Bryant Surfaces with Smooth Ends

G. Paul Peters
(joint work with Christoph Bohle [2])

Bryant surfaces—the surfaces of constant mean curvature 1 in hyperbolic space—have been studied intensively since Robert Bryant's influential paper [5], cf. [14] and the surveys [9, 11]. The following observation led us to introduce the notion of Bryant surfaces with smooth ends: the simplest nontrivial Dirac sphere [8] which is a surface of revolution related to a 1–soliton solution of the mKdV equation [13, 7, 3], is an immersed sphere that, besides the two points on the axis of rotation, is a Bryant surface in the Poincaré ball model of hyperbolic space (see the left surface in Figure 1).

Definition. *A Bryant surface E in the Poincaré ball model $\mathbf{B}^3 \subset \mathbb{R}^3$ of hyperbolic space is a* smooth Bryant end *if there is a point $p_\infty \in \partial\mathbf{B}^3$ on the asymptotic boundary such that $E \cup \{p_\infty\}$ is a conformally immersed open disc in $\mathbb{R}^3$.*

From the Möbius geometric point of view, smooth Bryant ends correspond to planar minimal ends in $\mathbb{R}^3$: both can be smoothly extended through the ideal boundary, i.e., the 2–sphere at infinity in the case of hyperbolic space and the point at infinity in the case of $\mathbb{R}^3 = S^3 \setminus \{\infty\}$. We obtain the following analog to the theorem that a planar minimal end may be parametrized by the real part of a holomorphic $\mathbb{C}^3$–valued map with a pole at the end.

Theorem 1. *A surface E in the Poincaré ball is a smooth Bryant end if and only if there exists a holomorphic null immersion $F\colon \Delta \setminus \{0\} \to \mathrm{SL}(2,\mathbb{C})$ with a pole at 0 such that $F'F^{-1}$ has a pole of order 2 and E is parametrized by Bryant's representation formula*

$$\text{(BRF)} \qquad f = \frac{1}{x_0+1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \begin{pmatrix} x_0 + x_3 & x_1 + x_2 i \\ x_1 - x_2 i & x_0 - x_3 \end{pmatrix} = F\bar{F}^t.$$

A deeper analogy becomes apparent when one considers compact surfaces in S^3 obtained by extending Bryant surfaces with smooth ends or minimal surfaces with planar ends through the respective ideal boundary. As Robert Bryant proved in [4, 6], the inversion of a complete finite total curvature minimal surfaces with planar ends extends to a compact Willmore surface (critical point of the Willmore energy $W = \int H^2 dA$) whose Willmore energy is 4π times the number of ends. Moreover, all Willmore spheres in S^3 are extended minimal surfaces with planar ends and the possible Willmore energies of Willmore spheres are $W \in 4\pi(\mathbb{N}^* \setminus \{2,3,5,7\})$. For Bryant surfaces, we prove:

Theorem 2. *Compact Bryant surfaces with smooth ends have Willmore energy $W = 4\pi n$, where $n \in \mathbb{N}^*$ is the total pole order of the Bryant representation, which is greater or equal to the number of Bryant ends. The possible Willmore energies of Bryant spheres with smooth ends are $4\pi(\mathbb{N}^* \setminus \{2,3,5,7\})$.*

Note that the Willmore energy W of the compact surface of genus g and the total curvature of the minimal or Bryant surface are related by $W + \int K dA = 4\pi(1-g)$.

In order to prove Theorem 2 we use the fact that Bryant spheres with smooth ends are related to rational null immersions into the non–degenerate quadric $\mathbf{Q}^3 \subset \mathbb{CP}^4$ via Bryant's representation formula (BRF) and

$$\begin{aligned} \mathrm{SL}(2,\mathbb{C}) &= \left\{ F = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1 \right\} \\ &\cong \mathbf{Q}^3 \setminus \{e = 0\} = \{ \Phi = [a,b,c,d,e] \in \mathbb{CP}^4 \mid ad - bc - e^2 = 0,\ e \neq 0 \}. \end{aligned}$$

With this identification Theorem 1 becomes:

Theorem 1'. *If E is a smooth Bryant end, then there exists a holomorphic null immersion $\Phi\colon \Delta \to \mathbf{Q}^3$ that parametrizes E via (BRF). Conversely, if $\Phi\colon \Delta \to \mathbf{Q}^3$ is a holomorphic null immersion that intersects $\{e = 0\}$*

(1) *transversely at 0, then F and F^{-1} (restricted to $\Delta \setminus \{0\}$) parametrize smooth horospherical Bryant ends, or*

(2) *non–transversely at 0, then either F or F^{-1} (restricted to $\Delta \setminus \{0\}$) parametrizes a smooth catenoidal Bryant end.*

We then show that (similar to minimal surfaces with planar ends) the degree d of the null immersion Φ into $\mathbf{Q}^3$ and the Willmore energy of the corresponding Bryant surface with smooth ends are related by $W = 4\pi d$. This proves the first statement of Theorem 2. The second statement follows from the fact that rational null immersions into $\mathbf{Q}^3$ exist for every degree, except $2, 3, 5$, and 7, cf. [6] .

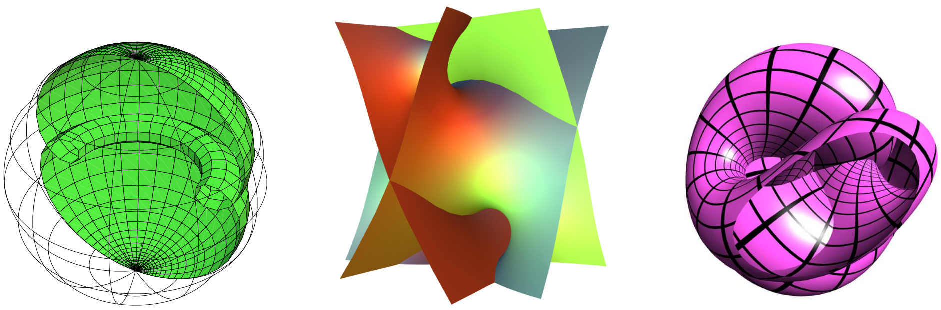

FIGURE 1.

Figure 1 shows three surfaces obtained from one degree 4 rational null immersion into $\mathbf{Q}^3$: a Bryant sphere with smooth ends and Willmore energy $W = 16\pi$ (which is a catenoid cousin), a minimal sphere with 4 planar ends, and its inversion, which is a Willmore sphere with $W = 16\pi$.

Figure 2 shows two Bryant spheres with smooth ends that are obtained applying an orthogonal transformation to the rational null immersion used for the surfaces in Figure 1. For the first surface the orthogonal transformation fixes the hyperplane $\{e = 0\}$ at infinity. Such transformations were also studied by Wayne Rossman, Masaaki Umehara, and Kotaro Yamada [10, 12]. The transformation for the second

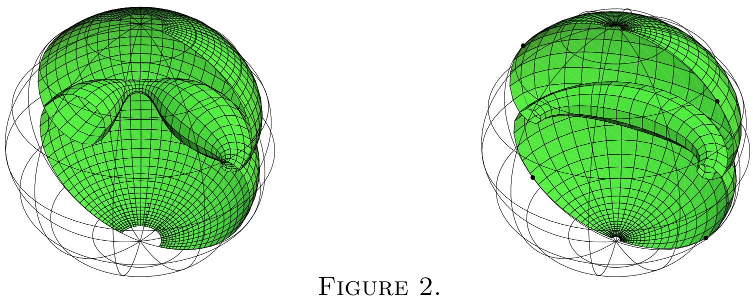

FIGURE 2.

surface does not fix the hyperplane at infinity, such that both catenoidal ends open up, and one gets a surface with 4 horospherical ends (marked points).

Both Bryant spheres with smooth ends and Willmore spheres in S^3 are examples of a more general class of surfaces in S^3 which we call soliton spheres, cf. [3]. Generalizing Robert Bryant's result about Willmore spheres and the second statement of Theorem 2, we prove that the quantization $W \in 4\pi\mathbb{N}^* \setminus \{2, 3, 5, 7\}$ holds for all soliton spheres in S^3.

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