Large deviation theory and applications

Application I: The parabolic Anderson model

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This presentation is based on [dH00, Chapter VIII]. The parabolic Anderson model is given by the heat equation on the lattice $\mathbb{Z}^d$ with a random potential, i.e. we consider the solution $u : [0,\infty) \times \mathbb{Z}^d \to [0,\infty)$ of the Cauchy problem

$$\begin{array}{lcll} \frac{\partial}{\partial t} u(t,z) & = & \Delta u(t,z) + \xi(z) u(t,z)\,, & (t,z) \in (0,\infty) \times \mathbb{Z}^d\,, \\ u(0,z) & = & 1\,, & z \in \mathbb{Z}^d\,. \end{array} \tag{1}$$

Here Δ is the discrete Laplacian

$$\Delta u(t,x) = \sum_{y \in \mathbb{Z}^d : y \sim x} \big(u(t,y) - u(t,x) \big)\,,$$

where $y \sim x$ means that y is a nearest-neighbour of site x. The potential $\xi = (\xi(z) : z \in \mathbb{Z}^d)$ is a collection of independent, identically distributed random variables. We denote by $\langle \cdot \rangle$ the expectation with respect to the random potential.

The solution u is influenced by the competition between the Laplacian Δ which has a smoothing effect, and the random potential, which makes the solution spatially irregular. In general, it is believed that there is a small number of *relevant island* where the potential takes especially large values, which are important for the large times asymptotics of the solution. This effect is called *intermittency.* For our purposes we will say that the model is intermittent if for any integers $p > q \geq 1$

$$\lim_{t\to\infty} \log \frac{\langle u(t,0)^p \rangle^{1/p}}{\langle u(t,0)^q \rangle^{1/q}} = \infty\,.$$

Our main aim will be to show that, for a special class of potentials, the parabolic Anderson model is intermittent.

Throughout, we will assume that the logarithmic moment generating function $H(t)$ is finite for all t, i.e.

$$H(t) = \log \left\langle e^{t\xi(0)} \right\rangle < \infty \quad \text{for all } t \geq 0\,. \tag{2}$$

To represent the solution of (1) in a way that makes it accessible to large deviations techniques, we consider a continuous-time simple random walk $(X_s)_{s\geq 0}$ on $\mathbb{Z}^d$ jumping at rate $2d$, i.e. the Markov process with generator Δ.

Lemma 1. *Under assumption (2), the heat equation (1) has a unique non-negative solution* $u : [0, \infty) \times \mathbb{Z}^d \to [0, \infty)$ *with* Feynman-Kac representation

$$u(t, z) = \mathbb{E}_z \Big[\exp \Big\{ \int_0^t \xi(X_s) ds \Big\} \Big] .$$

For all $t \geq 0$*, the random field* $(u(t, z) \, : \, z \in \mathbb{Z}^d)$ *is stationary and ergodic under translations.*

Proof. A direct calculation using the Markov property shows that u defined in this way is a solution. For the uniqueness of the solution see [GM90]. Stationarity follows since $u(0, \cdot) \equiv 1$, ξ is stationary and since the increments of the random walk are i.i.d. Similarly, $(u(t, z) \, : \, z \in \mathbb{Z}^d)$ is ergodic, since ξ is ergodic. □

The Feynman-Kac representation tells that $u(t, z)$ can be expressed as some functional of a simple random walk, whose large deviation behaviour we know. We are now in the continuous-time setting, so let us state the equivalent of Sanov's theorem for continous-time Markov chains.

Let $\mathcal{X}$ be a finite set and let $(X_t)_{t \geq 0}$ be a $\mathcal{X}$-valued continuous-time Markov chain with an irreducible generator $G = (G_{ij})_{i,j \in \mathcal{X}}$. Define the *empirical measure* (or occupation time measure)

$$L_t = \frac{1}{t} \int_0^t \delta_{X_s} ds \, .$$

In order to obtain a nice rate function, we assume additionally that the generator G is symmetric.

Theorem 2. *Under the above assumptions, the empirical measures* L_t *satisfy a large deviation principle on* $\mathcal{M}_1(\mathcal{X})$ *with speed* t *and rate function*

$$I_G(\nu) = - \sum_{x,y \in \mathcal{X}} \sqrt{\nu(x)} G_{xy} \sqrt{\nu(y)} = \big\langle \sqrt{\nu}, (-G) \sqrt{\nu} \big\rangle \, .$$

Intuitively, we expect that the field $(u(t, z) \, : \, z \in \mathbb{Z}^d)$ develops peaks where the potential ξ takes large values. So it is clear that the asymptotic behaviour of these peaks depends on the right tail of the distribution of ξ. We will consider a special class of potentials, namely those which satisfy

$$\lim_{t \to \infty} \frac{1}{t} (H(ct) - cH(t)) = \rho c \log c \quad \text{for all } c \in (0, 1) \, , \tag{3}$$

for some parameter $\rho \in (0, \infty)$. The most basic example satisfying this condition is when $\xi(0)$ has the double exponential distribution

$$\mathrm{Prob}(\xi(0) > s) = \exp\{-e^{s/\rho}\} \, , \quad s \in \mathbb{R} \, .$$

In this case, the probability that $\xi(0)$ takes values larger than $s = \rho$ drops quickly to zero, so ρ indicates the degree of disorder in the underlying potential ξ.

In order to show intermittency, we need to evaluate the asymptotics of the integer moments of $u(0,t)$ for large t.

Theorem 3. *Assume (2) and (3). Then, for any $p \in \mathbb{N}$,*

$$\langle u^p(0,t)\rangle = \exp\{H(pt) - 2dpt\chi(\rho) + o(t)\}\,,$$

where

$$\chi(\rho) = \frac{1}{2d} \inf_{\nu \in \mathcal{M}_1(\mathbb{Z}^d)} \{I(\nu) + \rho J(\nu)\}\,,$$

with

$$\begin{aligned} I(\nu) &= \langle \sqrt{\nu}, (-\Delta)\sqrt{\nu}\rangle = \tfrac{1}{2}\textstyle\sum_{x,y\in\mathbb{Z}^d\,:\,x\sim y} \left(\sqrt{\nu(x)} - \sqrt{\nu(y)}\right)^2, \\ J(\nu) &= -\textstyle\sum_{x\in\mathbb{Z}^d} \nu(x)\log\nu(x)\,. \end{aligned}$$

Remark 4. (a) It can be shown that the infimum in the definition of χ factorizes as d times the equivalent expression for the one-dimensional model, so χ does not depend on the dimension d. Moreover, χ can be linked to the non-linear difference equation

$$\Delta v + 2\rho v \log v = 0\,, \quad v : \mathbb{Z} \to (0,\infty)\,.$$

Namely, this equation has a ground state v_ρ, i.e. a solution with minimal ℓ^2 norm, and

$$\chi = \rho \log \|v_\rho\|_2\,.$$

(b) Note, $I(\nu)$ is the (weak) rate function for the local times of the Markov chain on $\mathbb{Z}^d$ with generator Δ. The level sets of I are not compact, since $\mathbb{Z}^d$ is infinite.

As a direct consequence of Theorem 3 we can shown that the model is intermittent. Indeed, using the scaling assumption (3) on the tails

$$\begin{aligned} \frac{\langle u^p(0,t)\rangle^{\frac{1}{p}}}{\langle u^q(0,t)\rangle^{\frac{1}{q}}} &= \exp\left\{\tfrac{1}{p}H(pt) - \tfrac{1}{q}H(qt) + o(t)\right\} \\ &= \exp\left\{\tfrac{1}{p}H(pt) - H(\tfrac{1}{p}pt) - (\tfrac{1}{q}H(qt) - H(\tfrac{1}{q}qt)) + o(t)\right\} \\ &= \exp\left\{-pt\rho\tfrac{1}{p}\log\tfrac{1}{p} + qt\rho\tfrac{1}{q}\log\tfrac{1}{q} + o(t)\right\} \\ &= \exp\left\{t\rho\log\tfrac{p}{q} + o(t)\right\}. \end{aligned}$$

The physical interpretation is that each moment is carried by higher and higher peaks that are localized on random islands, which are far apart and occupy a vanishing fraction of the lattice. Indeed, if we write $\Lambda_p(t) = \log \langle u^p(t,0)\rangle$, then

$$\frac{1}{p+1}\Lambda_{p+1}(t) - \frac{1}{p}\Lambda_p(t) = \log \frac{\langle u^{p+1}(t,0)\rangle^{1/(p+1)}}{\langle u^p(t,0)\rangle^{1/p}} = \rho t \log \frac{p+1}{p} + o(t)\,.$$

Therefore, we can choose level functions $\ell_p(t)$ such that

$$\frac{\Lambda_p}{p} \ll \ell_p(t) \ll \frac{\Lambda_{p+1}}{p+1}\,,$$

where $f(t) \ll g(t)$ means that $g(t) - f(t) \to \infty$ as $t \to \infty$. Then consider the event

$$E_p(t) = \{u(t,0) > e^{\ell_p(t)}\}\,.$$

Note that by Chebyshev's inequality,

$$\mathrm{Prob}(E_p(t)) \le \exp\{\Lambda_p(t) - p\ell_p(t)\} \to 0 \qquad \text{as } t \to \infty\,,$$

so that by ergodicity the sites $x \in \mathbb{Z}^d$ where $u(t,x) > e^{\ell_p(t)}$ occupy a vanishing fraction of the lattice. But on the other hand,

$$\left\langle u(t,0)^{p+1}\mathbf{1}_{\Omega\setminus E_p(t)}\right\rangle \le e^{(p+1)\ell_p(t)} = e^{(p+1)\ell_p(t)-\Lambda_{p+1}(t)}\left\langle u(t,0)^{p+1}\right\rangle = o(\left\langle u(t,0)^{p+1}\right\rangle)\,.$$

Hence, we can deduce that

$$\left\langle u(t,0)^{p+1}\right\rangle \sim \left\langle u(t,0)^{p+1}\mathbf{1}_{E_p(t)}\right\rangle\,,$$

so that the main contribution to the $(p+1)$st moment is carried by those sites where the field $(u(t,x))$ takes values larger than $e^{\ell_p(t)}$.

Proof of Theorem 3

We start by sketching the proof in the case $p = 1$, later on we see how to fix the gaps and also how to extend it to $p \ge 2$.

Define the local time of the simple random walk

$$\ell_t(z) = \int_0^t \mathbf{1}_{\{X_s = z\}} ds\,, \quad z \in \mathbb{Z}^d, t \ge 0\,.$$

Then, by the Feynman-Kac formula in Lemma 1, we have that

$$u(t,0) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)ds\Big\}\Big] = \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z)\Big\}\Big]\,.$$

Hence taking the expectation and using the independence of the $\xi(z)$, we obtain

$$\langle u(t,0)\rangle = \mathbb{E}_0\Big[\prod_{z\in\mathbb{Z}^d}\langle e^{\ell_t(z)\xi(z)}\rangle\Big] = \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\}\Big].$$

Recall that $L_t(z) = \frac{1}{t}\ell_t(z)$ and use $\sum_{z\in\mathbb{Z}^d}\ell_t(z) = t$ to write

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d} H(\ell_t(z)) &= H(t) + t\sum_{z\in\mathbb{Z}^d}\frac{1}{t}\Big[H(L_t(z)t) - L_t(z)H(t)\Big]\\ &= H(t) + t\sum_{z\in\mathbb{Z}^d}[\rho L_t(z)\log L_t(z) + o(1)]\\ &= H(t) - t\rho J(L_t) + o(t),\end{aligned}$$

where the second step is *plausible* by our assumption (3) on H. Therefore, we obtain that

$$\langle u(t,0)\rangle = e^{H(t)+o(t)}\mathbb{E}_0\big[e^{-t\rho J(L_t)}\big],$$

which is an expression that is of the right form for Varadhan's lemma. Thus (if we ignore the problem that Z_t is a Markov chain with an infinite state space), it becomes *plausible* that

$$\begin{aligned}\langle u(t,0)\rangle &= \exp\Big\{H(t) + t\sup_{\nu\in\mathcal{M}_1(\mathbb{Z}^d)}\{-\rho J(\nu) - I(\nu)\}\Big\}\\ &= \exp\Big\{H(t) - 2dt\chi(\rho) + o(t)\Big\}.\end{aligned}$$

The two problems in the above argument can be fixed by restricting the problem to a large box $T_N = (-N,N]^d \cap \mathbb{Z}^d$ and finally letting $N\to\infty$.

For the *upper bound*, we assume that the simple random walk moves on T_N with periodic boundary conditions. Then, if $\{\ell_t^N(z) : z\in T_N\}$ denote the local times of the wrapped random walk, we can write

$$\ell_t^N(z) = \sum_{y\in(2N\mathbb{Z})^d}\ell_t(z+y), \quad z\in T_N.$$

Since $H(0)=0$ and H is convex, H is super-additive, and thus

$$\sum_{z\in\mathbb{Z}^d} H(\ell_t(z)) = \sum_{z\in T_N}\sum_{y\in(2N\mathbb{Z})^d} H(\ell_t(z+y)) \le \sum_{z\in T_N} H(\ell_t^N(z)).$$

On the finite set T_N, the above calculations become rigorous, so that we get by Theorem 2 and Varadhan's lemma

$$\langle u(0,t)\rangle \le \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in T_N} H(\ell_t^N(z))\Big\}\Big] = \exp\big\{H(t) - 2dt\chi^N(\rho) + o(t)\big\},$$

where χ^N is defined as

$$\chi^N(\rho) = \frac{1}{2d} \inf_{\nu\in\mathcal{M}^1(T_N)} \{I^N(\nu) + \rho J^N(\nu)\}$$

where I^N and J^N are defined as before, but they are now restricted to the box T_N with periodic boundary conditions. See [GM98] for a proof of the fact that $\chi^N \to \chi$ as $N \to \infty$.

For the *lower bound*, we kill the random walk when it hits ∂T_N. This means that we have to include the indicator of the event that $\ell_t(z) = 0$ for all $z \notin T_N \setminus \partial T_N$ under the expectation. Thus, we get the lower bound,

$$\langle u(t,0)\rangle \geq \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in T_N} H(\ell_t^N(z))\Big\}\mathbf{1}\{\ell_t^N(z) = 0\ \forall z \in \partial T_N\}\Big].$$

As before, we can now apply our large deviation results and obtain for the corresponding χ

$$\widehat{\chi}^N(\rho) = \frac{1}{2d} \inf_{\substack{\nu\in\mathcal{M}^1(T_N)\\ \nu(z)=0\ \forall z\in\partial T_N}} \{I^N(\nu) + \rho J^N(\nu)\}.$$

Generalization to $p \geq 2$. Consider p independent copies of our simple random walk $\{X_t^i : t \geq 0\}$, $i = 1,\dots,p$ with corresponding local times $\{\ell_t^i\}$. Then by Lemma 1, we find

$$\begin{aligned}\langle u(t,0)^p\rangle &= \Big\langle \mathbb{E}_0^{\otimes p}\Big[\prod_{i=1}^p \exp\Big\{\int_0^t \xi(X_s^i)ds\Big\}\Big]\Big\rangle \\ &= \mathbb{E}_0^{\otimes p}\Big[\Big\langle \exp\Big\{\sum_{z\in\mathbb{Z}^d} \xi(z)\sum_{i=1}^p \ell_t^i(z)\Big\}\Big\rangle\Big] \\ &= \mathbb{E}_0^{\otimes p}\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} H\Big(\sum_{i=1}^p \ell_t^i(z)\Big)\Big\}\Big].\end{aligned}$$

Following the same argument as for $p = 1$ and using that $\sum_{z\in\mathbb{Z}^d}\sum_{i=1}^p \ell_t^i(z) = pt$, we obtain

$$\langle u^p(t,0)\rangle = \exp\{H(pt) - 2dpt\chi_p(\rho) + o(t)\},$$

with

$$\chi_p(\rho) = \frac{1}{2d} \inf_{\nu^1,\dots,\nu^p\in\mathcal{M}_1(\mathbb{Z}^d)} \Big\{\frac{1}{p}\sum_{i=1}^p I(\nu^i) + \rho J\Big(\frac{1}{p}\sum_{i=1}^p \nu^i\Big)\Big\}.$$

Here, $\frac{1}{p}\sum_{i=1}^p I(\nu^i)$ is the weak rate function in the large deviations principle for $(L_t^1,\dots,L_t^p)$. Since $\nu \mapsto J(\nu)$ is strictly concave, the infimum reduces to the diagonal $\nu^1 = \dots = \nu^p$, so that $\chi_p(\rho) = \chi(\rho)$, which completes the proof of Theorem 3.

Appendix: Large deviations for continuous-time Markov chains

In this section, we will give a very rough sketch of how to obtain Theorem 2 from the corresponding theorem for discrete-time Markov chains. For more details see [dH00, Section IV].

Before we move to the continuous-time setting, let us recall the result for discrete-time Markov chains and reformulate the expression for the rate function. Let $\mathcal{X}$ be a finite set and suppose $Z_1, Z_2, \ldots$ is a Markov chain on $\mathcal{X}$ with strictly positive transition matrix P. Then, we know that the empirical measures are defined as

$$L_n^Z(y) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{Z_i = y\}\,, \quad \text{for } y \in \mathcal{X}\,.$$

Applying the contraction principle to the result about the LDP for the empirical pair measures, we obtain that the empirical measures L_t^Z satisfy a large deviation principle on $\mathcal{M}_1(\mathcal{X})$ with speed n and good rate function given by

$$I_P(\nu) = \inf_{\mu \in \mathcal{M}_1(\mathcal{X}\times\mathcal{X}) :\, \nu_1 = \mu} I_P^2(\mu)\,,$$

where I_P^2 is the rate function for the empirical pair measures, i.e.

$$I_P^2(\mu) = \begin{cases} H(\mu \| \mu_1 \otimes P) & \text{if } \mu_1 = \mu_2\,, \\ \infty & \text{otherwise.} \end{cases}$$

We can express I_P in a slightly different form, which will be useful for the continuous-time setting.

Lemma 5. *We can write*

$$I_P(\nu) = \sup_{u>0} \Big[-\sum_{x\in\mathcal{X}} \nu(x) \log\Big(\frac{(Pu)_x}{u_x}\Big)\Big]\,, \tag{4}$$

where the supremum runs over all $u : \mathcal{X} \to (0,\infty)$ *and* $(Pu)_x = \sum_y P_{xy} u_y$.

Proof. Fix $\nu \in \mathcal{M}_1(\mathcal{X})$. Let

$$f_\nu(u) = -\sum_x \nu(x) \log\Big(\frac{(Pu)_x}{u_x}\Big)\,.$$

Assume that $\nu > 0$ (otherwise restrict all the sums to the support of ν). Then, one can check that the supremum in (4) is attained for u^* satisfying

$$0 = -\sum_x \nu(x) \Big(\frac{P_{xy}}{(Pu^*)_x} - \frac{\delta_{xy}}{u_y^*}\Big) \quad \forall y\,.$$

Rearranging yields that

$$\nu(y) = \sum_x \nu_x Q^{u^*}_{xy} \quad \text{where} \quad Q^{u^*}_{xy} = \frac{P_{xy} u^*_y}{(Pu^*)_x} > 0\,,$$

so that ν is the stationary distribution of the stochastic matrix Q^{u^*}. Define, $\mu^* \in \mathcal{M}_1(\mathcal{X} \times \mathcal{X})$ by

$$\mu^*(x,y) = \nu(x) Q^{u^*}_{xy}\,.$$

Then for $\mu \in \mathcal{M}_1(\mathcal{X} \times \mathcal{X})$ with $\mu_1 = \mu_2 = \nu$, we can calculate using that $\mu^*_1 = \mu^*_2 = \nu$

$$\begin{aligned} I^2_P(\mu) = H(\mu \| \nu \otimes P) &= H(\mu\|\mu^*) + \sum_{x,y} \log \frac{\mu^*(x,y)}{\nu(x) P_{xy}} \\ &= H(\mu\|\mu^*) + \sum_{x,y} \mu(x,y) \log \frac{u^*_y}{(Pu^*)} = H(\mu\|\mu^*) + f_\nu(u^*)\,. \end{aligned}$$

The infimum of the last term over the set $\{\mu \in \mathcal{M}_1(\mathcal{X} \times \mathcal{X}) \,:\, \mu_1 = \nu\}$ is zero, which completes the proof. □

Now, we move to the continuous time setting, so let $(X_t \,:\, t \geq 0)$ be a continuous-time Markov chain with an irreducible generator $G = (G_{xy})_{x,y \in \mathcal{X}}$. First, we will indicate how the general rate function can be derived and finally we see how it simplifies when G is symmetric. Again, denote by L_t the empirical measures.

Theorem 6. *(a) Under the above assumptions, the empirical measures L_t satisfy a large deviation principle on $\mathcal{M}_1(\mathcal{X})$ with speed t and rate function*

$$I_G(\nu) = \sup_{u>0} \Big\{ - \sum_{x \in \mathcal{X}} \nu(x) \frac{(Gu)_x}{u_x} \Big\}\,,$$

where the supremum runs over all $u : \mathcal{X} \to (0,\infty)$.

(b) If, in addition, G is symmetric, then the rate function is given by

$$I_G(\nu) = - \sum_{x,y \in \mathcal{X}} \sqrt{\nu(x)} G_{xy} \sqrt{\nu(y)} = \big\langle \sqrt{\nu}, (-G)\sqrt{\nu} \big\rangle\,.$$

Proof. (a) We will only sketch the proof, which is based on a discrete approximation of the Markov chain. Fix $\delta > 0$ and let

$$L^\delta_t = \lfloor t/\delta \rfloor^{-1} \sum_{k=1}^{\lfloor t/\delta \rfloor} \delta_{X_{k\delta}}$$

be the empirical measure after $\lfloor t/\delta \rfloor$ steps of the $\mathcal{X}$-valued discrete-time Markov chain with strictly positive transition matrix $P^\delta = e^{\delta G}$. Thus, by Lemma 5 we find that L_t^δ satisfies a large deviation principle with speed t and rate function

$$\tfrac{1}{\delta} I_{P^\delta}(\nu) = \tfrac{1}{\delta} \sup_{u>0} \Big[-\sum_{x\in\mathcal{X}} \nu(x) \log \Big(\frac{(P^\delta u)_x}{u_x} \Big) \Big] .$$

Then, we have to show that L_t^δ approximates L_t sufficiently well in the limit $\delta \to 0$, and also that

$$\lim_{\delta} \tfrac{1}{\delta} I_{P^\delta}(\nu) = I_G(\nu) .$$

To get a rough idea we will show that $\frac{1}{\delta} I_{P^\delta}(\nu) \le I_G(\nu)$. Indeed, using that $\frac{d}{d\delta} P^\delta = G P^\delta$, we obtain that

$$\begin{aligned}
\tfrac{1}{\delta} I_{P^\delta} &= \sup_{u>0} \Big[-\sum_{x\in\mathcal{X}} \nu(x) \tfrac{1}{\delta} \int_0^\delta d\varepsilon \Big(\frac{(GP^\varepsilon u)_x}{(P^\varepsilon u)_x} \Big) \Big] \\
&\le \tfrac{1}{\delta} \int_0^\delta d\varepsilon \sup_{u>0} \Big[-\sum_{x\in\mathcal{X}} \nu(x) \Big(\frac{(GP^\varepsilon u)_x}{(P^\varepsilon u)_x} \Big) \Big] \\
&\le \tfrac{1}{\delta} \int_0^\delta d\varepsilon I_G(\nu) = I_G(\nu) .
\end{aligned}$$

For more details, see [dH00, Thm. IV.14].

(b) Write the rate function I_G as

$$I_G(\nu) = -\sum_{x,y} \sqrt{\nu(x)} G_{xy} \sqrt{\nu(y)} - \inf_{u>0} \sum_{x,y} G_{xy} \Big(\nu(x) \frac{u_y}{u_x} - \sqrt{\nu(x)}\sqrt{\nu(y)} \Big) .$$

If G is symmetric, then the last term can be rewritten as

$$-\inf_{u>0} \frac{1}{2} \sum_{x\neq y} G_{xy} \left(\sqrt{\frac{\nu(x) u_y}{u_i}} - \sqrt{\frac{\nu(y) u_x}{u_y}} \right)^2 ,$$

which is zero, since the generator satisfies $G_{xy} \ge 0$ for $x \neq y$. $\square$

References

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