Ageing in the parabolic Anderson model

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Ageing is a phenomenon that can be observed in many physical systems that are out of equilibrium. Vaguely speaking, if starting at time t we observe for how long the system stays in the same state, then if the answer depends on t, the system ages. Mathematical treatments have focused mainly on the dynamical properties of spin glasses and related trap models, see [1] for an overview. Also in [2], the occurrence of ageing is investigated for different models of interacting diffusion processes. In our project we show that ageing occurs in the parabolic Anderson model.

The parabolic Anderson model is given by the heat equation on the lattice $\mathbb{Z}^d$ with a random potential, i.e. we consider the solution $u : [0,\infty) \times \mathbb{Z}^d \to [0,\infty)$ of the Cauchy problem

$$\begin{aligned} \tfrac{\partial}{\partial t} u(t,z) &= \Delta u(t,z) + \xi(z) u(t,z)\,, && (t,z) \in (0,\infty) \times \mathbb{Z}^d\,, \\ u(0,z) &= \mathbf{1}_0(z)\,, && z \in \mathbb{Z}^d\,. \end{aligned}$$

Here Δ is the discrete Laplacian

$$\Delta f(x) = \sum_{y \in \mathbb{Z}^d : y \sim x} (f(y) - f(x))\,,$$

where $y \sim x$ means that y is a nearest-neighbour of site x. The potential $\xi = (\xi(z) : z \in \mathbb{Z}^d)$ is a collection of independent, identically distributed random variables, which we assume to be Pareto-distributed for some $\alpha > d$, i.e.

$$\mathrm{Prob}\{\xi(z) \le x\} = 1 - x^{-\alpha}\,, \quad \text{for } x \ge 1\,.$$

As our main result we show that the ageing phenomenon occurs in this model in the following form.

Theorem 1. *Let $0 < \varepsilon < \frac{1}{2}$. For any $c > 0$,*

$$\lim_{t \to \infty} \mathrm{Prob}\Big\{ \sup_{z \in \mathbb{Z}^d} \sup_{t \le s \le t(1+c)} \Big| \frac{u(t,z)}{\sum_{x \in \mathbb{Z}^d} u(t,x)} - \frac{u(s,z)}{\sum_{x \in \mathbb{Z}^d} u(s,x)} \Big| < \varepsilon \Big\} = I(c)\,,$$

for a constant $I(c) \in (0,1)$ given explicitly by

$$I(c) = \frac{1}{B(\alpha - d + 1, d)} \int_0^1 v^{\alpha - d} (1-v)^{d-1} \phi_c(v) \mathrm{d}v\,,$$

where the weight $\phi_c(v)$ is defined as

$$\phi_c(v) = \Big(1 - \tilde{B}(v, \alpha - d, d) + (1+c)^{\alpha} (\frac{c}{v} + 1)^{d-\alpha} \tilde{B}\Big(\frac{v+c}{1+c}, \alpha - d, d \Big) \Big)^{-1}\,.$$

Here B is the Beta function and $\tilde{B}$ the incomplete Beta function

$$\tilde{B}(x,a,b) = \frac{1}{B(a,b)} \int_0^x v^{a-1} (1-v)^{b-1} \mathrm{d}t\,.$$

Note that as one would expect, if $c \to \infty$, then $I(c) \to 0$ and if $c \to 0$, then $I(c) \to 1$.

Our proofs rely on the techniques developed in [4], where the authors show that at most times the solution u is localized in one point. After a certain time another point becomes more attractive and the solution relocalizes. Now, if $Z_t = \text{argmax}\{u(t,z) : z \in \mathbb{Z}^d\}$ denotes the point where most of the mass is localized, then the authors show that as $t \to \infty$

$$\frac{u(t,Z_t)}{U(t)} \to 1 \quad \text{in probability, and} \quad \frac{u(t,Z_t) + u(t,Z_t^{(2)})}{U(t)} \to 1 \quad \text{almost surely},$$

where $Z_t^{(2)}$ is an auxiliary process, which is only important for the relative short time intervals when u relocalizes from one point to another. Using the point process technique developed in [3], we can show that for any $c > 0$,

$$\text{Prob}\{Z_t = Z_{t(1+c)}\} \to I(c) \quad \text{as } t \to \infty,$$

from which we can then deduce Theorem (1).

If we define the "remaining lifetime" function R as

$$R(t) = \sup\{s \geq 0 : Z_t = Z_{t+s}\}.$$

then we have shown that $R(t)/t$ converges weakly to a random variable with law given by $1 - I$. In addition to these results about the weak convergence, we can also describe the almost sure asymptotics of the function R. Let $(\tau_n)_{n \geq 0}$ denote the jump times of the process $(Z_t)_{t \geq t_0}$, where t_0 can be chosen such that Z_t never returns to the same point in $\mathbb{Z}^d$. Then the function $(R(t), t \geq t_0)$ is of a very simple structure, see also Figure 1. At time τ_n, R jumps from zero to $\tau_{n+1} - \tau_n$, then it decreases linearly with slope -1 during the interval $[\tau_n, \tau_{n+1})$. Therefore, in order to describe the almost sure asymptotics of R it suffices to describe the asymptotics of $\{\tau_n\}$.

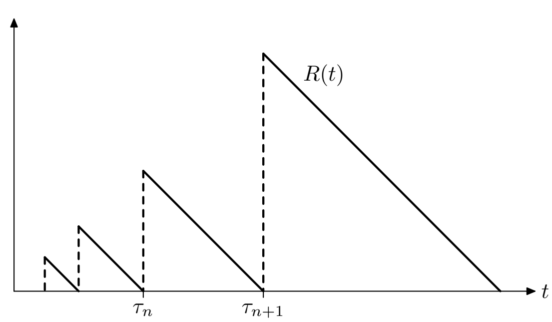

FIGURE 1. A schematic sketch of the remaining lifetime function R.

Theorem 2. *For all n sufficiently large, for $\beta, \beta' \gg 1$,*

$$(\log \tau_n)^{-\beta} \leq \frac{R(\tau_n)}{\tau_n} \leq (\log \tau_n)^{\beta'}.$$

As it is known that one sees qualitatively different localization behaviour depending on the tail of the distribution of the potential, it is a natural question to ask at what point the ageing behaviour in the form of Theorem 1 changes qualitatively as the tails of the potential get lighter. Since our proofs rely on the precise estimates developed in [4] that only work for polynomial tails, further research that might reveal a more complete picture requires the development of more general techniques.

References

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