Speed of biased random walks on Galton-Watson Trees

presenting Sections 1 - 4 in: R. Lyons, R. Pemantle, Y. Peres: Biased Random Walks on Galton-Watson Trees

Marcel Ortgiese

University of Bath

December 1st 2006

Set-up

- Consider a supercritical GW tree T with generating function $f(s) = \sum_{k=0}^{\infty} p_k s^k$, $p_k \neq 1 \forall k$ and $m := f'(1) \in (1, \infty)$. Denote its root by ρ.
- For $\lambda \geq 0$, consider the λ-biased random walk RW_λ, i.e. the Markov chain $(X_n, n \geq 0)$ on the vertices of T with the usual transition rules.
- The **speed** of the random walk is the a.s. limit of $\frac{|X_n|}{n}$ (provided it exists).

Recall RW_λ is transient for $0 \leq \lambda < m$.

Main results:

Recall, for $\lambda = 1$: the speed is $\mathbb{E}\left[\frac{Z_1-1}{Z_1+1}\right]$, where Z_1 is the number of children of the root.

- if $p_0 = 0$ and $\lambda < 1$, then the speed exists a.s. and is a positive constant depending only on λ and f.
- if $1 < \lambda < m$, then the same result holds for a.e. Galton-Watson tree conditioned on non-extinction.
- if $p_0 \neq 0$ and $\lambda < 1$ and T is conditioned on non-extinction: if $\lambda > f'(q)$ then the random walk escapes with positive speed, otherwise with zero speed. Here q is the extinction probability of the GW tree.

Outward bias and $p_0 = 0$

In this section, we assume $p_0 = 0$ and $\lambda < 1$. We structure the proof as follows:

- show the range grows linearly,
- discuss regeneration epochs,
- show that there are infinitely many regeneration epochs,
- deduce that the speed exists and is a positive constant.

Linear Growth of range

Let $d(x)$ denote the number of children of a vertex x. Define R_n to be the number of distinct vertices visited by RW_λ by time n. Then we have for $x \neq \rho$,

$$\mathbb{E}\left[|X_{k+1}| - |X_k| \,|\, X_k = x\right] = \frac{d(x) - \lambda}{d(x) + \lambda}\,.$$

Hence,

$$\begin{aligned}\mathbb{E}[R_n] &\geq 1 + \mathbb{E}|X_n| \geq 1 + \mathbb{E}\left[\sum_{k=0}^{n-1} \frac{d(X_k) - \lambda}{d(X_k) + \lambda}\right] \\ &\geq 1 + \mathbb{E}\left[\sum_{k=0}^{n-1} \frac{d(X_k) - \lambda}{2d(X_k)}\right] \\ &\geq 1 + n\frac{1-\lambda}{2}\end{aligned}$$

Regeneration epochs

Given a path $(X_0, X_1, \ldots)$ we call $n > 0$

- a **fresh epoch** if $X_n \neq X_k$ for all $k < n$,
- a **regeneration epoch** if in addition $X_{n-1} \neq X_k$ for all $k > n$.

Let T' be the tree obtained by adjoining a new vertex to the root of T, denote the new vertex ρ', so it is the root of T'. Let $\gamma(T)$ be the probability that RW_λ on T' never returns to ρ'.

Construction of tree via regeneration epochs

We construct our tree and random walk as follows,

- Generate a normal GW tree and the RW_λ as usually. If there is no regeneration epoch, the process is finished. If there is a regeneration epoch τ_1, then remove the tree $T(X_{\tau_1})$ (including the random walk) - the remainder is called a **slab**. Replace it by an independently generated GW tree combined with RW_λ on that new tree and continue with the next step.
- Now look at the new tree: If there is no regeneration epoch, then process is finished. If there is a second regeneration epoch τ_2, replace the tree $T(X_{\tau_2})$ by an independently generate GW tree combined with RW_λ on that tree. Continue with next step.
- and so on...

We can then prove that the tree and the random walk constructed in this way coincides with our usual random walk RW_λ on a GW tree.

There are infinitely many regeneration epochs

Lemma 1
Let $0 \leq \lambda < m$. For a.e. Galton-Watson tree T and a.e. sample path of RW_λ, there are infinitely many regeneration epochs.

By the independence of the previously constructed slabs, it suffices to show that for any N,

$$\mathbb{P}\{\exists \text{ a regeneration epoch } \geq N\} > 0\,.$$

Since the event that there is a regeneration epoch $\geq N$ has either probability zero or one.

Proof of Lemma 1

Let $\mathcal{F}_n$ be the σ-field generated by the events $\{X_i \neq X_j\}$ for $0 \leq i < j \leq n$. Since T is infinite and RW_λ transient, there is a.s. a fresh epoch $n \geq N$, let α be first such. Hence,

$$\mathbb{P}\{\exists \text{ a regeneration epoch } \geq N \,|\, \mathcal{F}_N\}$$
$$\geq \mathbb{P}\{\alpha \text{ is a regeneration epoch} \,|\, \mathcal{F}_N\} = \mathbb{E}[\gamma(T)]\,.$$

Denote by $\mathcal{F}_\infty$ the σ-algebra generated by the union of the $\mathcal{F}_n$. Since the regeneration epochs are $\mathcal{F}_\infty$-measurable, it follows from Lévy's upward theorem, that

$$\mathbb{P}\{\exists \text{ a regeneration epoch } \geq N\}$$
$$= \lim_{k\to\infty} \mathbb{P}\{\exists \text{ regeneration } \geq N \,|\, \mathcal{F}_{N+k}\}$$
$$\geq \mathbb{E}[\gamma(T)]\,.$$

□

Differences of successive regeneration epochs

Let the regeneration epochs be $0 < \tau_1 < \tau_2 < \dots$.
As an easy consequence of the above construction, we get

Corollary 2

For $0 \leq \lambda < m$ the differences between successive regenerations epochs $\{\tau_{n+1} - \tau_n\}_{n \geq 1}$ are iid as are the increments $\{|X_{\tau_{n+1}}| - |X_{\tau_n}|\}_{n \geq 1}$.

Proposition 3

If $\mathbb{E}\left[\frac{R_n}{n}\right] \geq c$, for some constant $c > 0$ depending only on λ and f, then the differences between successive regeneration epochs $\{\tau_{n+1} - \tau_n\}_{n \geq 1}$ have finite means.

Proof of Proposition 3

Let $N(n)$ be the number or regeneration epochs in $\{1, \ldots, n\}$.

$$\begin{aligned}\mathbb{E}N(n) &= \sum_{k=1}^{n} \mathbb{P}\{k \text{ is a regeneration epoch }\} \\ &= \sum_{k=1}^{n} \mathbb{E}[\gamma(T)]\mathbb{P}\{k \text{ is a fresh epoch }\} \\ &= \mathbb{E}[\gamma(T)]\mathbb{E}[R_n] \geq c\mathbb{E}[\gamma(T)]n\end{aligned}$$

Hence, by the strong law of large numbers,

$$\frac{\tau_n}{n} = \frac{1}{n}\sum_{k=1}^{n}(\tau_k - \tau_{k-1}) \to \mathbb{E}[\tau_2 - \tau_1] < \infty \ a.s..$$

□

Proof of main theorem for $\lambda < 1$ and $p_0 = 0$

By the strong law of large numbers and Proposition 3: $\frac{\tau_n}{n} \to \mathbb{E}[\tau_2 - \tau_1] < \infty$ a.s. and $|X_{\tau_n}|/n \to \mathbb{E}[|X_{\tau_2}| - |X_{\tau_1}|] \le \mathbb{E}[\tau_2 - \tau_1] < \infty$ a.s. Therefore,

$$\frac{|X_{\tau_n}|}{\tau_n} \to \frac{\mathbb{E}[|X_{\tau_2}| - |X_{\tau_1}|]}{\mathbb{E}[\tau_2 - \tau_1]} \; a.s..$$

Since $\lim \frac{\tau_n}{n}$ exists, we have $\frac{\tau_{n+1}}{\tau_n} \to 1$. Now for $\tau_k \le n < \tau_{k+1}$, we have

$$|X_{\tau_k}| \le |X_n| \le |X_{\tau_k}| + n - \tau_k \le |X_{\tau_k}| + \tau_{k+1} - \tau_k \,.$$

Proof of main theorem for $\lambda < 1$ and $p_0 = 0$-continued

Hence,

$$\frac{\tau_k}{\tau_k}\frac{|X_{\tau_k}|}{\tau_{k+1}} \leq \frac{|X_n|}{n} \leq \frac{X_{\tau_k} + \tau_{k+1} - \tau_k}{n} \leq \frac{|X_{\tau_k}|}{\tau_k} + \frac{\tau_{k+1}}{\tau_k} - 1$$

Therefore, taking limits,

$$\lim_{n\to\infty} \frac{|X_n|}{n} = \lim_{k\to\infty} \frac{|X_{\tau_k}|}{\tau_k} = \frac{\mathbb{E}[|X_{\tau_2}| - |X_{\tau_1}|]}{\mathbb{E}[\tau_2 - \tau_1]} > 0 \, a.s. \qquad \square$$

Inward-biased random walk

Suppose $1 < \lambda < m$ and p_0 arbitrary. Then all the previous results still hold. Firstly, we can show that

$$\frac{\mathbb{E}R_n}{n} \geq \frac{1}{n} + \frac{\lambda - 1}{2\lambda G(\rho,\rho) + (\lambda - 1)},$$

where G is the associated Green function, i.e. $G(x,y)$ is the expected number of visits to y when starting in x. Provided we condition on non-extinction, we can see that a.s. there exist infinitely many regeneration epochs, with the same properties as before. So Proposition 3 holds and so does the final proof, always provided we condition on non-extinction.

Main theorem for outward biased random walk

Theorem 4

Assume $p_0 > 0$. Let T be a GW tree conditioned on non-extinction. Then, the speed of RW_λ exists and is constant almost surely. It is positive if $f'(q) < \lambda < 1$ and zero if $0 \leq \lambda \leq f'(q)$.

Denote by $\mathbb{P}_{non}$ and $\mathbb{E}_{non}$ the probability and expectation conditional on non-extinction.

Since the case $\lambda = 0$ is obvious, we can assume $\lambda > 0$.

Generating the tree

Define

$$g(s) = \frac{f(s) - f(qs)}{1 - q} \quad \text{and} \quad h(s) = \frac{f(qs)}{q}$$

Generate an f-Galton-Watson tree T_f conditioned on non-extinction as follows:

- generate a g-Galton-Watson Tree T_g
- append to each vertex x of T_g a random number N_x of h-Galton-Walton **shrubs**, s.t. the distribution of N_x depends only on $d_{T_g}(x)$ and given T_g and the numbers N_x the shrubs are iid.

Note: $g(0) = 0$ and $h'(1) = f'(q) < 1$.
Call the union of the N_x shrubs at x a **bush**. Observe RW_λ at time σ_n when it makes a transition along an edge of T_g, set $Y_n = X_{\sigma_n}$. Between these observations, the random walk makes an excursion of random length.

Expected length of excursion into a shrub

If Γ is a fixed finite tree, with Γ_n vertices in the n^{th} generation, then the expected length of time that RW_λ takes to return to the root is:

$$2\sum_{n\geq 1}\Gamma_n\lambda^{1-n}/\Gamma_1\,,$$

since it is precisely the reciprocal of the stationary probability of the root. In particular for h-Galton-Watson trees, this sum has expectation

$$2\sum_{n\geq 1} h'(1)^{n-1}\lambda^{1-n} = \begin{cases} \frac{2}{(1-f'(q)\lambda^{-1})} & \text{if } \lambda > f'(q) \\ \infty & \text{if } 0 < \lambda \leq f'(q) \end{cases} \tag{1}$$

Speed is zero for $\lambda \leq f'(q)$

By the previous result, the expected time between regenerations epochs on T_f is infinite. So by the strong law of large numbers, $\frac{\tau_n}{n} \to \infty$, but the expected distance between successive regenerations loci on T_f is the same as on T_g, so

$$\frac{|X_{\tau_n}|}{n} \to \mathbb{E}\left[|X_{\tau_2}| - |X_{\tau_1}|\right] < \infty \, .$$

It follows that $\lim_{n\to\infty} \frac{|X_{\tau_n}|}{\tau_n} = 0$, which implies that the speed is zero.

The case $f'(q) < \lambda < 1$

Between σ_n and σ_{n+1}, the walk (X_k) makes a random number of excursions into the bush at Y_n, which has a geometric distribution with mean

$$\frac{d_{T_f}(Y_n) - d_{T_g}(Y_n)}{\lambda + d_{T_g}(Y_n)}$$

By (1)

$$\begin{aligned}\mathbb{E}_{non}[\sigma_{n+1} - \sigma_n \mid Y_n] &= \frac{2}{(1 - f'(q)\lambda^{-1})} \frac{d_{T_f}(Y_n) - d_{T_g}(Y_n)}{\lambda + d_{T_g}(Y_n)} \\ &\leq c d_{T_f}(Y_n)\,,\end{aligned}$$

where c is a constant depending only on λ and f. Hence,

$$\mathbb{E}_{non}\left[\frac{\sigma_n}{n}\right] = \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}_{non}[\sigma_{j+1} - \sigma_j \mid Y_j] \leq c\mathbb{E}_{non}\left[\sum_{j=0}^{n-1} d_{T_f}(Y_n)\right] \qquad (2)$$

The case $f'(q) < \lambda < 1$ - continued

Let $Z_1, Z_2, \ldots, Z_{K_n}$ be the distinct vertices among $Y_1, \ldots, Y_n$. Let $U_i = \sum_{j=1}^{\infty} 1\{Y_j = Z_i\}$. Then,

$$\mathbb{E}_{non}\left[\sum_{i=1}^{n} d_{T_f}(Y_i)\right] \leq \mathbb{E}_{non}\left[\sum_{k=1}^{K_n} U_k d_{T_f}(Z_k)\right]$$

Comparing to an asymmetric simple random walk gives

$$\begin{aligned}\mathbb{E}_{non}[U_k d_{T_f}(Z_k)] &= \mathbb{E}_{non}[d_{T_f}(Z_k)\mathbb{E}_{non}[U_k \mid d_{T_f}(Z_k)]] \\ &\leq \mathbb{E}_{non}\left[d_{T_f}(Z_k)\frac{1+\lambda}{1-\lambda}\right] = \frac{m}{1-q}\frac{1+\lambda}{1-\lambda}.\end{aligned}$$

Therefore,

$$\mathbb{E}_{non}\left[\sum_{i=1}^{n} d_{T_f}(Y_i)\right] \leq n\frac{m}{1-q}\frac{1+\lambda}{1-\lambda}.$$

The case $f'(q) < \lambda < 1$ - final part

Hence by (2), we have

$$\mathbb{E}_{non}[\sigma_n/n] \leq \frac{cm}{1-q}\frac{1+\lambda}{1-\lambda}$$

So by Fatou's lemma, $\liminf_{n\to\infty} \sigma_n/n < \infty$. On T_g there is an infinite number of generation epochs, so we have $\liminf_{k\to\infty} \tau_k/k < \infty$ a.s. By the strong law of large numbers, it follows that $\mathbb{E}[\tau_{k+1} - \tau_k] < \infty$, and the above liminf is a limit a.s. satisfying

$$\lim_{k\to\infty} \frac{\tau_k}{k} \to \mathbb{E}[\tau_2 - \tau_1]\,.$$

Since $\lim_{k\to 1} \tau_{k+1}/\tau_k = 1$, it follows that

$$\lim_{n\to\infty} \frac{|X_n|}{n} = \lim_{k\to\infty} \frac{|X_{\tau_k}|}{\tau_k} \geq \lim_{k\to\infty} \frac{k}{\tau_k} > 0.$$

□