f\colon U \to \mathbb{H}$ is a conformal immersion if and only if f is the quotient

$$f = \tilde{\lambda}^{-1}\lambda$$

of two solutions $\lambda, \tilde{\lambda}\colon U \to \mathbb{C}^2 \cong \mathbb{C} \oplus \mathbf{k}\mathbb{C} = \mathbb{H}$ to the Dirac equation

$$\begin{pmatrix} q & \partial \\ -\bar{\partial} & \bar{q} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{$\mathbb{C}$–D}$$

for some function $q\colon U \to \mathbb{C}$. The functions $q, \lambda, \tilde{\lambda}$ are not unique, as in the Weierstrass representation of conformal immersions into $\mathbb{R}^3$. This is due to the fact that in contrast to the Weierstrass representation in $\mathbb{R}^3$ the conformal coordinate does not uniquely (up to sign) determine a trivialization of the corresponding quaternionic holomorphic line bundle, cf. Section 8.
The equation ($\mathbb{C}$–D) is associated to the Davey–Stewartson I or DS_3 flow:

$$\begin{aligned} q_t &= q_{zzz} + q_{\bar{z}\bar{z}\bar{z}} + 3(p q_z + \bar{p} q_{\bar{z}}) + 3(w + \tilde{w})q, \\ p_{\bar{z}} &= (|q|^2)_z, \quad w_{\bar{z}} = (\bar{q} q_z)_z, \qquad \tilde{w}_z = (\bar{q} q_{\bar{z}})_{\bar{z}}, \end{aligned} \tag{DS_3}$$

which for real q reduces to the mNV flow.
The equation ($\mathbb{C}$–D) also governs the Weierstrass representation of surfaces in $\mathbb{R}^4$ as introduced in [15]. And again the representation is not uniquely determined by the coordinate. However, Konopelchenko [12] suggested to investigate the effect of the DS_3 flow to surfaces. Later, Taimanov [19], although showing that there are due to the non uniqueness of the representation many DS_3 flows, proves (making appropriate choices) that the DS_3 flow preserves tori and their Willmore energy. Moreover Burstall, Pedit, and Pinkall introduced an mNV flow that is Möbius invariant by definition in [7]. One clearly hopes that the three flows are the same or at least strongly related. We will see now that this is true in the 1–dimensional reduction to surfaces of revolution.

4 Reduction to Surfaces of Revolution

Let $\gamma\colon I \supset \mathbb{R} \to \mathbb{R}_{>0} \oplus \mathbb{R}\mathbf{k}$ be a curve in the 1–$\mathbf{k}$–plane with positive real part. Then

$$f(x,y) = e^{\mathbf{i}\frac{y}{2}}\mathbf{j}\gamma(x)e^{-\mathbf{i}\frac{y}{2}}$$

defines a surface of revolution, cf. Figure 1. This surface is conformally parametrized if and only if the meridian curve γ is parametrized by arc length with respect to the hyperbolic metric $g_p(X,Y) = \mathrm{Re}^{-2}(p)\langle X, Y\rangle = \mathrm{Re}^{-2}(p)\,\mathrm{Re}(X\bar{Y})$. Then $\mathrm{Re}(\gamma) = |\gamma'| = |df|(dx^2+dy^2)^{-\frac{1}{2}}$. The Weierstrass representation gets the following form: let $u\colon I \to \mathbb{R}$ then every solution $\lambda\colon I \to \mathbb{R}^2$ of

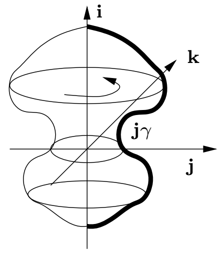

Figure 1:
surface of revolution

$$\begin{pmatrix} u & \frac{\partial}{\partial x} - \frac{1}{2} \\ \frac{\partial}{\partial x} + \frac{1}{2} & -u \end{pmatrix}\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = 0$$

defines via $\gamma' = (\lambda_2 + \mathbf{k}\lambda_1)^2$, $\mathrm{Re}(\gamma(x_0)) = |\lambda(x_0)|^2$, $x_0 \in I$ a curve parametrized by hyperbolic arc length (i.e., $\mathrm{Re}(\gamma) = |\gamma'|$) such that $u = H|\lambda|^2$, where H is the mean curvature of the corresponding surface of revolution. The mNV equation for $u(x,t) = 2q(x+iy, 4t)$ reduces to the modified Korteweg–de Vries equation

$$u_t = u_{xxx} + 6u_x u^2. \qquad \text{(mKdV)}$$

The Möbius invariant representation now reads: let $\kappa\colon I \to \mathbb{R}$ then every two solutions $\lambda, \tilde{\lambda}\colon I \to \mathbb{R}^2$ of

$$\begin{pmatrix} -\frac{1}{2}\kappa & \frac{\partial}{\partial x} - \frac{1}{2} \\ \frac{\partial}{\partial x} + \frac{1}{2} & \frac{1}{2}\kappa \end{pmatrix}\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = 0$$

define via $\gamma = (\tilde{\lambda}_1 - \mathbf{k}\tilde{\lambda}_2)^{-1}(\lambda_2 + \mathbf{k}\lambda_1)$, a curve γ parametrized by hyperbolic arc length with hyperbolic curvature κ. Substituting $\kappa(x,t) = 4q(x+iy, 4t)$ into (DS_3) one gets the mKdV flow for $\frac{1}{2}\kappa$, i.e.,

$$\kappa_t = \kappa_{xxx} + \frac{3}{2}\kappa_x\kappa^2.$$

The corresponding evolution of the curve γ is given by

$$\gamma_t = \kappa_x \mathbf{i}\gamma_x + \frac{1}{2}(\kappa^2 - 1)\gamma_x,$$

which preserves arc length parametrization. This flow was studied by Langer and Perline [14]. Moreover, Garay and Langer [9] noticed that *both* u and $\frac{1}{2}\kappa$ solve the mKdV equation, which shows that the mNV and DS_3 flow coincide in the case of surfaces of revolution, and that $u \longrightarrow \frac{1}{2}\kappa$ forms a Bäcklund transformation of the mKdV equation. If one for example starts with the meridian curve of the unit sphere ($\kappa = 0$, $u = \mathrm{sech}$) one gets the meridian curve of the catenoid ($\kappa = 2\,\mathrm{sech}$, $u = 0$).

5 Elastic Curves and mKdV Flow

The 1–solitons of the mKdV equation are its traveling wave solutions, i.e., $u_t = cu_x$ for some $c \in \mathbb{R}$. The hyperbolic curvature $\frac{1}{2}\kappa$ is an mKdV 1–soliton if and only if there exists $d \in \mathbb{R}$ such that

$$2\kappa_{xx} + \kappa^3 - 2c\kappa + d = 0.$$

This is the Euler–Lagrange equation of the hyperbolic elastic energy $E = \int \kappa^2 ds$, cf. [13]. More precisely, the hyperbolic curve whose curvature κ satisfies this equation is

- for $c = 1$ and $d = 0$ a free elastic curve, i.e., a critical point of E under all closed curves or curves with fixed first order boundary data,
- for $c \in \mathbb{R}$ and $d = 0$ an elastic curve, i.e., a critical point of E under all variations as above with fixed infinitesimal arc length,
- for $c, d \in \mathbb{R}$ a critical point of E under all variations as above with fixed infinitesimal arc length and enclosed area.

If we look for asymptotically geodesic solutions, i.e., $\kappa \to 0$ as $x \to \pm\infty$ one gets $d = 0$, $c > 0$, and

$$\kappa = 2\sqrt{c}\,\mathrm{sech}(\sqrt{c}x + \tilde{c}), \quad \tilde{c} \in \mathbb{R}.$$

These curvatures are the hyperbolic curvatures of the meridian curves of the famous catenoid cousins [6]. These are surfaces of revolution of mean curvature one in the Poincaré ball bounded by the smallest sphere that intersects the axes of revolution orthogonally and contains the surface, cf. Figure 2, where $c = (2\mu + 1)^2$ and μ is the parameter used by Bryant in [6].

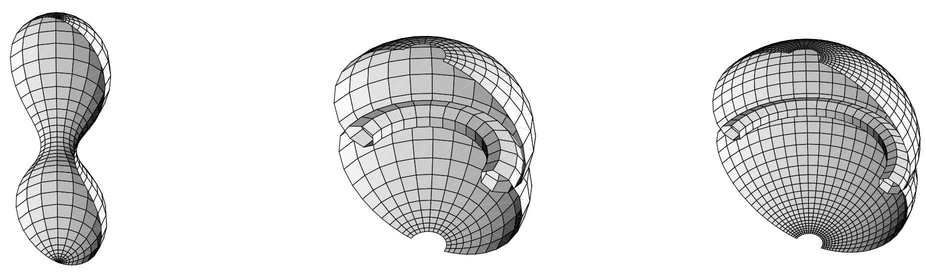

Figure 2a: $\mu = \frac{1}{9}$ Figure 2b: $\mu = 1$ Figure 2c: $\mu = 2$

In [5] it is shown that these surfaces extend through the axis of revolution to immersed spheres if and only if $\mu \in \mathbb{N}$.

6 Isothermic Darboux Transformation and mKdV Flow

A theorem of Hertrich–Jeromin, Musso, and Nicolodi [10] states that one gets the catenoid cousins as isothermic Darboux transformations of the round sphere, which forms the ideal boundary of the Poincaré ball in which they have mean curvature one. Here two surfaces are said to be *isothermic Darboux transformations* of each other if at each point there is an oriented sphere that touches both surfaces in orientation such that the point correspondence is conformal. So the Darboux transformation produce all mKdV 1–solitons from the trivial solution $\kappa = 0$, except $\kappa = 2\,\mathrm{sech}(x+\tilde{c})$, which is obtained from the Bäcklund transformation $u \longrightarrow \frac{1}{2}\kappa$ described in Section 4.

In terms of the meridian curves $\gamma, \hat{\gamma}\colon I \to \mathbb{R} \oplus \mathbb{R}\mathbf{k}$ of surfaces of revolution the isothermic Darboux transformation has the following form:

$$\hat{\gamma}'\gamma' = \tau^2, \quad \hat{\gamma} = \gamma + \rho\tau, \quad \rho \in \mathbb{R}. \tag{DT}$$

If γ is parametrized by hyperbolic arc length, i.e., $\mathrm{Re}(\gamma) = |\gamma'|$, then this equation is compatible with $\mathrm{Re}(\gamma + \rho\tau) = |\gamma' + \rho\tau'|$. Hence, there is a 1–parameter family of initial conditions such that the transformation preserves hyperbolic arc length, and one gets a Darboux transformation for arc length parametrized curves in the hyperbolic plane. The corresponding surfaces of revolution are then Darboux transforms of each other.

The mKdV flow is in the following sense an infinitesimal Darboux transformation. Suppose there exists a family of Darboux transforms

$$\hat{\gamma}(x,\rho) = \gamma(x) + \rho\tau(x,\rho),$$

that depends analytically on ρ. Writing $\hat{\gamma}'\gamma' = \tau^2$ as

$$\frac{\tau'}{\tau} = \frac{1}{\rho}\left(\frac{\tau}{\gamma'} - \frac{\gamma'}{\tau}\right),$$

one gets

$$\hat{\gamma} = \gamma + \rho\gamma' + O(\rho^2).$$

So the infinitesimal Darboux transformation is just reparametrization. This may be eliminated taking $\tilde{\gamma}(x,\rho) = \hat{\gamma}(x-\rho,\rho)$. Then also the second order term vanishes and one gets

$$\tilde{\gamma} = \gamma + 3\rho^3\left[\kappa_x \mathbf{i}\gamma_x + \frac{1}{2}(\kappa^2-1)\right]\gamma_x + O(\rho^4).$$

The fact that the Darboux transformation commutes with the mKdV flow may now be deduced from Bianchi's parmutability theorem: let $\hat{\gamma}_{1,2}$ be two Darboux transforms of γ with respect to two parameters $\rho = \rho_{1,2}$ in (DT) then

$$\hat{\gamma} = \gamma_1 + \rho_2\tau_{12}, \quad \tau_{12} = \frac{\rho_1\tau_1 - \rho_2\tau_2}{\rho_1\tau_2 - \rho_2\tau_1}\tau_1,$$

is a Darboux transform of *both* $\hat{\gamma}_1$ and $\hat{\gamma}_2$ with parameter ρ_2 and ρ_1, respectively.

7 Generalized Darboux Transformation

A fundamental feature of the isothermic Darboux transformation is expressed by the word "isothermic": it exists if and only if one (and then both) surfaces are isothermic, i.e., admit a conformal curvature line parametrization. In order to get a transformation that applies to a larger class of surfaces Pinkall suggested to relax the touching condition [2, 3]: a conformal immersion $f^\sharp\colon M \to S^4$ of Riemann surface into the 4–sphere is called to be a *generalized Darboux transformation* of a conformal immersion $f\colon M \to S^4$ if at each point $p \in M$ there is an oriented sphere that touches f and *left–touches* $f^\sharp$ at p with the right orientation. Two oriented 2–dimensional subspaces of $\mathbb{R}^4$ are left–touching if their oriented intersection great circles in the Lie group $S^3 \subset \mathbb{R}^4$ are right translates of each other. If one restricts to conformal immersions into S^3 then the generalized Darboux transformation reduces to the isothermic, since left–touching 2–dimensional oriented subspaces in $\mathbb{R}^4$ that only span a 3–space coincide.

8 Soliton Surfaces and Darboux Transformation

In order to formulate the result we aim for, we need the global Weierstrass and Möbius invariant representations of conformal immersions into S^4 (for details we refer to [15, 8]).
Let M be a Riemann surface. The *global Weierstrass representation* then reads: if $f\colon M \to \mathbb{R}^4$ is a conformal immersion then there exists a quaternionic holomorphic line bundle L over M and holomorphic sections φ of KL^{-1} and ψ of L such that

$$df = \varphi(\psi).$$

The data L, φ, ψ is up to bundle isomorphism uniquely determined by f. Quaternionic multiples of φ and ψ correspond to similarity transforms of f. The Willmore energy of L satisfies $W(L) =$

$\int|\mathcal{H}|^2$, where $\mathcal{H}$ is the mean curvature vector of f. If f is $\mathbb{R}^3$–valued then $\varphi \mapsto \psi$ defines an isomorphism of KL^{-1} and L so one might write $df = \psi^2$, which is the coordinate free version of $df = \bar{\lambda}\mathbf{j}dz\lambda$ of Section 1.
The *global Möbius invariant representation* reads: If $f\colon M \to \mathbb{R}^4$ is a conformal immersion then there exists a quaternionic holomorphic line bundle L over M and holomorphic sections φ, ψ (ψ without zeros) of L such that f is the quotient of φ and ψ, i.e.,

$$\varphi = \psi f.$$

The data L, φ, ψ are up to bundle isomorphism uniquely determined by f. Quotients from different elements in $\mathrm{span}_{\mathbb{H}}\{\varphi, \psi\}$ are the Möbius transforms of f. The Willmore energy of L satisfies $W(L) = \int(|\mathcal{H}|^2 - K + K^{\perp})$, where K and $K^{\perp}$ are the curvatures of the tangent and normal bundle of f.
A fundamental theorem in the quaternionic holomorphic geometry is the *Plücker estimate* [8]: Let M be a compact Riemann surface of genus g, L a quaternionic holomorphic line bundle over M, H a quaternionic $(n+1)$–dimensional linear space of holomorphic sections of L, d the degree of L. Then one has the following lower estimate for the Willmore energy of L:

$$\frac{1}{4\pi}W(L) \geq (n+1)(n(1-g)-d) + \operatorname{ord} H,$$

where $\operatorname{ord} H$ takes account of the common zeros of the elements of H and their derivatives.
If L is the quaternionic holomorphic line bundle of the Weierstrass representation of a sphere of revolution then equality holds in the Plücker estimate if and only if the hyperbolic curvature of the potential u of the meridian curve is a soliton of the mKdV equation, i.e., a reflectionless potential of the operator $\begin{pmatrix} u & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -u \end{pmatrix}$. Taimanov [18] studied these *soliton spheres* and provided explicit formulas. The immersed Taimanov soliton spheres for which $\frac{1}{2}\kappa$ is a 1–soliton are the catenoid cousins for $\mu \in \mathbb{N}$ and H is 2–dimensional in this case.
Generalizing Taimanov's definition we call a conformal immersion $f\colon \mathbb{C}P^1 \to \mathbb{R}^4$ a *soliton sphere* if one of the holomorphic sections φ or ψ of its Weierstrass representation is contained in a linear space of holomorphic sections (of dimension k) with equality in the Plücker estimate. Then one can show [4] that f is a soliton sphere if and only if the holomorphic sections φ and ψ of the Möbius invariant representation of f or its quaternionic conjugate $\bar{f}$ is contained in a linear space of holomorphic sections (of dimension $k+1$) with equality in the Plücker estimate.
Two important classes of examples of soliton spheres are provided by the Bryant spheres with smooth ends [5] (immersed spheres that are besides finitely many ends surfaces with mean curvature one in hyperbolic space, e.g., the catenoid cousins for $\mu \in \mathbb{N}$) and the Willmore spheres [4] (critical points of the Willmore energy). In both cases the holomorphic sections φ and ψ of the Möbius invariant representation are contained in a 3–dimensional linear system with equality in the Plücker estimate.
Let $f\colon M \to S^4$ be a conformal immersion, and L, φ, ψ the data of its Möbius invariant representation. Then holomorphic sections of L that are linear independent of φ and ψ and have no zeros give (in a unique way) rise to generalized Darboux transforms of f [2, 3]. Suppose that L contains a 3–dimensional linear space of holomorphic sections H which contains the sections φ and ψ of the representation. Let $f^{\sharp}\colon M \to S^4$ be a generalized Darboux transformation corresponding to a nowhere vanishing holomorphic section contained in H.

Theorem. *Then equality holds in the Plücker estimate for H if and only if $f^{\sharp}$ is the twistor projection of a holomorphic map $h\colon M \to \mathbb{C}P^3$.*

The soliton spheres with 3–dimensional linear systems in their Möbius invariant representation might be called 1–solitons (generalizing the fact that the surfaces of revolution where $\frac{1}{2}\kappa$ is a 1–soliton of the mKdV equation are examples). Then the theorem states that the 1–soliton spheres have a twistor holomorphic generalized Darboux transformation. Note that the only twistor holomorphic map $f^\sharp\colon \mathbb{C}P^1 \to S^3 \subset S^4$ is the round sphere, which is an isothermic Darboux transformation of the rotationally symmetric 1–solitons spheres (Section 6). As explained in Section 5 these 1–soliton spheres are the spheres of revolution with an elastic meridian curve. In the general case the elastic spheres (Willmore spheres) are 1–solitons but there are more, e.g., the Bryant spheres with smooth ends.

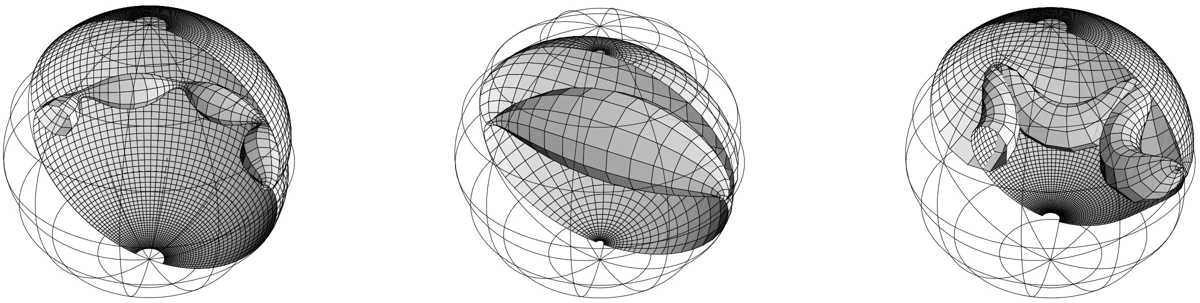

Figure 3: Bryant spheres with 2, 4, 6 smooth ends

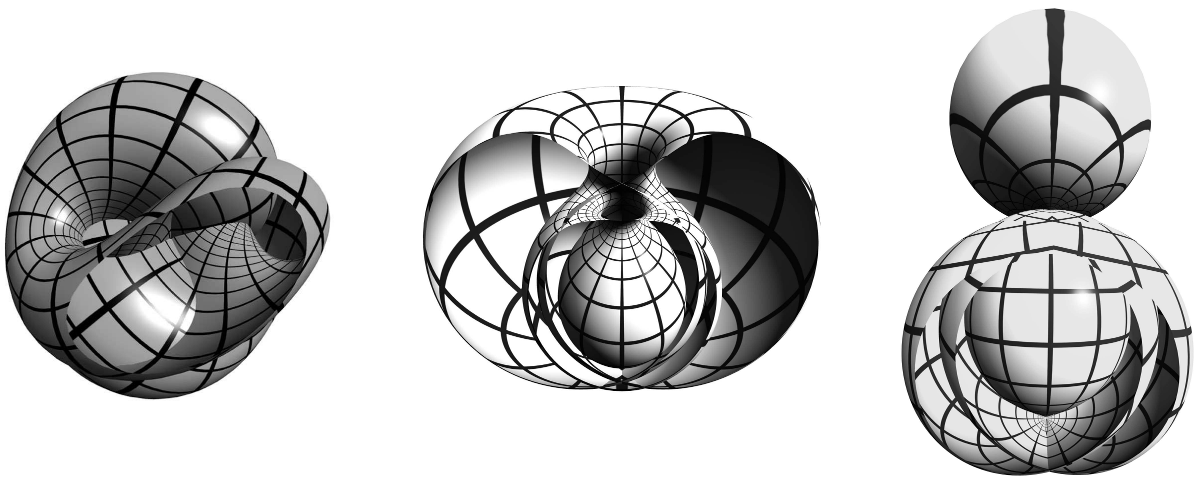

Figure 4: Willmore spheres with $W = 16\pi$.

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