Directed Polymers in a Random Environment

Localisation in the Weak Disorder Phase

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The Model

A directed polymer in $\mathbb{Z}^{1+d}$ is represented as $(j,\omega_j)_{j=1}^n$, where $(\omega_j)_{j\geq 1}$ is a simple, symmetric random walk. Given a random environment $\mathcal{V} = (V(j,x) : j \in \mathbb{Z}, x \in \mathbb{Z}^d)$ of i.i.d. random variables, the energy associated to such a polymer is

$$-\beta \sum_{j=1}^n V(j,\omega_j)\,,$$

where $\beta > 0$ is the inverse temperature. The polymer measure $\mu_n^{(\beta)}$ favours polymers with low energy, and it is defined as

$$\mu_n^{(\beta)}(d\omega) = \frac{1}{Z_n(\beta)} \exp\Big(\beta \sum_{j=1}^n V(j,\omega_j)\Big) \mathbf{P}(d\omega)\,,$$

where $Z_n(\beta)$ is a normalising constant, the partition function.

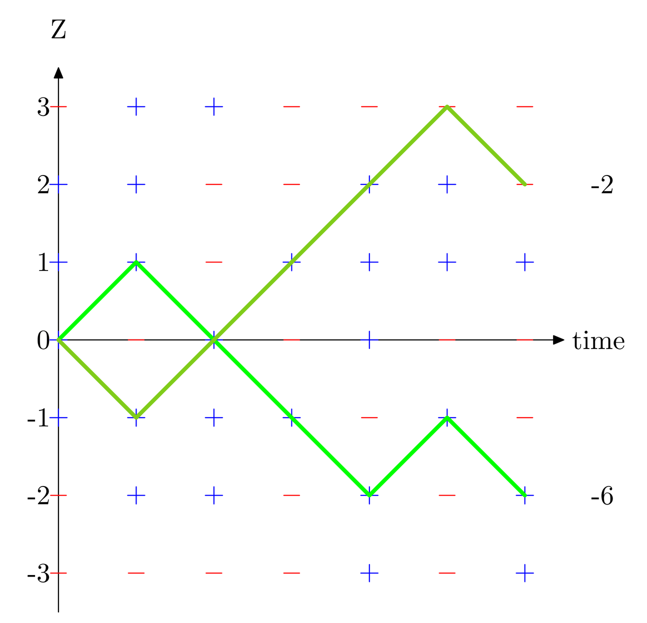

Figure 1: The model for $d=1$ and the environment taking values 1 or -1 with equal probability. The figure shows two particular polymers with energies -2β and -6β respectively.

A Phase Transition

Suppose that V is distributed as $V(j,x)$ and satisfies

$$\varphi(\beta) = \mathbb{E}e^{\beta V} < \infty \quad \forall \beta > 0\,.$$

Then, the normalised partition function

$$M_n^{(\beta)} := \frac{1}{(\varphi(\beta))^n} Z_n(\beta)$$

is a martingale with respect to the filtration $\mathcal{G}(n) = \sigma(V(j,x) : j \leq n, x \in \mathbb{Z}^d)$. By positivity, its $\mathbb{P}$-almost sure limit $M^{(\beta)}$ exists. Comets and Yoshida (2006) show that there exists a $0 \leq \beta_c \leq \infty$ such that

$$\mathbb{P}\{M^{(\beta)} > 0\} = 1 \text{ if } \beta < \beta_c \quad \text{and} \quad \mathbb{P}\{M^{(\beta)} = 0\} = 1 \text{ if } \beta > \beta_c\,.$$

For $\beta < \beta_c$ we say that we are in the weak disorder phase and for $\beta > \beta_c$ in the strong disorder phase. It is known that

$$\begin{array}{lll} d = 1,2 & \Longrightarrow & \beta_c = 0 \\ d \geq 3 & \Longrightarrow & 0 < \beta_c \leq \infty\,, \end{array}$$

but in the latter case β_c is not known explicitly.

Comets and Yoshida (2006) also show that a central limit theorem holds in the full weak disorder regime.

For $\beta < \beta_c$, the free energy is

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta) = \log \varphi(\beta) + \lim_{n\to\infty} \frac{1}{n} \log M_n^{(\beta)} = \log \varphi(\beta)\,.$$

The Tree Structure of the Path Space

The collection of polymers has a natural structure of a $(2d)$-ary tree T with polymers of length n forming the nth generation and the last common ancestor of two polymers given as the longest shared initial substring.

Denoting by T_n the vertices in the nth generation, we can write the partition function as

$$Z_n(\beta) = \frac{1}{(2d)^n} \sum_{v \in T_n} \exp(\beta \sum_{j=1}^n V(j,v_j))\,.$$

Localisation in the Weak Disorder Phase

Define $f : (0,\beta_c) \to [0,\infty)$ by

$$f(\beta) := \log(2d\varphi(\beta)) - \frac{\varphi'(\beta)}{\varphi(\beta)}\beta\,.$$

We obtain a localisation effect in the sense that a tree of growth rate strictly less than $\log 2d$ supports the free energy.

Theorem. *Let $\beta < \beta_c$ and $\mathbb{E}[M^{(\beta)} \log M^{(\beta)}] < \infty$.*

(a) Almost surely, there exists a tree $\tilde{T} \subset T$ with $\lim_{n\to\infty} \frac{1}{n} \log |\tilde{T}_n| = f(\beta)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{(2d)^n} \sum_{v \in \tilde{T}_n} e^{\beta \sum_{j=1}^n V(j,v_j)} = \log \varphi(\beta)\,.$$

(b) Almost surely, for every tree $\tilde{T} \subset T$ with $\limsup_{n\to\infty} \frac{1}{n} \log |\tilde{T}_n| < f(\beta)$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{(2d)^n} \sum_{v \in \tilde{T}_n} e^{\beta \sum_{j=1}^n V(j,v_j)} < \log \varphi(\beta)\,.$$

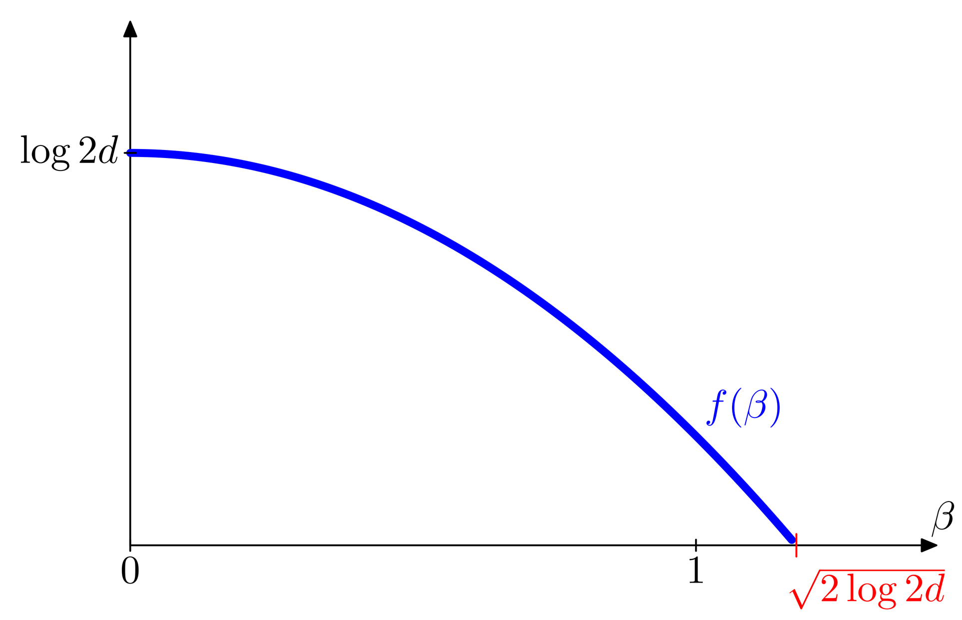

Figure 2: $f(\beta)$ for V standard normally distributed.

Open Problems

Let β' be such that $f(\beta') = 0$. Assuming $\beta' < \infty$ then it is known that $\beta > \beta'$ implies $M^{(\beta)} = 0$ $\mathbb{P}$-a.s.

Question: Can a localisation effect in the above sense be observed in this case or is the free energy only supported by a tree of growth rate $2d$?

Question: Is $\beta' = \beta_c$ or is there an intermediate phase?

References

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- F. Comets and N. Yoshida. Directed polymers in random environment are diffusive at weak disorder. *Ann. Probab.*, 34(5):1746 - 1770, 2006.

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