Convex integral functionals of regular processes

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Abstract

This article gives dual representations for convex integral functionals on the linear space of regular processes. This space turns out to be a Banach space containing many more familiar classes of stochastic processes and its dual is identified with the space of optional Radon measures with essentially bounded variation. Combined with classical Banach space techniques, our results allow for systematic treatment of a large class of optimization problems from optimal stopping to singular stochastic control and financial mathematics.

Keywords. regular process; integral functional; conjugate duality

AMS subject classification codes. 46N10, 60G07

1 Introduction

This article studies convex integral functionals of the form

$$EI_h(v) = E \int_0^T h_t(v_t) \mu_t$$

defined on the linear space $\mathcal{R}^1$ of regular processes in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. Here $\mu$ is a positive optional measure on $[0, T]$ and $h$ is a convex normal integrand on $\Omega \times [0, T] \times \mathbb{R}^d$. An optional cadlag process $v$ of class (D) is regular if $Ev_{\tau_\nu} \to Ev_\tau$ for every increasing sequence of stopping times $\tau_\nu$ converging to a finite stopping time $\tau$ or equivalently (see [DM82, Remark 50d]), if the predictable projection and the left limit of $v$ coincide. Regular processes is quite a large family of stochastic processes containing e.g. continuous adapted processes, Levy processes and Feller processes as long as they are of class (D).

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A semimartingale is regular if and only if it is of class \((D)\) and the predictable
BV part of its Doob–Meyer decomposition is continuous.

Inspection of [Bis78] reveals that \(R^1\) is a Banach space under a suitable
norm and its dual may be identified with the space \(\mathcal{M}^\infty\) of optional random
measures with essentially bounded variation. Our main result characterizes the
conjugate and subdifferential of \(EI_h\). This allows for functional
analytic treatment of many stochastic optimization problems studied in the
literature. As examples, we establish the existence of solutions and optimality
conditions for optimal stopping, the finite fuel problem and a class of singular
stochastic control problems. Our approach unifies and complements earlier
results obtained e.g. in [BS77, EK81, BSW81, LS86]. Applications to mathematical
finance will be given in a separate article.

We employ the duality theory of integral functionals on the space of continuous
functions developed by [Roc71b]. This is combined with Bismut’s character-
ization of regular processes as optional projections of continuous stochastic
processes; see [Bis78]. Our main result states that if the conjugate \(h^*\) of \(h\) is the
optimal projection of a convex normal integrand that allows for Rockafellar’s
dual representation of \(I_h\) scenariowise, then under mild integrability conditions,
the dual representation of \(EI_h\) is given simply as the expectation of that of \(I_h\).

Section 6 studies the classical problems of optimal stopping, the finite fuel
problem and singular stochastic control by embedding them in the conjugate
duality framework of Rockafellar [Roc74]. Our results allow for explicit expres-
sions for the dual problem and optimality conditions while existence of solutions
follows from the classical Banach-Alaoglu theorem applied in the dual \(\mathcal{M}^\infty\) of the
space \(R^1\) of regular processes.

2 Integral functionals and duality

This section collects some basic facts about integral functionals defined on the
product of a measurable space \((\Xi, \mathcal{A})\) and a Suslin locally convex vector space
\(U\). Recall that a Hausdorff topological space is \textit{Suslin} if it is a continuous image
of a complete separable metric space. We will also assume that \(U\) is a countable
union of Borel sets that are Polish spaces in their relative topology. Examples
of such spaces include separable Banach spaces as well as their topological duals
when equipped with the weak*-topology. Indeed, such dual spaces are Suslin
[Trè67, Proposition A.9] and their closed unit balls are metrizable in the weak*
topology by [DS88, Theorem V.5.1], compact by the Banach–Alaoglu theorem,
and thus separable by [DS88, Theorem I.6.25].

A set valued mapping \(S: \Xi \rightrightarrows U\) is \textit{measurable} if the inverse image \(S^{-1}(O) := \{\xi \in \Xi \mid S(\xi) \cap O \neq \emptyset\}\) of every open \(O \subseteq S\) is in \(\mathcal{F}\). An extended real-valued
function \(f : U \times \Xi \to \mathbb{R}\) is said to be a \textit{normal integrand} if the \textit{epigraphical mapping}
\[
\xi \mapsto \text{epi} f(\cdot, \xi) = \{(u, \alpha) \in U \times \mathbb{R} \mid f(u, \xi) \leq \alpha\}
\]
is closed-valued and measurable. A normal integrand \(f\) is said to be \textit{convex}
if \(f(\cdot, \xi)\) is a convex function for every \(\xi \in \Xi\). A normal integrand is always
$\mathcal{B}(U) \otimes \mathcal{A}$-measurable, so $\xi \mapsto f(u(\xi), \xi)$ is $\mathcal{A}$-measurable whenever $u : \Xi \to U$ is $\mathcal{A}$-measurable. Conversely, if $(\Xi, \mathcal{A})$ is complete with respect to some $\sigma$-finite measure $m$, then any $\mathcal{B}(U) \otimes \mathcal{A}$-measurable function $f$ such that $f(\cdot, \xi)$ is lsc, is a normal integrand; see Lemma 22 in the appendix. Note, however, that the optional $\sigma$-algebra on $\Omega \times [0, T]$ is not complete [Ran90], so we cannot always use this simple characterization when studying integral functionals of optional stochastic processes.

Given a normal integrand and a nonnegative measure $m$ on $(\Xi, \mathcal{A})$ the associated integral functional

$$I_f(u) := \int_{\Xi} f(u(\xi), \xi)dm(\xi)$$

is a well-defined extended real-valued function on the space $L^0(\Xi, \mathcal{A}, m; U)$ of equivalence classes of $U$-valued $\mathcal{A}$-measurable functions. Here and in what follows, we define the integral of a measurable function as $+\infty$ unless the positive part of the function is integrable. This convention is not arbitrary but specifically suited for studying minimization problems involving integral functionals. The function $I_f$ is called the integral functional associated with the normal integrand $f$. If $f$ is convex, $I_f$ is a convex function on $L^0(\Xi, \mathcal{A}, m; U)$.

Normal integrands are quite general objects and they arise naturally in various applications. We list below some useful rules for checking whether a given function is a normal integrand. When $U$ is a Euclidean space, these results can be found e.g. in [Roc76, RW98]. For Suslin spaces, we refer to the appendix.

A function $f : U \times \Xi \to \mathbb{R}$ is a Carathéodory integrand if $f(\cdot, \xi)$ is continuous for every $\xi$ and $f(u, \cdot)$ is measurable for every $u \in U$. Carathéodory integrands are normal; see Proposition 23 in the appendix. If $S : \Omega \to U$ is a measurable closed-valued mapping then its indicator function

$$\delta_{S(\xi)}(u) = \begin{cases} 0 & \text{if } u \in S(\xi) \\ +\infty & \text{otherwise}, \end{cases}$$

is a normal integrand. Many algebraic operations preserve normality. In particular, pointwise sums, recession functions and conjugates of proper normal integrands are again normal integrands; see Lemma 24 in the appendix. Recall that the recession function of a closed proper convex function $g$ is defined by

$$g^\infty(u) = \sup_{\alpha > 0} \frac{g(\bar{u} + \alpha u) - g(\bar{u})}{\alpha},$$

where the supremum is independent of the choice of $\bar{u} \in \text{dom } g$. When $U$ is in separating duality with another linear space $Y$, the conjugate of $g$ is the extended real valued function $g^*$ on $Y$ defined by

$$g^*(y) = \sup_{u \in U} \{ \langle u, y \rangle - g(u) \}.$$
2.1 Integral functionals on decomposable spaces

A space $U \subseteq L^0(\Xi, \mathcal{A}, m; U)$ is decomposable if

$$1_{A}u + 1_{\Omega \setminus A}u' \in U$$

whenever $u \in U$, $A \in \mathcal{F}$ and $u' \in L^0(\Xi, \mathcal{A}, m; U)$ is such that the closure of the range of $u'$ is compact. The following result combines the results of Rockafellar [Roc68, Roc71a] with their reformulation to Suslin spaces by Valadier [Val75].

**Theorem 1** (Interchange rule). Assume that $U = \mathbb{R}^d$ or that $\mathcal{A}$ is $m$-complete. Given a normal integrand $f$ on $U$, we have

$$\inf_{u \in U} f(u) = \int_{\Xi} \inf_{u \in U} f(u, \xi) dm(\xi)$$

as long as the left side is less than $+\infty$.

The interchange rule is convenient for calculating conjugates of integral functionals on decomposable spaces. Assume that $Y$ is a Suslin space in separating duality with $U$ and assume that $Y \subseteq L^0(\Xi, \mathcal{A}, m; Y)$ is a decomposable space in separating duality with $U$ under the bilinear form

$$\langle u, y \rangle := \int_{\Xi} \langle u(\xi), y(\xi) \rangle dm(\xi).$$

The following theorem is Valadier’s extension of Rockafellar’s conjugation formula to Suslin-valued function spaces; see [Val75]. For a convex function $g$ on $U$, $y \in Y$ is a subgradient of $g$ at $u$ if

$$g(u') \geq g(u) + \langle u' - u, y \rangle \quad \forall u' \in U.$$ 

The set $\partial g(u)$ of all subgradients is known as the subdifferential of $g$ at $u$. We often use the fact $y \in \partial g(u)$ if and only if

$$g(u) + g^*(y) = \langle u, y \rangle.$$ 

For a normal integrand $f$ and for any $u \in L^0(\Xi, \mathcal{A}, m; U)$, we denote by $\partial f(u)$ the set-valued mapping $\xi \mapsto \partial f(u(\xi), \xi)$, where the subdifferential is with respect to the $u$-argument.

**Theorem 2.** Assume that $U = \mathbb{R}^d$ or that $\mathcal{A}$ is $m$-complete. Given a normal integrand $f$ on $U$, the integral functionals $I_f$ and $I_{f^*}$ on $U$ and $Y$ are conjugates of each other as soon as they are proper and then $y \in \partial I_f(u)$ if and only if

$$y \in \partial I_f(u) \quad a.e.$$ 

**Proof.** Valadier [Val75] did not give the subdifferential formula but it follows from the conjugate formula just like in [Roc68]: when $I_f$ and $I_{f^*}$ are conjugates of each other, then $y \in \partial I_f(u)$ if and only if

$$I_f(u) + I_{f^*}(y) = \langle u, y \rangle.$$
which, by the Fenchel inequality $f(u) + f^*(y) \geq \langle u, y \rangle$, is equivalent to

$$f(u) + f^*(y) = \langle u, y \rangle \text{ a.e.}$$

which in turn is equivalent to $y \in \partial f(u)$ a.e. \hfill \square

### 2.2 Integral functionals of continuous functions

Consider now the case where $U = \mathbb{R}^d$ equipped with the Euclidean topology and $\Xi$ is a compact interval $[0, T] \subset \mathbb{R}$ equipped with the Borel sigma-algebra and a nonnegative Radon measure $\mu$ with full support, i.e., $\text{supp} \mu = [0, T]$. This section reviews conjugation of convex integral functionals on the space $C$ of $\mathbb{R}^d$-valued continuous functions on an interval $[0, T]$. Recall that under the supremum norm, $C$ is a Banach space whose dual can be identified with the linear space $M$ of (signed) Radon measures $\theta$ through the bilinear form

$$\langle v, \theta \rangle := \int vd\theta.$$

Here and in what follows, the domain of integration is $[0, T]$ unless otherwise specified.

Given a normal integrand $h$ on $\mathbb{R}^d \times [0, T]$, consider the integral functional $I_h$ on $C$. The space $C$ is not decomposable so one cannot directly apply the interchange rule to calculate conjugate of $I_h$. Rockafellar [Roc71b] and more recently Perkkiö [Per14] gave conditions under which

$$(I_h)^* = J_h^*,$$  \hfill (1)

where for a normal integrand $f$ on $\mathbb{R}^d \times [0, T]$, the functional $J_f : M \to \mathbb{R}$ is defined by

$$J_f(\theta) = \int f(d\theta/d\mu)d\mu + \int f^\infty(d\theta/d|\theta^s|)d|\theta^s|,$$

where $\theta^s$ is the singular part of $\theta$ with respect to $\mu$ and $|\theta^s|$ is the total variation of $\theta^s$.

The validity of (1) depends on the behavior of the set-valued mapping

$$\text{dom } h_t = \{x \in \mathbb{R}^d \mid h_t(x) < \infty\}$$

as a function of $t$. Recall that a set-valued mapping $S$ from $[0, T]$ to $\mathbb{R}^d$ is inner semicontinuous (isc) if $\{t \mid S_t \cap A \neq \emptyset\}$ is an open set for any open $A$; see [RW98, Section 5B]. Given a normal integrand $h$, we will use the notation $\partial h_t^* := \partial\delta_{\text{cl dom } h_t}$, that is, $\partial h_t^*(v)$ is the normal cone to $\text{cl dom } h_t$ at $v$. We denote the relative interior of a set $A$ by rint $A$.

**Theorem 3.** Assume that $\text{dom } h$ is isc and that for every $v \in C$,

$$v \in \text{rint dom } h_t \quad \forall t \quad \Longrightarrow \quad v \in \text{dom } I_h \quad \Longrightarrow \quad v \in \text{cl dom } h_t \quad \forall t.$$
Then $I_h : C \to \mathbb{R}$ and $J_{h^*} : M \to \mathbb{R}$ are conjugates of each other as soon as $J_{h^*}$ is proper and then $\theta \in \partial I_h(v)$ if and only if
\[
d\theta/d\mu \in \partial h(v) \quad \mu\text{-a.e.},
\]
\[
d\theta^* / d|\theta^*| \in \partial h^*(v) \quad |\theta^*|\text{-a.e.}.
\]

**Proof.** That $I_h$ and $J_{h^*}$ are conjugates of each other, follows from [Per14, Theorem 3]. Indeed, since $\text{cl } B = \text{cl } \text{rint } B$ for any convex $B \subseteq \mathbb{R}^d$ (see [Roc70, Theorem 6.3]), the inner semicontinuity of $\text{dom } h$ implies that of $t \mapsto \text{rint } \text{dom } h_t$. Michael’s selection theorem [Mic56, Theorem 3.1”] thus gives the existence of a $v \in C$ such that $v_t \in \text{rint } \text{dom } h_t$ for all $t$. Our assumptions then imply $v \in \text{dom } I_h$. All the assumptions of [Per14, Theorem 3] are thus met except the outer $\mu$-regularity of $\text{dom } h$. An inspection of the proof of [Per14, Theorem 3] reveals that this condition was only used to show that $v \in \text{dom } I_h$ implies $v_t \in \text{cl } \text{dom } h_t$ for all $t$, which is the last assumption above.

As to the subdifferential, for $v \in \text{dom } I_h$ and any $\theta \in M$, we have the Fenchel inequalities
\[
h(v) + h^*(d\theta/d\mu) \geq v \cdot (d\theta/d\mu) \quad \mu\text{-a.e.},
\]
\[
(h^*)^\infty(d\theta^*/d|\theta^*|) \geq v \cdot (d\theta^*/d|\theta^*|) \quad |\theta^*|\text{-a.e.},
\]
so $\theta \in \partial I_h(v)$ if and only if $I_h(v) + J_{h^*}(\theta) = \langle v, \theta \rangle$ which is equivalent to having the Fenchel inequalities satisfied as equalities which in turn is equivalent to the pointwise subdifferential conditions.

Rockafellar [Roc71b, Theorem 5] gave a local condition for the normal integrand $h$ so that the first implication
\[
v_t \in \text{rint } \text{dom } h_t \quad \forall t \implies v \in \text{dom } I_h
\]
in Theorem 3 is satisfied. This condition also implies that $\text{int } \text{dom } h_t \neq \emptyset$ for all $t$. The latter implication
\[
v \in \text{dom } I_h \implies v_t \in \text{cl } \text{dom } h_t \quad \forall t
\]
holds when $\text{dom } h$ is *fully lower semicontinuous*, see [Roc71b]. This was extended in [Per14] to normal integrands whose domain mappings that are *outer $\mu$-regular*. When $\text{int } \text{dom } h_t \neq \emptyset$ for all $t$ and the first implication holds, then outer $\mu$-regularity is also necessary for the theorem to hold [Per14, Corollary 1].

We also note that, for continuous $\text{dom } h$, the latter implication in Theorem 3 is met. Recall that an isc set-valued mapping $S$ is *continuous* if its graph $\{(x,t) | x \in S_t\}$ is closed.

**3 Integral functionals of continuous processes**

For the remainder of this paper, we fix a complete probability space $(\Omega, F, P)$. This section studies integral functionals on the Banach space $L^1(C)$ of random
continuous functions \( v \) with the norm

\[
\|v\|_{L^1(C)} := \mathbb{E} \sup_{t \in [0,T]} |v_t|.
\]

Here and in what follows, \( \mathbb{E} \) denotes the integral with respect to \( P \) (expectation).

The results of this section will be used to derive our main results on integral functionals of regular processes.

We endow the space \( M \) of Radon measures with the Borel sigma-algebra associated with the weak*-topology and we denote by \( L^\infty(M) \) the linear space of \( M \)-valued random variables \( \theta \) with essentially bounded variation. The total variation of a \( \theta \in M \) will be denoted by \( \| \cdot \|_{TV} \). The first part of the following is from [Bis78, Theorem 2].

**Theorem 4.** The Banach dual of \( L^1(C) \) may be identified with \( L^\infty(M) \) through the bilinear form

\[
\langle v, \theta \rangle := \mathbb{E} \int v \, d\theta.
\]

The dual norm on \( L^\infty(M) \) can be expressed as

\[
\|\theta\|_{L^\infty(M)} = \text{ess sup} \|\theta\|_{TV}.
\]

**Proof.** By Lagrangian duality,

\[
\|\theta\|_{L^\infty(M)} = \sup \{ \langle \theta, v \rangle \mid \mathbb{E} \|v\|_C \leq 1 \} \\
= \inf_{\lambda \in \mathbb{R}^+} \sup_v \{ \langle \theta, v \rangle - \lambda \mathbb{E} \|v\|_C \} \\
= \inf_{\lambda \in \mathbb{R}^+} \{ \mathbb{E} \delta_{\mathbb{B}}(\theta/\lambda) + \lambda \} \\
= \text{ess sup} \|\theta\|_{TV},
\]

where \( \mathbb{B} \) is the closed unit ball of the total variation norm and the third equality follows from Theorem 2.

We will study integral functionals associated with normal integrands that are defined for each \( \omega \in \Omega \) as integral functionals on \( C \) and \( M \) as in Section 2.2. We allow both the integrand \( h \) and the measure \( \mu \) to be random. More precisely, we will assume that \( \mu \) is a nonnegative random Radon measure with full support almost surely and that \( h \) is a convex normal integrand on \( \mathbb{R}^d \times \Xi \), where \( \Xi = \Omega \times [0,T] \) is equipped with the product sigma-algebra \( \mathcal{F} \otimes \mathcal{B}([0,T]) \). We define \( I_h : C \times \Omega \to \mathbb{R} \) and \( J^{\ast}_h : M \times \Omega \to \mathbb{R} \) by

\[
I_h(v, \omega) := I_h(\cdot, \omega)(v),
\]
\[
J^{\ast}_h(\theta, \omega) := J^{\ast}_h(\cdot, \omega)(\theta),
\]

where the right sides are defined as in Section 2.2.

\[\text{By usual monotone class arguments, the elements of } L^\infty(M) \text{ are random Radon measures in the sense of [DM82].}\]
Theorem 5. Assume that $h$ satisfies the assumptions of Theorem 3 almost surely. Then $I_h$ and $J_h^*$ are normal integrands conjugate to each other.

Proof. By Theorem 3, $I_h(\cdot,\omega)$ and $J_h^*(\cdot,\omega)$ are conjugate to each other almost surely. The uniform topology on $C$ satisfies both (a) and (b) of Theorem 25 in the appendix, so $I_h$ is a normal integrand. By Lemma 24, $J_h^*$ is a normal integrand as well.

Note that both $C$ and $M$ are countable unions of Polish spaces; see the beginning of Section 2. By Theorem 5 and Lemma 22, the integral functionals $E I_h : L^1(C) \to \mathbb{R}$ and $E J_h^* : L^\infty(M) \to \mathbb{R}$ are well defined. An application of the interchange rule Theorem 1 and Theorem 5 now gives expressions for the conjugate and subdifferential of $E I_h$.

Theorem 6. Assume that $h$ satisfies the assumptions of Theorem 3 almost surely. Then the convex functions $E I_h : L^1(C) \to \mathbb{R}$ and $E J_h^* : L^\infty(M) \to \mathbb{R}$ are conjugate to each other whenever they are proper and then $\theta \in \partial E I_h(v)$ if and only if

$$
\frac{d\theta}{d\mu} \in \partial h(v) \quad \mu\text{-a.e.},
$$

$$
\frac{d\theta^*}{d|\theta^*|} \in \partial h^*(v) \quad |\theta^*|\text{-a.e.}
$$

almost surely.

Proof. Theorems 2 and 5 give

$$(E I_h)^*(\theta) = E I_h^*(\theta) = E J_h^*(\theta).$$

The subgradient characterization now follows from Theorems 2 and 3.

4 Regular processes

Let $(\mathcal{F}_t)_{t \geq 0}$ be an increasing sequence of $\sigma$-algebras on $\Omega$ that satisfies the usual hypotheses that $\mathcal{F}_t = \bigcap_{t' \geq t} \mathcal{F}_{t'}$ and $\mathcal{F}_0$ contains all the $P$-null sets. We denote by $\mathcal{T}$ the set of stopping times, that is, functions $\tau : \Omega \to [0,T] \cup \{+\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0,T]$. A process is optional if it is measurable with respect to the $\sigma$-algebra generated by right-continuous adapted processes. If $v$ is $\mathcal{T}$-integrable in the sense that $v_\tau$ is integrable for every $\tau \in \mathcal{T}$, then, by [HWY92, Theorem 5.1], there exists a unique (up to indistinguishability) optional process $\circ v$ such that

$$
E[v_\tau 1_{\{\tau < \infty\}} \mid \mathcal{F}_\tau] = \circ v_\tau 1_{\{\tau < \infty\}} \quad P\text{-a.s.} \text{ for all } \tau \in \mathcal{T}.
$$

The process $\circ v$ is called the optional projection of $v$. In particular, every $v \in L^1(C)$ has a unique optional projection.
We will denote by $\mathcal{R}^1$ the space of regular processes, i.e., the optional càdlàg processes $v$ of class $(D)$ such that $E v_{\tau^v} \to E v_{\tau}$ for every increasing sequence of stopping times $\tau^v$ converging to a finite stopping time $\tau$ or equivalently (see [DM82, Remark 50d]), such that the predictable projection and the left limit of $v$ coincide. Recall that a process $v$ is of class $(D)$ if $\{v_{\tau} \mid \tau \in \mathcal{T}\}$ is uniformly integrable. By [Bis78, Theorem 3], the optional projection is a linear surjection of $L^1(C)$ to $\mathcal{R}^1$.

**Remark 1.** An optional càdlàg process $v$ of class $(D)$ is in $\mathcal{R}^1$, in particular, if it is quasi left-continuous in the sense that $\lim v_{\tau^v} = v_{\tau}$ a.s. for any strictly increasing sequence of stopping times $(\tau^v)_{v=1}^{\infty}$ with $\tau^v \nearrow \tau$. The converse holds when the the filtration $(\mathcal{F}_t)_{t \geq 0}$ is quasi-left continuous; see [DM82, Remark 50.(d)] and [HWY92, Theorem 4.34]. Continuous adapted processes, Levy processes ([HWY92, Theorem 11.36]) and Feller processes ([Kal02, Proposition 22.20]) are quasi left-continuous. A semimartingale $v$ is regular if and only if it is of class $(D)$ and has a decomposition $v = m + a$ where $m$ is a local martingale and $a$ is a continuous BV process. Indeed, a semimartingale of class $(D)$ is special so the claim follows from [DM82, Remark VII.24(e)].

We will denote by $\mathcal{M}^\infty \subseteq L^\infty(M)$ the space of essentially bounded optional Radon measures on $\mathbb{R}^d$, i.e. the elements $\theta \in L^\infty(M)$ such that
\[ E \int v \, d\theta = E \int_o v \, d\theta \quad \forall v \in L^1(C). \]

The following result, essentially proved already in Bismut [Bis78], shows that $\mathcal{M}^\infty$ may be identified with the Banach dual of $\mathcal{R}^1$.

**Theorem 7.** The space $\mathcal{R}^1$ is a Banach space under the norm
\[ \|v\|_{\mathcal{R}^1} := \sup_{\tau \in \mathcal{T}} E|v_{\tau}| \]
and its dual may be identified with $\mathcal{M}^\infty$ through the bilinear form
\[ \langle v, \theta \rangle_{\mathcal{R}^1} = E \int v \, d\theta. \]

The dual norm can be expressed as
\[ \|\theta\|_{\mathcal{M}^\infty} = \text{ess sup} \|\theta\|_{TV}. \]

**Proof.** Since the optional projection is a surjection from $L^1(C)$ to $\mathcal{R}^1$, it defines a linear bijection from the quotient space $L^1(C)/K$ to $\mathcal{R}^1$. Here $K$ denotes the kernel of the projection. For any $v \in L^1(C)$, Jensen’s inequality gives
\[ \|v\|_{\mathcal{R}^1} = \sup_{\tau \in \mathcal{T}} E|v_{\tau}| \leq \sup_{\tau \in \mathcal{T}} E|v_{\tau} | \leq \|v\|_{L^1(C)}. \]
\[^{2}\text{right-continuous with left limits}\]
So the optional projection is continuous. In particular, $K$ is closed in $L^1(C)$ so $L^1(C)/K$ is a Banach space under the quotient space norm

$$\|v\|_{L^1(C)/K} := \inf_{v' \in K} \|v + v'\|_{L^1(C)}.$$ 

On the other hand, for each $w \in \mathcal{R}^1$ and $\varepsilon > 0$, [Bis78, Theorem 3] gives the existence of a $v \in L^1(C)$ such that $w = {}^*v$ and $\|v\|_{L^1(C)} \leq \|w\|_{\mathcal{R}^1} + \varepsilon$. The optional projection is thus an isometric isomorphism from the quotient space $L^1(C)/K$ to $\mathcal{R}^1$. It follows that $\mathcal{R}^1$ is Banach and, by [Bis78, Proposition 2], its dual may be identified with $\mathcal{M}\mathcal{I}$. As to the dual norm,

$$\|\theta\|_{\mathcal{M}\mathcal{I}} = \sup_{v \in \mathcal{R}^1} \{\langle v, \theta \rangle \mid \|v\|_{\mathcal{R}^1} \leq 1\} = \sup_{v \in L^1(C)} \{\langle v, \theta \rangle \mid \|v\|_{L^1(C)} \leq 1\},$$

where the second equality comes from the isomorphism of $\mathcal{R}^1$ and $L^1(C)/K$. 

Theorem 7 complements the results of [DM82, Section 7.1.4] on Banach duals of adapted continuous functions and adapted cádlág functions under the supremum norm. The dual space of adapted continuous functions consists of predictable random measures with essentially bounded variation whereas the dual of adapted cádlág functions is given in terms of pairs of optional and predictable random measures with essentially bounded variation; see [DM82, Theorem VII.67]. In the deterministic case, Theorem 7 reduces to the familiar Riesz representation of continuous linear functionals on the space of continuous functions (the duality between $C$ and $M$).

The norm $\|\cdot\|_{\mathcal{R}^1}$ in Theorem 7 is studied in [DM82, Section VI], where a general measurable process $v$ is said to be “bounded in $L^1$” if $\|v\|_{\mathcal{R}^1} < \infty$; see [DM82, Definition VI.20]. It is observed on p. 82–83 of [DM82] that a sequence converging in the $\mathcal{R}^1$-norm has a subsequence that converges almost surely in the supremum norm. Moreover, by [DM82, Theorem VI.22], the space of optional cadlag processes with finite $\mathcal{R}^1$-norm is Banach. Theorem 7 implies that regular processes form a closed subspace of this space.

\section{Integral functionals of regular processes}

This section gives the main result of this paper. It characterizes the conjugate of a convex integral functional of the form

$$EI_h(v) = E \int h(v) d\mu$$

on the space $\mathcal{R}^1$ of regular processes. The functional $EI_h$ is well-defined on $\mathcal{R}^1$ since $h(v)$ is an extended real-valued measurable process for every $v \in \mathcal{R}^1$, so
$I_h(v)$ is $\mathcal{F}$-measurable by Lemma 26 in the appendix. The random measure $\mu$ is as in Section 3 with the additional assumption that it is optional, i.e.

$$E \int v d\mu = E \int o v d\mu$$

for every nonnegative bounded process $v$. Our main result involves the notion of an optional projection of a normal integrand that we now recall; see [KP15].

If $h$ is a convex normal integrand such that $h^+(x)$ is $\mathcal{T}$-integrable for some $\mathcal{T}$-integrable $x$ then, by [KP15, Theorem 13 and Lemma 9], there exists a unique optional convex normal integrand $\partial h$ such that

$$\partial h(v) = \partial [h(v)]$$

for every bounded optional process $v$. Here we use the notion of optional projection of an extended real-valued process; see the appendix. The normal integrand $\partial h$ is called the optional projection of $h$. Note that, if $h(v, \omega) = x_t(\omega) \cdot v$ for a measurable $\mathcal{T}$-integrable process $x$, then $\partial h_t(v, \omega) = \partial x_t(\omega) \cdot v$.

The following is the main result of this paper. Its proof will be given in the appendix. Here and in what follows, the abbreviation a.s.e. stands for “$P$-almost surely everywhere on $[0, T]$”, that is, outside an evanescent set.

**Theorem 8.** Assume that $h$ is a convex normal integrand such that $v \in \mathbb{R}^1 \cap \text{dom} EI_h$ implies $v \in \text{cl dom} h$ a.s.e. and $h^* = \partial (\tilde{h}^*)$ for a convex normal integrand $\tilde{h}$ that satisfies the assumptions of Theorem 3 almost surely and

$$\tilde{h}(v) \geq v \cdot \bar{x} - \alpha,$$

$$\tilde{h}^*(x) \geq \bar{v} \cdot x - \alpha$$

for some $\bar{v} \in L^1(C)$, bounded optional process $\bar{x}$, and some nonnegative $\mathcal{T}$-integrable $\alpha$ with $E \int \alpha d\mu < \infty$. Then $EI_h : \mathbb{R}^1 \to \mathbb{R}$ and $EJ_{h^*} : \mathcal{M}^\infty \to \mathbb{R}$ are proper and conjugate to each other, and, moreover, $\theta \in \partial EI_h(v)$ if and only if

$$d\theta / d\mu \in \partial h(v) \quad \mu \text{-a.e.,}$$

$$d\theta^* / d|\theta^*| \in \partial h^*(v) \quad |\theta^*| \text{-a.e.}$$

almost surely.

If $v$ is a regular process, then $h^*_t(x, \omega) = v_t(\omega) \cdot x$ satisfies the assumptions of Theorem 8. Indeed, in this case $h^* = \partial (\tilde{h}^*)$, where $\tilde{h}_t^*(x, \omega) = \tilde{v}_t(\omega) \cdot x$ for some $\tilde{v} \in L^1(C)$ with $v = \partial \tilde{v}$. The converse holds by Lemma 10 below. Note that in this linear case, $h_t(\cdot, \omega)$ is the indicator function of $\{v_t(\omega)\}$. Theorem 8 simplifies also in the other extreme where $h$ is real-valued.

**Corollary 9.** Assume that $h$ is a real-valued optional convex normal integrand such that $EI_h$ is finite on $L^1(C)$ and

$$h(v) \geq v \cdot \bar{x} - \alpha,$$

$$h^*(x) \geq \bar{v} \cdot x - \alpha$$
for some \( \tilde{v} \in L^1(C) \), bounded optional process \( \tilde{x} \), and some nonnegative \( T \)-integrable \( \alpha \) with \( E \int \alpha \, d\mu < \infty \). Then \( EI_h \) is finite on \( \mathcal{R}^1 \), \( EI_{h^\ast} \) is proper on \( \mathcal{M}^{\infty} \), they are conjugates of each other, and, moreover, \( \theta \in \partial EI_h(v) \) if and only if \( \theta \ll \mu \) and

\[
d\theta/d\mu \in \partial h(v) \quad \mu \text{-a.e.}
\]

almost surely.

**Proof.** As in the proof of Theorem 8 in the appendix, we see that \( EI_h(\circ v) \leq EI_{h}(v) \) for every \( v \in L^1(C) \). Thus, since \( EI_h \) is finite on \( L^1(C) \), \( EI_h \) never takes the value \(+\infty\) on \( \mathcal{R}^1 \).

By finiteness of \( EI_h \) on \( L^1(C) \), \( I_h(v) \) is almost surely finite for every continuous \( v \), and consequently the first implication in Theorem 3 holds almost surely, whereas the second implication holds by finiteness of \( h \). Thus we may apply Theorem 8 with \( \tilde{h} = h \) from which the result follows, since \( \partial h^\ast(v) = \{0\} \) when \( h \) is finite.

Note that the assumptions in the Theorem 8 do not imply that \( J_{h^\ast} \) is a normal integrand on \( M \times \Omega \). Indeed, in the linear case \( h^\ast_t(x, \omega) = v_t(\omega) \cdot x \), the function \( J_{h^\ast}(-, \omega) \) is lower semicontinuous on \( M \) only if \( v \) has continuous paths. Nevertheless, \( EJ_{h^\ast} \) is a well-defined convex functional on \( \mathcal{M}^{\infty} \) and it coincides with an integral functional associated with a normal integrand; see Lemma 30 in the appendix.

The rest of this section specializes Theorem 8 to the case where \( h \) or \( h^\ast \) is the indicator of a general closed convex-valued mapping. Given a measurable closed convex-valued mapping \( S \), there exists an optional closed convex-valued mapping \( \circ S \) such that \( \circ v \in \circ S \) a.s.e. for every \( T \)-integrable \( v \in S \) a.s.e. and \( \circ S \subseteq S \) a.s.e. for any \( S \) with the same properties. The mapping \( \circ S \) is called the *optional projection* of \( S \). If \( S \) has a \( T \)-integrable selection, then \( \circ S \) exists and it is unique [KP15, Theorem 17]. Clearly, if \( S_t(\omega) = \{v_t(\omega)\} \) for a \( T \)-integrable process \( v \), then \( \circ S_t(\omega) = \{\circ v_t(\omega)\} \).

The *support function* \( \sigma_{S_t}(x, \omega) := \sup_{v \in S_t(\omega)} x \cdot v \) of a measurable closed-valued mapping \( S \) is a normal integrand since it is the conjugate of the normal integrand \( \delta_S \).

**Lemma 10.** The support function of a closed convex-valued mapping \( S \) satisfies the assumptions of Theorem 8 if and only if \( v \in S \) a.s.e. for every \( v \in \mathcal{R}^1 \) with \( v \in S \) \( \mu \)-a.e. almost surely, \( S \) is the optional projection of a closed convex-valued measurable isc mapping \( \tilde{S} \) such that \( \text{dom} \, EI_{\delta_{\tilde{S}}} \cap L^1(C) \neq \emptyset \), and \( v \in \tilde{S} \) a.s.e. for every \( v \in L^1(C) \) with \( v \in \tilde{S} \) \( \mu \)-a.e. almost surely.

**Proof.** Sufficiency is obvious. To prove the necessity, let \( \tilde{h} \) be a convex normal integrand in Theorem 8 so that \( \circ h^\ast = \sigma_S \). By [KP15, Theorem 14], \( \circ [\circ (\tilde{h}^\ast)]^{\infty} = \circ (\tilde{h}^\ast)^{\infty} \). Since \( \circ (\tilde{h}^\ast) \) positively homogeneous, \( \circ (\tilde{h}^\ast) \) is also the optional projection of \( (\tilde{h}^\ast)^{\infty} \). But \( (\tilde{h}^\ast)^{\infty} = \sigma_{\text{cl dom} \tilde{h}} \), so we may choose \( \tilde{S} = \text{cl dom} \tilde{h} \). \( \square \)
We will say that a closed convex-valued mapping is regular if its support function satisfies the assumptions of Theorem 8. Given a stochastic process $v$, the above lemma implies that the set-valued mapping $S_t(\omega) = \{v_t(\omega)\}$ is regular if and only if $v \in \mathcal{R}^1$.

**Corollary 11.** Let $S$ be a regular set-valued mapping. Then

$$S = \{v \in \mathcal{R}^1 | v \in S \text{ a.s.e.}\}$$

is closed in $\mathcal{R}^1$,

$$\sigma_S(\theta) = E \int \sigma_S(d\theta/d|\theta|)d|\theta|$$

and $\theta \in N_S(v)$ if and only if $d\theta/d|\theta| \in N_S(v) |\theta|-\text{a.e. almost surely}$.

**Proof.** Since $S = \circ \tilde{S}$, we have $\circ v \in S$ for every $v \in EI_{\delta_\tilde{S}} \cap L^1(C)$, so $S \neq \emptyset$. Thus we may apply Theorem 8 to $h = \delta_\tilde{S}$.

**Corollary 12.** Let $S$ be a closed convex-valued optional continuous mapping that admits an $L^1(C)$ selection. Then

$$S = \{v \in \mathcal{R}^1 | v \in S \text{ a.s.e.}\}$$

is closed in $\mathcal{R}^1$,

$$\sigma_S(\theta) = E \int \sigma_{S_t}(d\theta/d|\theta|)d|\theta|$$

and $\theta \in N_S(v)$ if and only if $d\theta/d|\theta| \in N_S(v) |\theta|-\text{a.e. almost surely}$.

**Proof.** We apply Corollary 11 with $\mu(\omega)$ equal to the Lebesgue measure for every $\omega$. Applying [KP15, Theorem 15] to $\delta_\tilde{S}$, we see that $\circ \tilde{S} = S$. Thus $S$ is regular, since we may choose $\tilde{S} = \delta\tilde{S}$ in Lemma 10.

Recall that the support function of a convex cone is the indicator function of the polar cone. In the conical and continuous case, Corollary 12 reduces to the following.

**Corollary 13.** Let $K$ be a closed convex cone-valued optional continuous mapping. Then

$$K = \{v \in \mathcal{R}^1 | v \in K \text{ a.s.e.}\}$$

is closed in $\mathcal{R}^1$ and $\theta \in K^*$ if and only if $d\theta/d|\theta| \in K^* |\theta|-\text{a.e. almost surely}$.

In particular, the polar of the cone $\mathcal{R}^1_+$ of nonpositive regular processes is simply the cone $\mathcal{M}^\infty_+$ of nonnegative optional measures.

In the financial context, sets of the form $K^*$ are used to model self-financing trading strategies in the currency market model of Kabanov where the set $K^*$ describes the set of portfolios that are freely available in the market; see [KS09, Section 3.6.3]. By [RW98, Example 5.10 and Corollary 11.35], the assumptions in [KS09, Section 3.6.3] imply the conditions on $K$ in Corollary 13. The next result gives an extension to nonconical and discontinuous market models. The result follows by applying Theorem 8 to the case where $h^*$ is the indicator function of a convex set $S$. 

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**Corollary 14.** Let $S$ be an optional closed convex-valued mapping that admits a bounded optional selection. If $v \in (S^\infty)^*$ a.e. for every $v \in \mathbb{R}^1$ with $v \in (S^\infty)^*$ $\mu$-a.e. almost surely and $\delta_S = 0 \delta_{\tilde{S}}$ for a closed convex-valued measurable mapping $\tilde{S}$ such that $(\tilde{S}^\infty)^*$ is isc with $v \in (\tilde{S}^\infty)^*$ a.e. for every $v \in L^1(C)$ with $v \in (\tilde{S}^\infty)^*$ $\mu$-a.e. almost surely, then the set

$$C = \{\theta \in M^\infty | d\theta/d\mu \in S \mu$-a.e., $d\theta^\mu/d|\theta^\mu| \in S^\infty |\theta^\mu|$-a.e. $P$-a.s.}$$

is closed in $M^\infty$ and its support function has the integral representation $\sigma_C = EI_{\sigma_2}.$

### 6 Applications

The space $M$ of Radon measures may be identified with the space $X_0$ of $\mathbb{R}^d$-valued left-continuous functions of bounded variation on $\mathbb{R}$ which are constant on $(T, \infty]$ and $x_0 = 0.$ Indeed, for every $x \in X_0$, there exists a unique $Dx \in M$ such that $x_t = Dx([0, t])$ for all $t \in \mathbb{R}$. Thus $x \mapsto Dx$ defines a linear isomorphism between $X_0$ and $M.$ The value of $x$ for $t > T$ will be denoted by $x_{T^+}.$ Similarly, the space $M^\infty$ may be identified with the space $N_0^\infty$ of adapted processes $x$ with $x \in X_0$ almost surely and $Dx \in M^\infty$.

In the applications below, we will use the general results of Rockafellar [Roc74] on duality and optimality in convex parametric optimization problems. The optimization problems studied below will have the general form

$$\text{minimize } F(x, u) \text{ over } x \in N_0^\infty, \quad (P)$$

where the parameter $u$ belongs to a LCTVS $U$ in separating duality with a LCTVS $Y$ and $F$ is a proper convex function on $N_0^\infty \times U$ such that $F(x, \cdot)$ is closed for every $x \in N_0^\infty$. This fits the general duality framework of [Roc74] where the associated Lagrangian

$$L(x, y) = \inf \{F(x, u) - \langle u, y \rangle\}$$

plays an important role. The Lagrangian is an extended real-valued function on $N_0^\infty \times U$, convex in $x$ and concave in $y$. Clearly,

$$\varphi^*(y) = -\inf_{x \in N_0^\infty} L(x, y).$$

The following result is obtained by combining Theorem 17 and Corollary 15A of [Roc74] suffices in the applications studied below.

**Theorem 15.** Assume that the optimal value function

$$\varphi(u) = \inf_{x \in N_0^\infty} F(x, u)$$

is proper and continuous on $U$. Then $\varphi = \varphi^{**}$ and an $x \in N_0^\infty$ solves $(P)$ if and only if there exists $y \in Y$ such that

$$0 \in \partial x L(x, y) \quad \text{and} \quad u \in \partial y[-L](x, y).$$
6.1 Optimal stopping

Let \( R \in \mathcal{R}^1 \) and consider the optimal stopping problem

\[
\text{maximize } \mathbb{E} R_\tau \quad \text{over } \tau \in \mathcal{T}.
\]

Existence of optimal stopping times has been established by Bismut and Skalli [BS77] and El Karoui [EK81, Section 2.18] for bounded regular processes. This section gives a functional analytic proof of the existence for general \( R \in \mathcal{R}^1 \).

In order to apply results of the previous sections, we first write the problem as

\[
\text{maximize } \langle R, Dx \rangle \quad \text{over } x \in \mathcal{C}_c,
\]

where \( \mathcal{C}_c := \{ x \in \mathcal{N}_0^\infty \mid Dx \in \mathcal{M}_\infty^c, \ x_t \in \{0, 1\} \} \). The equation \( \tau(\omega) = \inf \{ t \in \mathbb{R} \mid x_t(\omega) \geq 1 \} \) gives a one-to-one correspondence between the elements of \( \mathcal{T} \) and \( \mathcal{C}_c \). Consider also the convex relaxation

\[
\text{maximize } \langle R, Dx \rangle \quad \text{over } x \in \mathcal{C},
\]

where \( \mathcal{C} := \{ x \in \mathcal{N}_0^\infty \mid Dx \in \mathcal{M}_\infty^c, \ x_{T+} \leq 1 \} \). Clearly, \( \mathcal{C}_c \subset \mathcal{C} \) so the optimum value of optimal stopping is dominated by the optimum value of the relaxation.

The elements of \( \mathcal{C} \) are randomized stopping times in the sense of Baxter and Chacon [BC77, Section 2].

Recall that \( x \in \mathcal{C} \) is an extreme point of \( \mathcal{C} \) if it cannot be expressed as a convex combination of two points of \( \mathcal{C} \) different from \( x \).

Lemma 16. The set \( \mathcal{C} \) is convex, \( \sigma(\mathcal{N}_0^\infty, \mathcal{R}^1) \)-compact and \( \mathcal{C}_c \) is the set of its extreme points.

Proof. The set \( \mathcal{C} \) is a closed convex set of the dual ball that \( \mathcal{N}_0^\infty \) has as the dual of the Banach space \( \mathcal{R}^1 \). Thus, the compactness follows from Banach-Alaoglu. It is easily shown that the elements of \( \mathcal{C}_c \) are extreme points of \( \mathcal{C} \). It thus suffices to show that extreme points of \( \mathcal{C} \) are in \( \mathcal{C}_c \).

The optional measure associated with a randomized stopping time \( x \in \mathcal{C} \) can be represented pathwise as the inverse image under \( t \mapsto x_t(\omega) \) of the Lebesgue measure \( \lambda \) on \([0, 1] \). We have \( x \in \mathcal{C}_c \) if and only if the stopping times \( x_{-1}(s) \) are a.s. equal for all \( s \in (0, 1) \). If \( x \notin \mathcal{C}_c \), there is an \( \bar{s} \in (0, 1) \) such that \( x_{-1}(s) \neq x_{-1}(\bar{s}) \) for \( s > \bar{s} \). Defining \( \alpha = \lambda((0, \bar{s})) \) and

\[
x_1^s = \frac{1}{\alpha} \int_{[0, \bar{s}]} \mathbbm{1}_{[0, t]}(s) \, d\lambda(s),
\]

\[
x_2^s = \frac{1}{1 - \alpha} \int_{[\bar{s}, 1]} \mathbbm{1}_{[0, t]}(s) \, d\lambda(s)
\]

we have \( x_1^s, x_2^s \in \mathcal{C}, x_1^s \neq x_2^s \) and \( x = \alpha x_1^s + (1 - \alpha)x_2^s \). \( \square \)

Note that \( x \) solves the relaxed optimal stopping problem if and only if \( R \) is normal to \( \mathcal{C} \) at \( x \), i.e. if \( R \in \partial \mathbb{h}_\mathcal{C}(x) \) or equivalently \( x \in \partial \sigma_\mathcal{C}(R) \), where

\[
\sigma_\mathcal{C}(R) = \sup_{x \in \mathcal{C}} \langle R, Dx \rangle.
\]
If $R$ is nonnegative, we have $\sigma_C(R) = \|R\|_{R^1}$ (by Krein–Milman) and the optimal solutions of the relaxed stopping problem are simply the subgradients of the $R^1$-norm at $R$.

The following gives a dual expression for the optimum value as well as optimality conditions for the relaxed problem in terms of martingales that dominate the reward process $R$. The statement involves the random variable $r := \sup_{[0,T]} |Z_t|$, where $Z \in L^1(C)$ is such that $R = oZ$.

**Theorem 17.** Optimal stopping time exists for every $R \in R^1$ and the optimum value equals

$$\inf\{EM_0 \mid M \in K_+, M \geq R\},$$

where $K_+$ denotes the set of nonnegative martingales with $M_T \leq r$ almost surely and the optimum is attained. Moreover, a randomized stopping time $x \in C$ is optimal if and only if there exists $M \in K_+$ with $M \geq R$ and

$$\int (M - R)dx = 0,$$

$$M_T[x_T+ - 1] = 0.$$

**Proof.** By Krein-Milman theorem, a continuous linear functional attains its supremum over a compact convex set at an extreme point of the set. The first claim thus follows from Lemma 16.

For the second, we note that the optimal value of the convex relaxation coincides with

$$\max_{x \in K_+} E \left[ \int Rdx - r(x_T+ - 1)^+ \right] \text{ subject to } Dx \in M_+^\infty.$$

Indeed, if $x$ is feasible in this problem, then $\tilde{x} := x \wedge 1$ belongs to $C$ and it achieves an objective value at least as good as $x$. The above problem fits (P) with $U = L^\infty$, $Y = L^1$ and

$$F(x, u) = E \left[ - \int Rdx + r(x_T+ + u - 1)^+ \right] + \delta_{M_+^\infty}(Dx).$$

Clearly, $\varphi(u) \leq F(0, u) = E(u - 1)^+$, where, by [Roc74, Theorem 22], the last expression is Mackey-continuous on $L^\infty$. Since $\varphi(0)$ is finite, $\varphi$ is thus proper. By [Roc74, Theorem 8], $\varphi$ is then Mackey-continuous as well, so Theorem 15 applies.
Using the interchange rule in Theorem 1, we can write the Lagrangian as
\[
L(x, y) = \inf_{u \in L^\infty} \{ F(x, u) - E uy \} \\
= \begin{cases} 
+\infty & \text{if } Dx \notin \mathcal{M}_+^\infty, \\
E \left[ -\int R dx + \inf_{u \in \mathbb{R}} \{ r(x T^+ + u - 1)^+ - uy \} \right] & \text{otherwise} \\
+\infty & \text{if } Dx \notin \mathcal{M}^\infty, \\
E \left[ \int (y - R) dx - y + \inf_{u \in \mathbb{R}} \{ ru^+ - uy \} \right] & \text{otherwise} \\
+\infty & \text{if } Dx \notin \mathcal{M}_+^\infty, \\
E[\int (M - R) dx - Mt] & \text{if } Dx \in \mathcal{M}_+^\infty \text{ and } 0 \leq Mt \leq r, \\
-\infty & \text{otherwise},
\end{cases}
\]
where \( M = o(y \mathbb{1}) \). Thus
\[
\varphi^*(y) = -\inf_{x \in N_0^\infty} L(x, y) \\
= \begin{cases} 
EM_t & \text{if } 0 \leq Mt \leq r \text{ and } R - M \in (\mathcal{M}_+^\infty)^*, \\
+\infty & \text{otherwise},
\end{cases}
\]
where \((\mathcal{M}_+^\infty)^* = \mathcal{R}_{+}^1\), by Corollary 13. By Theorem 15,
\[
-\varphi(0) = -\varphi^{**}(0) \\
= \inf_{y \in L^1} \varphi^*(y) \\
= \inf\{ EM_0 \mid M \in \mathcal{K}, 0 \leq Mt \leq r, M \geq R \}.
\]
As to the optimality conditions, \( 0 \in \partial_x L(x, y) \) means that \( R - M \) belongs to the normal cone of \( \mathcal{M}_+^\infty \) at \( Dx \). This holds if and only if \( Dx \in \mathcal{M}_+^\infty, R - M \in (\mathcal{M}_+^\infty)^* \) and \( \langle R - M, Dx \rangle_{\mathcal{R}_+^1} = 0 \), where again \((\mathcal{M}_+^\infty)^* = \mathcal{R}_{+}^1\), by Corollary 13. By Theorem 2, \( 0 \in \partial_y [-L](x, y) \) means that
\[
\begin{cases} 
y = 0 & \text{if } xT^+ < 1, \\
y \in [0, r] & \text{if } xT^+ = 1, \\
y = r & \text{if } xT^+ > 1
\end{cases}
\]
almost surely. Thus \( x \in \mathcal{C} \) is primal optimal if and only it satisfies the conditions given in the statement. \( \square \)

If \( M \in \mathcal{K}_+^1 \) is such that \( M \geq R \), then
\[
E \sup_{t \in [0, T]} (R_t + Mt - Mt) \leq EM_0,
\]
so, by Theorem 17, the optimum value of the stopping problem can also be expressed as
\[
\inf_{M \in \mathcal{K}^1} E \sup_{t \in [0, T]} (R_t + Mt - Mt).
\]

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This is the dual problem derived in Davis and Karatsas [DK94] and Rogers [Rog02]. Note also that if $Y$ is the Snell envelope of $R$ (the smallest supermartingale that dominates $R$), then the martingale part $M$ in the Doob–Meyer decomposition $Y = M - A$ is dual optimal. These facts were obtained in [DK94] and [Rog02] under the assumptions that $\sup_t R_t$ is integrable. The integrability is neither necessary nor sufficient for $R$ to be regular. Regularity, however, gives the existence of optimal stopping strategies. This was observed already in Bismut and Skalli [BS77] and [EK81, Section 2.18] for bounded $R$.

### 6.2 Finite fuel problem

Consider the problem

\[
\text{minimize } \quad E I_y(x) \quad \text{over } \quad x \in \mathcal{N}_0 \infty
\]

\[
\text{subject to } \quad \int |Dx| \leq \alpha \quad \text{P-a.s.} \quad (FFP)
\]

where $g$ is a convex normal $\mathcal{F} \otimes \mathcal{B}([0,T])$-integrand on $\mathbb{R}^d$ and $\alpha > 0$. When $g_t(x,\omega) = \frac{1}{2} |W_t(\omega) - x|$ for a Brownian motion $W$, this becomes a finite time horizon version of the finite fuel problem from Beneš, Shepp and Witzenhausen [BSW81] who studied the structure of optimal strategies. We will use convex duality to derive a dual problem and optimality conditions for the above extended formulation.

We denote the space of (not necessarily optional) $p$-integrable $\mathbb{R}^d$-valued processes by

\[ L^p := L^p(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]), P \times \lambda; \mathbb{R}^d). \]

The spaces $L^p$ and $L^q$ (where $1/p = 1/q = 1$) are in separating duality under the bilinear form

\[ \langle u, y \rangle = E \int u \cdot y d\lambda. \]

We assume that $g$ is integrable in the sense that $g(x) \in L^1$ for every constant process $x \in \mathbb{R}^d$.

Given a $\mathcal{T}$-integrable process $w$, we will denote the associated potential (we borrow the term from [DM82] who studied potentials of increasing processes) by

\[ P(w) = o[w_T - w]. \]

**Theorem 18.** Assume that the optimum value is finite. Then an optimal solution exists and the optimum value equals

\[
- \inf_{y \in L^1} \left\{ E I_y^*(y) + \alpha \sup_{\tau \in \mathcal{T}} E |R(y)_{\tau}| \right\},
\]

where $R(y) \in R^1$ is the potential associated with

\[ I(y)_t = \int_0^t y_s ds. \]
Moreover, the infimum is attained and an \( x \in \mathcal{N}_0^\infty \) solves (FFP) if and only if 
\[
\|Dx\|_{\mathcal{M}^\infty} \leq \alpha \text{ and there exists } y \in \mathcal{L}^1 \text{ such that }
\]
\[
\begin{align*}
&x \in \partial g^*(y) \quad P \times \lambda\text{-a.e.}, \\
&\langle R(y), Dx \rangle_{\mathcal{R}^1} = \alpha \sup_{\tau \in T} E|R(y)_\tau|,
\end{align*}
\]
or equivalently,
\[
\begin{align*}
&y \in \partial g(x) \quad P \times \lambda\text{-a.e.}, \\
&E \int x \cdot y d\lambda = \alpha \sup_{\tau \in T} E|R(y)_\tau|.
\end{align*}
\]

**Proof.** Problem (FFP) fits (P) with \( U = \mathcal{L}^\infty, Y = \mathcal{L}^1 \) and
\[
F(x, u) = EI_g(i(x) + u) + \delta_{\mathcal{B}_\alpha}(Dx),
\]
where \( i \) is the natural embedding of \( \mathcal{N}_0^\infty \) to \( \mathcal{L}^\infty \) and \( \mathcal{B}_\alpha \) is the closed ball in \( \mathcal{M}^\infty \) with radius \( \alpha \). Integration by parts gives
\[
\begin{align*}
\langle i(x), y \rangle &= E \int y x d\lambda \\
&= E \int [I(y)_T - I(y)] dx \\
&= E \int \alpha[I(y)_T - I(y)] dx \\
&= \langle R(y), Dx \rangle_{\mathcal{R}^1},
\end{align*}
\]
so \( i \) is continuous from \( \sigma(\mathcal{N}_0^\infty, \mathcal{R}^1) \) to \( \sigma(\mathcal{L}^\infty, \mathcal{L}^1) \). The integrability condition of \( g \) implies that \( EI_g \) coincides with the integral functional associated with \( g \) on \( \mathcal{L}^\infty \). By [Roc71b, Corollary 2A], \( EI_g : \mathcal{L}^\infty \to \mathbb{R} \) and \( EI_{g^*} : \mathcal{L}^1 \to \mathbb{R} \) are proper and conjugate to each other so \( F \) is proper and closed. It then follows from Theorem 7 and Banach-Alaoglu that (FFP) always admits solutions.

Clearly
\[
\varphi(u) \leq F(0, u) = EI_g(u),
\]
where, by [Roc74, Theorem 22], the last expression is Mackey-continuous throughout \( U \). Since \( \varphi(0) \) is finite, \( \varphi \) is proper and we get from [Roc74, Theorem 8] that \( \varphi \) is Mackey-continuous as well. By the interchange rule,
\[
L(x, y) = \inf_u \{F(x, u) - \langle u, y \rangle\}
\]
\[
= \langle i(x), y \rangle - EI_{g^*}(y) + \delta_{\mathcal{B}_\alpha}(Dx)
\]
\[
= \langle R(y), Dx \rangle_{\mathcal{R}^1} - EI_{g^*}(y) + \delta_{\mathcal{B}_\alpha}(Dx)
\]
so, by Theorem 7,
\[
\varphi^*(y) = -\inf_{\mathcal{N}_0^\infty} L(x, y) = EI_{g^*}(y) + \alpha\|R(y)\|_{\mathcal{R}^1}.
\]
The optimality conditions now follow from Theorem 15 and Theorem 2. \( \square \)
Combining Theorems 18 and 17 gives the following.

**Corollary 19.** Assume that the optimum value of (FFP) is finite and that the filtration \((\mathcal{F}_t)_{t \geq 0}\) is quasi-left continuous. Then the optimum value of (FFP) can be written as

\[
- \inf_{y \in \mathcal{L}^1, M \in \mathcal{K}_r^+} \{ EI_{g^*}(y) + \alpha EM_0 | M \geq |R(y)| \}.
\]

**Proof.** The quasi-felt continuity of the filtration implies that \(R(y)\) is quasi-left continuous, so \(|R(y)|\) is quasi left-continuous and hence regular; see Remark 1. The expression now follows by applying Theorem 17 to the supremum term in Theorem 18. \(\square\)

### 6.3 Singular stochastic control

Let \(g\) and \(h\) be optional normal integrands on \(\mathbb{R}^d\) and consider the problem

\[
\min_{c \in \mathcal{N}_0^\infty} \mathbb{E}[I_g(\dot{z}) + e(z_T) + J_{h^*}(Dc)]
\]

subject to

\[
\begin{aligned}
z_0 &= \bar{z}_0, \\
\dot{z}_t &= Az_t + Bc_t + W_t,
\end{aligned}
\]  

(SCP)

where \(A, B \in \mathbb{R}^{d \times d}\) and \(W \in L^1(C)\) is optional. We will assume that the reference measure \(\mu(\omega)\) equals the Lebesgue measure \(\lambda\) on \([0, T]\) for all \(\omega \in \Omega\). In the one-dimensional case, where \(g(z) = \frac{1}{2}r|z|^2\) and

\[
h^*(c) = \begin{cases} 
  c^2 & \text{if } |c| \leq k/2, \\
  |c| - k^2/4 & \text{if } |c| \geq k/2
\end{cases}
\]

for some nonnegative constants \(r\) and \(k\), we recover a finite-horizon version of the singular stochastic control problem studied by Lehoczky and Shreve [LS86] (note that they wrote the problem in terms of the variables \(x = \dot{z}\)). Whereas [LS86] analyzed the Hamilton–Jacobi–Bellman equation associated with the above one-dimensional case, we use convex duality to derive a dual problem and optimality conditions in the general case.

In what follows, \(\mathcal{W}^{1,p}_0\) denotes the space of (not necessarily optional) processes \(w\) such that \(w_0 = 0\) almost surely and \(\dot{w} \in \mathcal{L}^p\). The space \(\mathcal{W}^{1,p}_0\) is in separating duality with \(\mathcal{W}^{1,q}_0\) (where \(1/p + 1/q = 1\)) under the bilinear form

\[
\langle w, w^* \rangle_{\mathcal{W}^{1,1}} := \mathbb{E} \int \dot{w}\dot{w}^* d\lambda.
\]

**Lemma 20.** For any \(c \in \mathcal{N}_0^\infty\), the solution of the system equation is given by

\[
z = Ac + a
\]

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where \( a_t = \int_t^0 e^{(t-s)A} W_s ds + e^{tA} z_0 \) and \( Ac \in W^{1,\infty}_0 \) is the unique pathwise solution of
\[
\begin{cases}
  z_0 = 0, \\
  \dot{z}_t = Az_t + Bc_t.
\end{cases}
\]

For each \( w \in W^{1,1}_0 \) and \( c \in \mathcal{N}_0^\infty \), we have \( \langle w, Ac \rangle_{W^{1,1}_0} = \langle B^T p, Dc \rangle_{\mathcal{R}^1} \), where \( p \) is the potential associated with the unique pathwise solution \( u \) of
\[
\begin{cases}
  u_T = 0, \\
  \dot{u}_t = -A^T u_t + \dot{w}_t.
\end{cases}
\]

**Proof.** Given \( c \in \mathcal{N}_0^\infty \) and \( w \in W^{1,1}_0 \), let \( z \) and \( u \) be the corresponding solutions to the system equations. Integration by parts gives
\[
\langle w, Ac \rangle_{W^{1,1}_0} = E \int \dot{z}(\dot{u} + A^T u) d\lambda
\]
\[
= E \int \dot{u}(\dot{z} - Az) d\lambda
\]
\[
= E \int \dot{u} Bcd\lambda
\]
\[
= -E \int B^T u d\lambda
\]
\[
= -E \int B^T o d\lambda
\]
\[
= \langle B^T p, Dc \rangle_{\mathcal{R}^1}.
\]

Using Lemma 20 we can write (SCP) as
\[
\text{minimize } E[I_{\bar{g}}(DAc) + \bar{e}((Ac)_T) + J_{h^*}(Dc)] \quad \text{over } c \in \mathcal{N}_0^\infty. \quad (4)
\]
where \( D : W^{1,\infty}_0 \to L^\infty \) is the differential operator, \( \bar{g}_t(x, \omega) = g_t(x + (Da)_t, \omega) \) and \( \bar{e}(x, \omega) = e(x + a_T) \). Since \( Da \) is an optional process, \( \bar{g} \) and \( \bar{e} \) are optional and \( \mathcal{F} \)-normal integrands, respectively.

We will relate problem (SCP) to the following dual problem
\[
\text{minimize } \{w^*, \eta^*\} \in L^1 \times L^1 E I_{\bar{g}^*}(w^*) + E \bar{e}^* (\eta^*) + E I_h(-B^T P(u))
\]
subject to
\[
\begin{cases}
  \dot{u}_t = -A^T u_t - o(w^* + \eta^*_t), \\
  u_T = 0,
\end{cases}
\]

where \( P(u) \) is the potential associated with \( u \). We say that a normal \( \mathcal{F} \)-integrand \( f \) is integrable if \( f(x, \cdot) \in L^1 \) for all \( x \in \mathbb{R}^d \).
Theorem 21. Assume that the optimal value is finite, \( \tilde{g} \) and \( \tilde{e} \) are integrable, \( h^* \) satisfies the assumptions of Theorem 8, and that \( J_{h^*}(0) \in L^1 \). Then \( \inf_{L^1}(SCP) = -\inf_{L^1}(DCP) \) and the infimum in (DCP) is attained. Moreover, \( c \in N_0^\infty \) attains the infimum in (SCP) if and only if there exist \( w^* \in L^1 \) and \( \eta^* \in L^1 \) such that, almost surely,

\[
dc/d\lambda \in \arg\min_c \{h^*(c) + P(u) \cdot Bc\} \quad \lambda\text{-a.e.,}
\]

\[
dc^*/|dc^*| \in \arg\min_c \{(h^*)^\infty(c) + P(u) \cdot Bc\} \quad |Dc^*|\text{-a.e.,}
\]

\[
\hat{z} \in \arg\min_z \{g(z) - w^* \cdot z\} \quad \lambda\text{-a.e.},
\]

\[
\hat{z}_T \in \arg\min_z \{e(z) - \eta^* \cdot z\},
\]

where \( z \) and \( u \) are the corresponding solutions of the primal and dual system equations.

Proof. Problem (4) fits (P) with

\[
F(c, u) = E[I_\tilde{g}(DAc + w) + \tilde{e}(Ac_T + \eta) + J_{h^*}(Dc)],
\]

where \( u = (w, \eta) \). Clearly

\[
\varphi(u) \leq F(0, u) = E[I_\tilde{g}(w) + \tilde{e}(\eta) + J_{h^*}(0)].
\]

Since \( \tilde{g} \) and \( \tilde{e} \) are integrable, the last expression is Mackey-continuous on \( U \); see [Roc74, Theorem 22]. By [Roc74, Theorem 8], \( \varphi \) is then Mackey-continuous as well. We may thus apply Theorem 15.

By the interchange rule, the Lagrangian can be expressed as

\[
L(c, y) = \inf_{w \in U^1} \left\{ E \int [\tilde{g}(DAc + w) - w \cdot w^*] d\lambda + E[\tilde{e}(Ac)_{T} + \eta - \eta^*] \right\} + EJ_{h^*}(Dc)
\]

\[
= E \int [DAc \cdot w^* - \tilde{g}^*(w^*)] d\lambda + E[(Ac)_{T} \cdot \eta^* - \tilde{e}^*(\eta^*)] + EJ_{h^*}(Dc)
\]

\[
= E \int [DAc \cdot w^* - \tilde{g}^*(w^*)] d\lambda + E[\int DAc \cdot \eta^* d\lambda - \tilde{e}^*(\eta^*)] + EJ_{h^*}(Dc)
\]

\[
= \langle w^* + \eta^*, Ac \rangle_{L^1} - EI_{\tilde{g}^*}(w^*) - E\tilde{e}^*(\eta^*) + EJ_{h^*}(Dc)
\]

\[
= (I(w^* + \eta^*), Ac)_{W^{1,1}} - EI_{\tilde{g}^*}(w^*) - E\tilde{e}^*(\eta^*) + EJ_{h^*}(Dc).
\]

By Lemma 20, \( (I(w^* + \eta^*), Ac)_{W^{1,1}} = (B^T P(u), Dc)_{R^1} \), where \( u \) is the solution to the dual system equations. By Theorem 8,

\[
\varphi^*(y) = \inf_{c \in N_0^\infty} L(c, y) = EI_{\tilde{g}^*}(w^*) + E\tilde{e}^*(\eta^*) + EI_{h^*}(B^T P(u)),
\]

so the first claim follows from Theorem 15.

Subdifferentiating the Lagrangian gives the optimality conditions

\[
\partial EJ_{h^*}(Dc) + B^T P(u) \ni 0,
\]

\[
\partial EI_{\tilde{g}^*}(w^*) - DAc \ni 0,
\]

\[
\partial E\tilde{e}^*(\eta^*) - (Ac)_{T} \ni 0.
\]
Since $g_t^*(w^*) = g_t^*(w^*) - w^* \cdot Da_t$ and $e^*(\eta^*) = e^*(\eta^*) - \eta^* \cdot a_T$, this can be written as

$$
\partial EJ_h^*(Dc) + B^T P(u) \ni 0,
\partial EI_{g^*}(w^*) - \dot{z} \ni 0,
\partial Ee^*(\eta^*) - z_T \ni 0,
$$

where $z$ the solution of the primal system equations. By the subdifferential formula in Theorem 8, this means that

$$
\partial h(-B^T P(u)) \ni dc/d\lambda \quad \lambda\text{-a.e.}
\partial h^*(-B^T P(u)) \ni dc^s/|dc| \quad |Dc^s|\text{-a.e.}
\partial g(\dot{z}) \ni w^* \quad \lambda\text{-a.e.}
\partial e(z_T) \ni \eta
$$

almost surely. That this system is equivalent to the given conditions follows from the definition of the subdifferential; see [Roc71a, Theorem 23.5].

Much like the classical Pontryagin maximum principle, the optimality conditions in Theorem 21 characterize optimal controls as pointwise minimizers. While the density $dc/d\lambda$ of the absolutely continuous part minimizes the “Hamiltonian”

$$
H_h(c, p) = h_h^*(c) + p \cdot Bc
$$

with $p$ equal to the “costate” variable $P(u)_t$, the density $dc^s/|dc^s|$ of the singular part minimizes the recession function $H_{\infty}^*(\cdot, p)$. However, problem (SCP) differs from classical formulations of optimal control so the optimality conditions are inevitably different as well.

**Remark 2.** Any feasible potential $p = P(u)$ in (DCP) solves the backward stochastic differential equation

$$
\begin{align*}
\begin{cases}
   dp_t = (A^T p_t - q_t) d\lambda + dm_t, \\
p_T = 0,
\end{cases}
\end{align*}
$$

where $m = \mathcal{O}\left(\int (A^T p_s - q_s) d\lambda\right)$ and $q = \mathcal{O}(w^* + \eta^*)$. 

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Proof. The claim follows by observing that, for every finite \( \tau \in T \),

\[
p_{\tau} = E[u_{\tau} | F_\tau] = E\left[\int^{T}_{\tau} u_\tau d\lambda \bigg| F_\tau\right] = E\left[\int^{T}_{\tau} (-A^{T} u - q) d\lambda \bigg| F_\tau\right]
\]

\[
= E\left[\int^{T}_{\tau} (-A^{T} u - q) d\lambda \bigg| F_\tau\right] = E\left[\int^{T}_{\tau} (A^{T} p - q) d\lambda \bigg| F_\tau\right]
\]

\[
= E\left[\int^{T}_{0} (A^{T} p + q) d\lambda + \int^{T}_{0} (A^{T} p - q) d\lambda \bigg| F_\tau\right] + m_\tau
\]

\[
= \int^{T}_{0} (A^{T} p - q) d\lambda + \int^{T}_{0} dm + p_0,
\]

where the fourth equality follows from [HWY92][Theorem 5.16] and the last equality by first going through the argument for \( \tau = 0 \). \qed

7 Appendix

7.1 Normal integrands on Suslin spaces

This section proves the claims made in Section 2 concerning criteria for checking that a function \( f : U \times \Xi \rightarrow \mathbb{R} \) is a normal integrand. For \( U = \mathbb{R}^{d} \), these results are well known and can be found e.g. in [Roc76] and [RW98, Chapter 14]. Various extensions exist beyond the finite-dimensional case. Below, we allow for a locally convex Suslin space \( U \) which covers the function spaces studied in this paper. Note that Suslin spaces are separable since the image of a countable dense set under a continuous surjection is dense.

Lemma 22. If \((\Xi, A)\) is complete with respect to some \( \sigma \)-finite measure \( m \), then any \( B(U) \otimes A \)-measurable function \( f \) such that \( f(\cdot, \xi) \) is lsc, is a normal integrand. The converse holds if \( U \) is a countable union of Borel sets that are Polish spaces in the relative topology.

Proof. Since \( f \) is measurable, \( \text{gph}(\text{epi} f) \) is \( F \otimes B(U) \otimes B(\mathbb{R}) \)-measurable, where \( B(U) \otimes B(\mathbb{R}) = B(U \times \mathbb{R}) \) [Bog07, Lemma 6.4.2]. The space \( U \times \mathbb{R} \) is also Suslin [Bog07, Lemma 6.6.5]. For any open \( O \subset U \times \mathbb{R} \), the set \( \text{gph}(\text{epi} f) \cap (\Omega \times O) \) is measurable, so by the projection theorem [CV77, Theorem III.23], \( \{ \omega | \exists (u, \alpha) : (\omega, u, \alpha) \in \text{gph}(\text{epi} f) \cap (\Omega \times O) \} \) belongs to \( \mathcal{F} \).

To prove the converse, denote by \( P^{\nu} \) the Borel sets in question. We have that \( \mathcal{B}(P^{\nu}) \) coincides with \( \mathcal{B}(U) \cap P^{\nu} \), so \( f \) is jointly measurable if and only if its restriction \( f^{\nu} := f|_{P^{\nu} \times \Omega} \) is jointly measurable for each \( \nu \). Since \( \text{epi} f^{\nu} \) is a measurable closed-valued mapping from \( \Omega \) to \( P^{\nu} \times \mathbb{R} \), its graph is measurable by [CV77, Theorem III.30], and thus, by [Val75, Lemma 7], \( f^{\nu} \) is jointly measurable. \( \Box \)
From now on we assume that $U$ is a countable union of Borel sets that are Polish spaces in the relative topology. A function $f$ satisfying the assumptions of the following proposition is known as a Carathéodory integrand. If $(\Xi, \mathcal{A})$ were complete w.r.t. some $\sigma$-finite measure $m$, the result would follow from [CRdFVM04, Lemma 1.2.3] and Lemma 22.

**Proposition 23.** Assume that $f: U \times \Xi \to \mathbb{R}$ is such that $f(\cdot, \xi)$ is continuous for every $\xi$ and that $f(u, \cdot)$ is measurable for every $u \in U$. Then $f$ is a normal integrand.

**Proof.** Let $\{u^\nu \mid \nu \in \mathbb{N}\}$ be a dense set in $U$ and define $\alpha^{\nu,q}(\xi) = f(u^\nu, \xi) + q$, where $q \in \mathbb{Q}_+$. Since $f(\cdot, \xi)$ is continuous, the set $\hat{O} = \{(u, \alpha) \mid f(u, \xi) < \alpha\}$ is open. For any $(u, \alpha) \in \text{epi} f(\cdot, \xi)$ and for any open neighborhood $O$ of $(u, \alpha)$, $O \cap \hat{O}$ is open and nonempty, and there exists $(u^\nu, \alpha^{\nu,q}(\xi)) \in O \cap \hat{O}$, i.e., $\{(u^\nu, \alpha^{\nu,q}(\xi)) \mid \nu \in \mathbb{N}, q \in \mathbb{Q}\}$ is dense in $\text{epi} f(\cdot, \xi)$. Thus for any open set $O \in U \times \mathbb{R}$,

$$\{\xi \mid \text{epi} f(\cdot, \xi) \cap O \neq \emptyset\} = \bigcup_{\nu,q} \{\xi \mid (u^\nu, \alpha^{\nu,q}(\xi)) \in O\}$$

is measurable.

Given a proper lower semicontinuous function $f$ and a nonnegative scalar $\alpha$, we define

$$(\alpha f)(x) := \begin{cases} \alpha f(x) & \text{if } \alpha > 0, \\ \delta_{\text{cl dom} f}(x) & \text{if } \alpha = 0. \end{cases}$$

**Lemma 24.** Let $f$, $f^i$, $i = 1, \ldots, n$ be proper normal integrands on $U$ and assume that $(\Xi, \mathcal{A})$ is complete w.r.t. a $\sigma$-finite measure or $U = \mathbb{R}^d$, then the functions

1. $(u, \xi) \mapsto \alpha(\omega)f(u, \xi)$, where $\alpha \in L^0(\Xi, \mathbb{R}_+)$,
2. $(u, \xi) \mapsto \sum_{i=1}^n f^i(u, \xi)$,
3. $f^\infty$ defined by $f^\infty(\cdot, \xi) = f(\cdot, \xi)^\infty$,
4. $f^*$ defined by $f^*(\cdot, \xi) = f(\cdot, \xi)^*$, where $Y$ is a Suslin space in separating duality with $U$

are normal integrands.

**Proof.** For $U = \mathbb{R}^d$, proofs can be found in [RW98, Chapter 14] except for part 1. In this case, using properness of $f(\cdot, \xi)$, it can be shown that $\alpha(\xi)f(\cdot, \xi)$ is the epi-graphical limit of $\alpha^{\nu}(\xi)$ as $\alpha^{\nu} \searrow \alpha(\omega)$ (see [RW98, Chapter 7]), so the result follows from [RW98, Proposition 14.53].

When $(\Xi, \mathcal{A})$ is complete, it suffices to verify, by Lemma 22, that the functions are jointly measurable and lower semicontinuous in the second argument. For 2, this follows from the fact that sums of measurable/lsc functions is again
measurable/lsc. For 3, it suffices to note that \( f^\infty \) is a pointwise increasing limit of jointly measurable functions that are lower semicontinuous in the second argument. Part 4 is from [Val75, Lemma 8]. To prove 1, note first that \( \{(u, \xi) \mid f(u, \xi) < \infty\} \) is measurable, so one may proceed as in the proof of Lemma 22 to show that \( \text{dom} f \) is a measurable set-valued mapping. It follows that \( \text{cl dom} f \) is a measurable mapping as well, so \( \delta_{\text{cl dom} f} \) is a normal integrand and thus jointly measurable by Lemma 22. Since \( \alpha f = 1_{\alpha=0} \delta_{\text{cl dom} f} + 1_{\alpha>0} \alpha f \), we get, using part 2, that \( f \) is jointly measurable.

7.2 Integral functionals as normal integrands

Let \( U \) be a Suslin subspace of Borel measurable functions on \([0, T]\) and consider the random integral functional \( I_h : U \times \Omega \to \mathbb{R} \) defined scenarioiswice by

\[
I_h(u, \omega) := I_h(\cdot, \omega).
\]

The following result was used in Theorem 5 for \( U = C \).

**Theorem 25.** Assume that \( U \) is a countable union of Borel sets that are Polish spaces in the relative topology. The function \( I_h \) is \( \mathcal{B}(U) \otimes \mathcal{F} \)-measurable under either of the following conditions:

(a) The sequential convergence in \( U \) implies pointwise convergence outside a countable set and point-evaluations in \( U \) are measurable.

(b) The topology on \( U \) be finer than the topology of convergence in \( \mu(\omega) \)-measure almost surely.

In such cases, \( I_h \) is a normal integrand on \( U \) whenever \( I_h(\cdot, \omega) \) is lower semicontinuous almost surely.

**Proof.** We only give proof for the first set of conditions, the second case is similar. We may assume without loss of generality that \( h \) is bounded from below. Indeed, If \( I_{h^\alpha} \) is \( \mathcal{B}(U) \otimes \mathcal{F} \)-measurable for \( h^\alpha = \sup \{h, \alpha\} \) and \( \alpha < 0 \), then, by the monotone convergence theorem (recall that by convention \( I_h(u, \omega) = +\infty \) unless \( I_{h^+}(u, \omega) < +\infty \)),

\[
I_h(u, \omega) = \lim_{\alpha \to -\infty} I_{h^\alpha}(u, \omega),
\]

and \( I_h \) is \( \mathcal{B}(U) \otimes \mathcal{F} \)-measurable as well. Assume first that \( \mu \) is an atomless random measure.

Case 1: Assume that \( \alpha \leq h_t(u, \omega) \leq \gamma \) for all \((\omega, t, u)\) and that \( h_t(\cdot, \omega) \) is continuous for all \((\omega, t)\). By the dominated convergence theorem and continuity of \( h_t(\cdot, \omega) \), \( I_h(\cdot, \omega) \) is continuous in \( \mu(\omega) \)-measure and thus continuous in \( U \). For every \( u \in U \), \( I_h(u, \cdot) \) is measurable, since \((t, \omega) \mapsto h_t(u_t, \omega)\) is measurable (being a composition of measurable mappings) and \( \mu \) is a random Radon measure. By Proposition 23 and Lemma 22, \( I_h \) is thus \( \mathcal{B}(U) \otimes \mathcal{F} \)-measurable.
Case 2: Assume that $\alpha \leq h_t(u, \omega)$ for all $(\omega, t, u)$ and that $h_t(\cdot, \omega)$ is continuous for all $(\omega, t)$. By Case 1, for $h^\gamma = \min\{h, \gamma\}$, $I_{h^\gamma}$ is $\mathcal{B}(U) \otimes \mathcal{F}$-measurable. By Monotone convergence theorem,

$$I_h(u, \omega) = \lim_{\gamma \to \infty} I_{h^\gamma}(u, \omega),$$

and therefore $I_h$ is $\mathcal{B}(U) \otimes \mathcal{F}$-measurable.

Case 3: Assume that $\alpha \leq h_t(u, \omega)$ for all $(\omega, t, u)$. By [RW98, p. 665], $h^\lambda_t(u, \omega) = \inf_{u'} \{h_t(u', \omega) + \frac{1}{\lambda} |u - u'|\}$ is a normal integrand, so $I_{h^\lambda}$ is $\mathcal{B}(U) \otimes \mathcal{F}$-measurable by Case 2. By [RW98, Example 9.11], $h^\lambda \to h$ as $\lambda \to \infty$ so, by the monotone convergence theorem,

$$I_h(u, \omega) = \lim_{\lambda \to 0} I_{h^\lambda}(u, \omega).$$

Thus, $I_h$ is $\mathcal{B}(U) \otimes \mathcal{F}$-measurable.

Consider now the case of a general nonnegative random Radon measure $\mu$. By [HWY92, Theorem 3.42], $\mu = \mu^c + \mu^d$, where $\mu^c$ is atomless and $\mu^d$ is supported on the union $\bigcup_i [\tau_i]$ of graphs of some random times $\tau_i$. We have the decomposition $I_h = I_h^c + I_h^d$ where the integral functionals are defined with respect to $\mu^c$ and $\mu^d$, respectively. Here $I_h^c$ is measurable by the first part. We have

$$I_h^d(u, \omega) = \sum_i h_{\tau_i}(u_{\tau_i}(\omega), \omega) \mu^d(\tau_i(\omega)).$$

By the monotone convergence theorem, it suffices to prove that each term in the sum defines a $\mathcal{B}(U) \otimes \mathcal{F}$-measurable function. The maps $(u, \omega) \mapsto (u, \tau(\omega), \omega)$, $(u, t, \omega) \mapsto (u_t, t, \omega)$ and $(x, t, \omega) \mapsto h_t(x, \omega)$ are measurable, so $(u, \omega) \mapsto h_{\tau_i}(u_{\tau_i}(\omega), \omega)$ is a composition of measurable mappings.

The last claim follows from Lemma 22.

### 7.3 Projections of extended-real valued processes

**Lemma 26.** Let $v$ be a measurable extended real-valued process and let $\mu$ be a nonnegative random Radon measure. Then

$$\omega \mapsto \int v_t(\omega)d\mu_t(\omega) := \int v^+_t(\omega)d\mu_t(\omega) - \int v^-d\mu_t(\omega)$$

is $\mathcal{F}$-measurable.

**Proof.** When $v$ is finite-valued and $\int |v_t|d\mu_t < \infty$ a.s., the claim follows from standard monotone class arguments. For a nonnegative $v$, the claim then follows from the monotone convergence theorem. For an arbitrary $v$, the integral is thus a sum of $\mathcal{F}$-measurable extended real-valued random variables. \(\square\)
We will need the notion of an optional projection of an extended real-valued processes; see [KP15]. For a nonnegative real-valued process \( v \), there exists a unique optional process \( o_v \) satisfying (2); see [DM82, Theorem VI.43]. The monotone convergence theorem then gives the existence of a unique optional projection for a nonnegative extended real-valued process as well. For an extended real-valued stochastic process \( v \) with \( T \)-integrable \( v^+ \) or \( v^- \), we define \( o_v = o(v^+) - o(v^-) \).

**Lemma 27.** Let \( \mu \) be a nonnegative optional Radon measure and \( v \) an extended real-valued process such that \( v^+ \) or \( v^- \) is \( T \)-integrable. If \( E \int v^- d\mu < \infty \), then

\[
E \int vd\mu = E \int o_v d\mu.
\]

**Proof.** By [DM82, Theorem VI.57],

\[
E \int vd\mu = E \int o_v d\mu
\]

for all nonnegative real-valued processes \( v \). For nonnegative extended real-valued process, the expression is then valid by the monotone convergence theorem. As to the general case,

\[
E \int vd\mu = E \int [v^+ - v^-] d\mu = E \int v^+ d\mu - E \int v^- d\mu = E \int o_v^+ d\mu - E \int o_v^- d\mu = E \int [o_v^+ - o_v^-] d\mu = E \int o_v d\mu,
\]

where the second equality holds since \( E \int v^- d\mu < \infty \) and the fourth since \( E \int o_v^- d\mu < \infty \), by [DM82, Theorem VI.57].

7.4 **Proof of Theorem 8**

**Lemma 28.** Let \( \mu \) be an optional random Radon measure and \( h \) a convex normal integrand such that \( h^*(\bar{x})^+ \) and \( h(\bar{y})^+ \) are \( T \)-integrable for some optional \( \bar{x} \) and \( T \)-integrable \( \bar{y} \). If \( x \) is an optional process such that \( E \int h^*(x)^- d\mu < \infty \), then

\[
E \int h^*(x)d\mu = E \int o(h^*)(x)d\mu.
\]

**Proof.** Let \( A = \{|x| \geq M\} \) for some strictly positive real number \( M \). We have

\[
E \int h^*(x)d\mu = E \int [1_A h^*(x)]d\mu + E \int [1_{A^c} h^*(x)]d\mu,
\]
since the negative parts of both terms are integrable.

Let \( \lambda = \mathbb{1}_A|x|^{-1} \), \( \alpha = \mathbb{1}_A x/|x| \), and \( d\hat{\mu} = |x|d\mu \) so that

\[
E \int [\mathbb{1}_A h^*(x)]d\mu = E \int \hat{h}^*(\alpha, \lambda)d\hat{\mu},
\]

where

\[
\hat{h}^*_\alpha(\lambda, \omega) = \begin{cases} 
\lambda h^*_\alpha(\alpha/\lambda, \omega) & \text{if } \lambda > 0, \\
\langle h^*_\alpha \rangle^\infty(\alpha, \omega) & \text{if } \lambda = 0, \\
+\infty & \text{otherwise}.
\end{cases}
\]

Since \( \lambda \) and \( \alpha \) are bounded optional processes, and since \( \hat{h}(\beta, \eta) = \delta_{\text{epi} \hat{h}}(\beta, -\eta) \) (see [Roc70, Corollary 13.5.1]), we get from Lemma 27 and (3) that

\[
E \int \hat{h}^*_\alpha(\alpha, \lambda)d\hat{\mu} = E \int [\mathbb{1}_A o^\alpha(h^*)(x)]d\mu.
\]

It is not difficult to verify from the definitions that \( o^\alpha(\hat{h}^\alpha(\beta, \eta)) = \delta_{\text{epi} \hat{h}^\alpha}(\beta, -\eta) \) [KP15, Theorem 15]. Thus

\[
E \int [\mathbb{1}_A h^*(x)]d\mu = E \int [\mathbb{1}_A o^\alpha(h^*)(x)]d\mu.
\]

As to the second term, \( \mathbb{1}_A C x \) is bounded, so \( \mathbb{1}_A C h^*(x) - \) is \( T \)-integrable by the Fenchel inequality, and thus, by Lemma 27 and (3),

\[
E \int [\mathbb{1}_A C h^*(x)]d\mu = E \int [\mathbb{1}_A o^\alpha(h^*)(x)]d\mu.
\]

Thus

\[
E \int h^*(x)d\mu = E \int [\mathbb{1}_A o^\alpha(h^*)(x)]d\mu + E \int [\mathbb{1}_A C o^\alpha(h^*)(x)]d\mu,
\]

which finishes the proof since the negative parts of both terms are again integrable. \( \square \)

**Lemma 29.** Under the assumptions of Theorem 8, \( \int \hat{h}(y)^-d\mu \), \( \int \hat{h}^*(d\theta/d\mu)^-d\mu \), and \( \int (\hat{h}^*)^\infty(d\theta/d\mu)^-d\theta^s \) are integrable for every \( y \in L^1(C) \) and \( \theta \in L^\infty(M) \). In particular, \( EI_{\hat{h}} \) and \( EJ_{\hat{h}^*} \) are proper.

**Proof.** We define \( \tilde{\theta} \in \mathcal{M}_\infty \) by \( d\tilde{\theta}/d\mu = \bar{x} \) and \( \tilde{\theta}^s = 0 \). The first lower bound in Theorem 8 implies

\[
E \int \hat{h}(y)^-d\mu \leq E \int [(d\tilde{\theta}/d\mu \cdot y) + \alpha]^+d\mu \leq E \int [||\tilde{\theta}||^y] + E \int \alpha d\mu.
\]

The other terms are handled similarly, where, for the recession function, the latter lower bound implies \( (\hat{h}^*)^\infty(x) \geq x \cdot \bar{v} \). The bounds also give that \( EI_{\hat{h}}(\tilde{\theta}) \leq E \int \alpha d\mu \) and \( EJ_{\hat{h}^*}(\tilde{\theta}) \leq E \int \alpha d\mu \), so \( EI_{\hat{h}} \) and \( EJ_{\hat{h}^*} \) are proper. \( \square \)
Lemma 30. Under the assumptions of Theorem 8, \(J_{h^*}(\theta)\) is \(\mathcal{F}\)-measurable and \(EJ_{h^*}(\theta) = EJ_h(\theta)\) for every \(\theta \in \mathcal{M}^\infty\).

Proof. By [HWY92, Theorem 5.14], the Radon–Nikodym densities \(d\theta/d\mu\) and \(d\theta^*/d\theta^*\) are optional processes, so \(h^*((d\theta/d\mu))\) and \((h^*)^\infty((d\theta^*/d\theta^*))\) are optional as well. By Lemma 26, \(J_{h^*}(\theta)\) is thus \(\mathcal{F}\)-measurable and \(EJ_{h^*}\) is well-defined. By Lemma 29, \(\int h^*(d\theta/d\mu) - d\mu \int (h^*)^\infty(d\theta^*/d\theta^*) - d\theta^*\) are integrable, so

\[
EJ_{h^*}(\theta) = E \left[ \int h^*(d\theta/d\mu)d\mu + \int (h^*)^\infty(d\theta^*/d\theta^*)d\theta^* \right]
\]

\[
= E \int h^*(d\theta/d\mu)d\mu + E \int (h^*)^\infty(d\theta^*/d\theta^*)d\theta^*
\]

where the last equality follows from Lemma 27. The lower bounds in Theorem 8 give that \(h^+(\bar{x}) + h^+(\tilde{x})\) are \(\mathcal{T}\)-integrable, so [KP15, Theorem 14] implies that \((h^+)^\infty = (\bar{h}^+)^\infty\). Consequently, \(EJ_{h^*}(\theta) = EJ_{h^*}(\theta)\) by Lemma 28.

Proof of Theorem 8. We get from the lower bounds that \(h^+(\bar{x}) + h^+(\tilde{x})\) are \(\mathcal{T}\)-integrable, so, by [KP15, Lemma 5 and Theorem 15], we have

\[
h(\bar{x}) \geq \bar{v} \cdot \bar{x} - \alpha \bar{\alpha},
\]

\[
h^+(\tilde{x}) \geq \tilde{v} \cdot \tilde{x} - \alpha \tilde{\alpha}.
\]

Thus \(EI_h(\bar{v}) \leq E \int \alpha d\mu\), where the right side is finite by Lemma 27. Similarly, \(EJ_{h^*}(\theta)\) is finite for \(\theta \in \mathcal{M}^\infty\) that is defined by \(d\theta/d\mu = \bar{x}\) and \(\tilde{\theta} = 0\). The lower bounds also imply that \(EI_h\) and \(EJ_h\) never take the value \(-\infty\) on \(\mathcal{R}^1\) and \(\mathcal{M}^\infty\). Thus \(EI_h\) and \(EJ_h\) are proper.

Since \(h^+(\bar{v})^\infty\) and \(h^+(\tilde{x})\) are \(\mathcal{T}\)-integrable, [KP15, Theorem 16] implies that \(h^+(\bar{y}) \leq \alpha^\infty h(y)\) for every \(y \in L^1(C)\). In conjunction with Lemma 29 and Lemma 27, we get \(EI_h(\bar{v}) \leq EI_h(\bar{y})\), so

\[
(EI_h)^*(\theta) = \sup_{v \in \mathcal{R}^1} \{ (v, \theta) - EI_h(v) \}
\]

\[
= \sup_{y \in L^1(C)} \{ (\bar{y}, \theta) - EI_{h^*}(\bar{y}) \}
\]

\[
\geq \sup_{y \in L^1(C)} \{ (y, \theta) - EI_{h^*}(y) \}
\]

\[
= EJ_{h^*}(\theta)
\]

where the last two lines follow from Theorem 6 and Lemma 30, respectively.

To prove the opposite inequality, let \(v \in \text{dom} EI_h\) and \(\theta \in \text{dom} EJ_{h^*}\). By assumption, \(v \in \text{cl dom} h\) a.s.e. Since \(\delta^\alpha_{\text{dom} h^*} = (h^*)^\infty\), we have, almost surely,
the Fenchel inequalities
\[
\begin{align*}
  h(v) + h^*(d\theta/d\mu) & \geq v \cdot (d\theta/d\mu) \quad \mu\text{-a.e.} \\
  (h^*)^\infty(d\theta^*/d|\theta^*|) & \geq v \cdot (d\theta^*/d|\theta^*|) \quad |\theta^*|\text{-a.e.,}
\end{align*}
\]
so \( I_h(v) + J_{h^*}(\theta) \geq \langle v, \theta \rangle \) a.s. and thus, \((EI_h)^* \leq EJ_{h^*}\). The subdifferential characterization follows now like in the proof of Theorem 3.

It remains to show that \( EI_h \) is lower semicontinuous. If \( v'' \to v \) in \( \mathcal{R}^1 \), we have (by passing to a subsequence if necessary) \( v'' \to v \) in the sup-norm almost surely (see [DM82, p. 82-83]) so, by Fatou’s lemma, the above Fenchel inequalities imply that
\[
\liminf E[I_h(v'') - \langle v'', \theta \rangle] \geq E[I_h(v) - \langle v, \theta \rangle],
\]
and thus, \( \liminf EI_h(v'') \geq EI_h(v) \). \( \square \)

References


