Small value probabilities via the branching tree heuristic

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joint work with Peter Mörters

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Overview

The *small value problem*: Given a non-negative random variable X, find the speed of decay of the left tail $\mathbb{P}\{X < \varepsilon\}$ as $\varepsilon \downarrow 0$.

- Martingale limit of a super-critical Galton-Watson tree. $\implies$ *branching tree heuristic*.
- Intersection local times of several Brownian motions.
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The martingale limit of a Galton-Watson tree

Consider a Galton-Watson branching process $(Z_n : n \geq 0)$ with offspring distribution $(p_k : k \geq 0)$ starting in a single ancestor ρ. Suppose the offspring variable N is non-degenerate and satisfies $\mu := \mathbb{E}N > 1$ and $\mathbb{E}[N \log N] < \infty$.

Kesten-Stigum theorem: the martingale limit

$$W := \lim_{n\to\infty} \frac{Z_n}{\mu^n}$$

exists and is non-trivial almost surely on survival. Without loss of generality, we will assume that $p_0 = 0$.

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Small value probabilities:

Two different cases:

- *Schröder case:* $p_1 > 0$.
- *Böttcher case:* $p_1 = 0$.

Theorem 1.1 (Dubuc 1971)

(a) *In the Schröder case, define* $\tau = -\log p_1 / \log \mu > 0$. *Then*

$$\mathbb{P}\{W < \varepsilon\} \asymp \varepsilon^{\tau}.$$

(b) *In the Böttcher case, define* $\nu := \min\{i \geq 0 : p_i \neq 0\} \geq 2$ *and* $\beta := \frac{\log \nu}{\log \mu} < 1$. *Then,*

$$-\log \mathbb{P}\{W < \varepsilon\} \asymp \varepsilon^{-\beta/(1-\beta)}.$$

$a(\varepsilon) \asymp b(\varepsilon) \iff$ there exist constants $0 < c < C$ such that $ca(\varepsilon) \leq b(\varepsilon) \leq Ca(\varepsilon)$ for all sufficiently small $\varepsilon > 0$.

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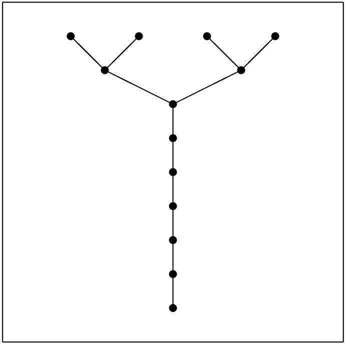

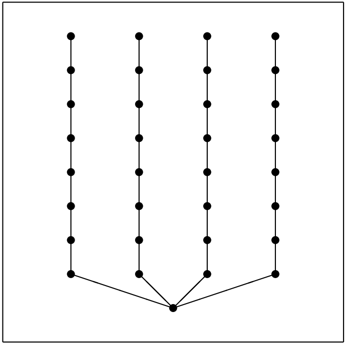

Sketch of proofs - Schröder case

Denote by $(Z_n(v) : n \geq 0)$ the generation sizes of the subtree consisting of the descendants of a vertex v.

Therefore, for the corresponding martingale limit $W(v) = \lim_{n\to\infty} Z_n(v)/\mu^n$ we get

$$W = \lim_{n\to\infty} \frac{Z_n}{\mu^n} = \mu^{-k} \lim_{n\to\infty} \sum_{i=1}^{Z_k} \frac{Z_{n-k}(v_k(i))}{\mu^{n-k}} \\ = \mu^{-k} \sum_{i=1}^{Z_k} W(v_k(i))$$

Hence, if $\varepsilon = \mu^{-k}$,

$$\mathbb{P}\{W < \varepsilon\} \geq \mathbb{P}\{W < \mu^{-k} | Z_k = 1\}\mathbb{P}\{Z_k = 1\} \\ = \mathbb{P}\{\mu^{-k} W(v_k(1)) < \mu^{-k}\} p_1^k = \mathbb{P}\{W < 1\}\varepsilon^\tau ,$$

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Mutual intersection local times of several Brownian motions

Let $W^{(1)}, \ldots, W^{(m)}$ be $m > 1$ independent (1-dimensional) Brownian motions started in 0, and denote by $(L_m(x,t) : t \geq 0)$ the local time of the mth Brownian motion at $x \in \mathbb{R}$. The *(mutual) intersection local time* is defined as

$$X(t_1, \ldots, t_m) := \int_{-\infty}^{\infty} \prod_{i=1}^{m} L_i^{q_i}(x, t_i) dx \,,$$

for some $q_1, \ldots, q_m \geq 1$.

If $q_1 = \ldots = q_m = 1$, $X(t_1, \ldots, t_m)$ measures the amount of intersection between the motions up to times $t_1, \ldots, t_m$.

Small value probabilities for intersection local times

Let $\sigma^{(1)}, \dots, \sigma^{(m)}$ be the first exit times of the Brownian motions from the interval $(-1, 1)$.

Theorem 2.1

If $q_1, \dots, q_m \geq 1$ *and* $q = \sum_{j=1}^{m} q_j$*, then*

(a)

$$\mathbb{P}\left\{ \int_{-\infty}^{\infty} \prod_{i=1}^{m} L_i^{q_i}(x, \sigma^{(i)}) dx < \varepsilon \right\} \asymp \varepsilon^{2/(1+q)} ,$$

(b)

$$\mathbb{P}\left\{ \int_{-\infty}^{\infty} \prod_{i=1}^{m} L_i^{q_i}(x, 1) dx < \varepsilon \right\} \asymp \varepsilon^{2/(1+q)} .$$

Analogy to Schröder case

Concentrate on $m = 2$ and consider two Brownian motions $W^{(1)}, W^{(2)}$ such that $W^{(1)}$ exits the interval $(-1, 1)$ at the upper end and $W^{(2)}$ at the lower end. Set for $\eta \in \mathbb{N}$,

$$\tau_k^{(1)} := \inf\{t \geq 0 : W^{(1)}(t) = \eta^{-k}\}\,,$$
$$\tau_k^{(2)} := \inf\{t \geq 0 : W^{(2)}(t) = -\eta^{-k}\}\,.$$

Construct the corresponding tree from its spine $v_0(1), \ldots, v_n(1)$ of leftmost particles in the first n generations:

- The kth individual $v_k(1)$ has more than one offspring if

$$W^{(1)}[\tau_{k+1}^{(1)}, \tau_k^{(1)}] \cap W^{(2)}[\tau_{k+1}^{(2)}, \tau_k^{(2)}] \neq \emptyset\,.$$

- If intervals intersect, the intersection local times of $W^{(j)}$ started at $\tau_{k+1}^{(j)}$ and stopped at $\tau_k^{(j)}$ gives rise to summand of the total intersection local time that is distributed like μ^{-k} times the intersection local time.

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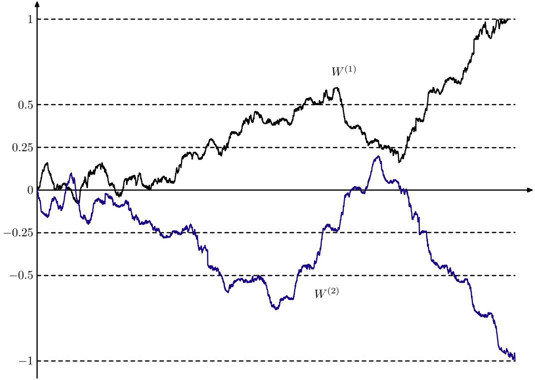
Picture: Constructing the tree

Consider $\eta = 2$.



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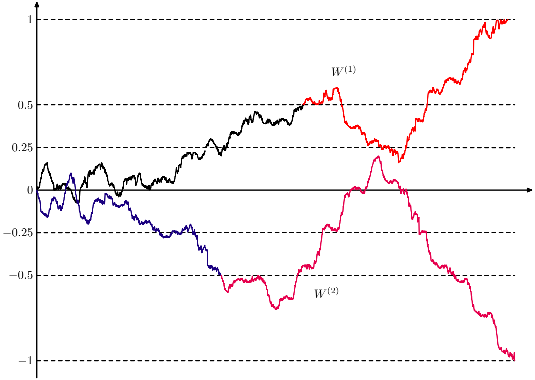

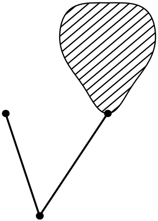

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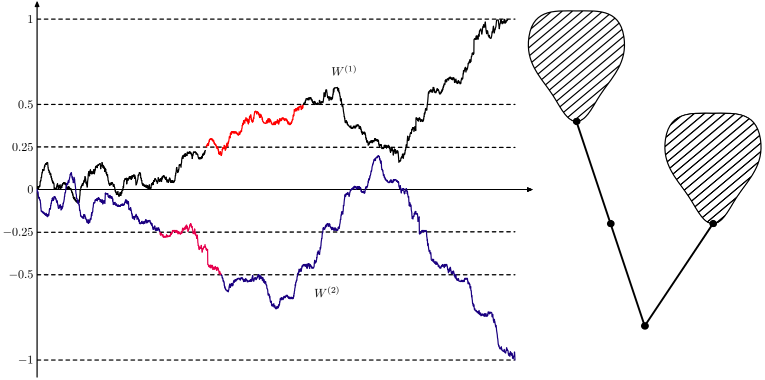

Identifying the parameters p_1 and μ

Recall, p_1 is the probability that a vertex has only one offspring, i.e.

$$p_1 = \mathbb{P}\{ W^{(1)}[\tau_1^{(1)}, \tau_0^{(1)}] \cap W^{(2)}[\tau_1^{(2)}, \tau_0^{(2)}] = \emptyset \} .$$

Gambler's ruin probability gives that $p_1 \approx \eta^{-2}$.

Define, for $L_j(x,s,t) = L_j(x,t) - L_j(x,s)$,

$$X_k := \int_{-\infty}^{\infty} \prod_{j=1}^{m} L_j^{q_j}(x, \tau_{k+1}^{(j)}, \tau_k^{(j)}) dx .$$

Recall, μ is defined via $X_k \overset{d}{=} \mu^{-k} X_0$. But Brownian scaling yields, for $q = \sum_{j=1}^m q_j$.

$$X_k \overset{d}{=} \eta^{-k(1+q)} X_0 .$$

Therefore, $\mu = \eta^{1+q}$.

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Putting it together

Recall, in the tree case we had for $\tau = -\log p_1 / \log \mu$,

$$\mathbb{P}\{W < \varepsilon\} \asymp \varepsilon^{\tau}\,,$$

Therefore, our analogy gives for the intersection local time, with $p_1 = \eta^{-2}$ and $\mu = \eta^{1+q}$,

$$\tau = \frac{2}{1+q}\,,$$

so that

$$\mathbb{P}\left\{\int_{-\infty}^{\infty}\prod_{i=1}^{m} L_i^{q_i}(x,\sigma^{(i)})dx < \varepsilon\right\} \asymp \varepsilon^{2/(1+q)}\,.$$

Self-intersection local times

Consider a single Brownian motion W and its q-fold *intersection local time*

$$X(t) := \int_{-\infty}^{\infty} L^q(x,t)\,dx\,,$$

for $(L(x,t) : t \geq 0)$ the local time of the Brownian motion at $x \in \mathbb{R}$.

Theorem 3.1

Let $\sigma = \inf\{t \geq 0 : |W(t)| = 1\}$. *For* $q \geq 1$,

$$-\log \mathbb{P}\{X(\sigma) < \varepsilon\} \asymp \varepsilon^{-1/q}\,.$$

Different behaviour for fixed time instead of stopping time, see Hofstad et al. (1997) ($q = 2$) and Chen and Li (unpublished).

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Analogy to Böttcher case

Construct a nested family of random walks embedded into Brownian paths: Let

$$\mathcal{D}_n := \{k2^{-n} : k \in \{-2^n, \ldots, 2^n\}\},$$

and set $\tau_0^{(n)} = 0$ and

$$\tau_j^{(n)} := \inf\{t > \tau_{j-1}^{(n)} : W(t) \in \mathcal{D}_n, W(t) \neq W(\tau_{j-1}^{(n)})\}.$$

Then define the nth embedded random walk $(X^{(n)} : 0 \leq j \leq N(n))$ by

$$X^{(n)}(j) := W(\tau_j^{(n)}).$$

Idea: a constant multiple of the total number of steps of embedded random walk approximates (for large n) the q-fold self-intersection times.

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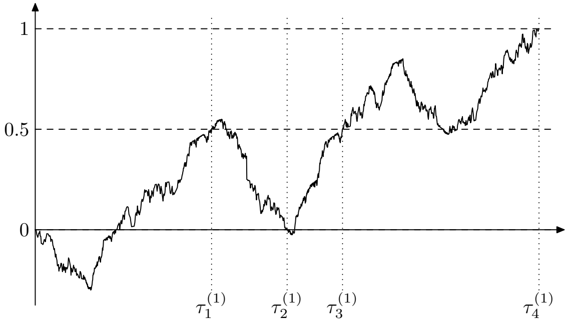
Picture: tree analogy

Assign $N(1)$ offspring to the root, i.e. vertices in first generation correspond to steps of height $1/2$ that the path takes to reach level 1 or -1. Then, the number of children of a vertex in the first generation is determined by number of steps of height $1/4$ that the path makes during the step of height $1/2$ corresponding to that vertex.



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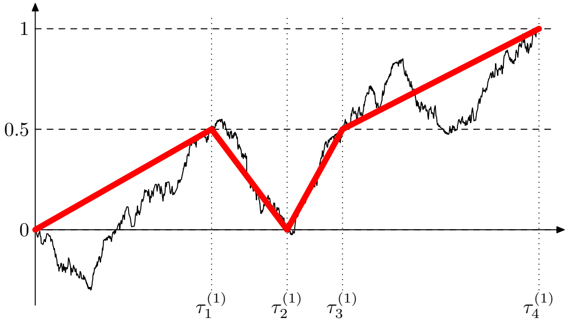

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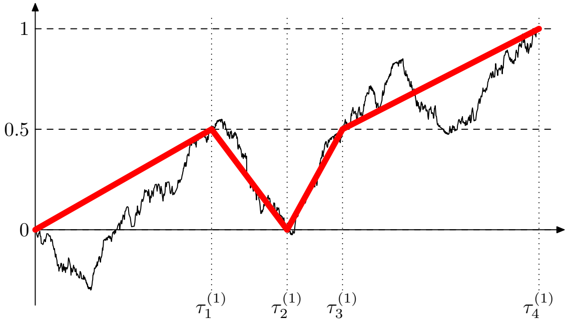

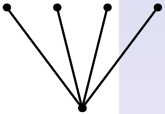

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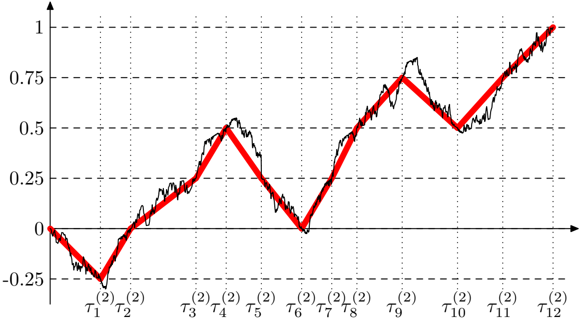

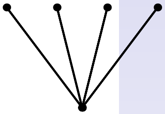

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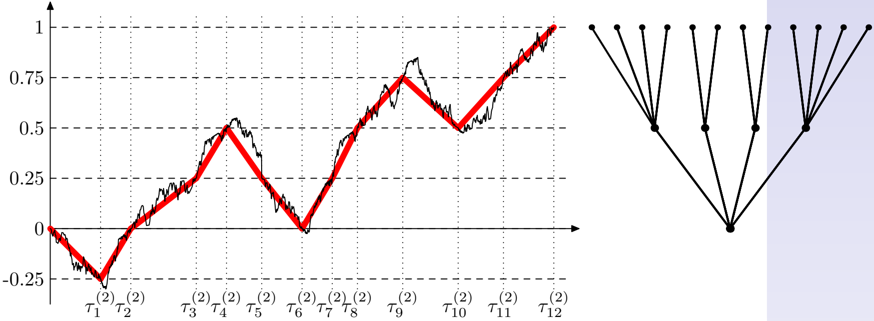

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