Ageing in the parabolic Anderson model

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Ageing: an example

The most classical example of a system that exhibits ageing are the trap models introduced by Bouchaud in 1992.

- Consider the complete graph G_n on vertices $\{1, \dots, n\}$.
- The traps are given by a collection $\tau = \{\tau_i : i = 1, \dots\}$ of independent random variables with polynomial tails.
- Let $X(t)$ be the continuous-time random walk that is started uniformly on G_n and if in state i it jumps to one of its neighours with rate τ_i^{-1}.

Theorem 1 (Bouchaud,Dean 1995)

There exists a constant $I(c)$ such that for almost every environment τ,

$$\lim_{t\to\infty} \lim_{n\to\infty} \mathrm{Prob}\{X_t^{(n)} = X_{t(1+c)}^{(n)} | \tau\} = I(c)$$

The constant $I(c)$ is given explicit by an arcsine law. This applies also to other trap models [Ben Arous, Černý]. These models were introduced as a toy model of the dynamics of spin glasses. In some simple case this can be made rigorous [Ben Arous, Bovier, Černý].

The parabolic Anderson model

The parabolic Anderson model is given by the *heat equation* on the lattice $\mathbb{Z}^d$ with a *random potential*, i.e. we consider the solution $u : [0,\infty) \times \mathbb{Z}^d \to [0,\infty)$ of the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t,z) &= \Delta\Delta u(t,z) + \xi(z)\xi(z)u(t,z)\,, \\ u(0,z) &= \mathbf{1}_0(z)\,, \end{aligned}$$

Here Δ is the discrete Laplacian

$$\Delta f(x) = \sum_{y \in \mathbb{Z}^d : y \sim x} (f(y) - f(x))\,,$$

where $y \sim x$ means that y is a nearest-neighbour of site x. The potential $\xi = (\xi(z) : z \in \mathbb{Z}^d)$ is a collection of independent, identically distributed random variables. At time 0, all the mass of u is localized at the origin. We assume that the potential is *Pareto-distributed* for some $\alpha > d$, i.e.

$$\mathrm{Prob}\{\xi(z) \le x\} = 1 - x^{-\alpha}\,, \quad \text{for } x \ge 1\,.$$

Ageing in the parabolic Anderson model

Theorem 2 ([Mörters,O.,Sidorova 09])

Let $0 < \varepsilon < \frac{1}{2}$. *For any* $c > 0$,

$$\lim_{t\to\infty} \mathrm{Prob}\Big\{ \sup_{z\in\mathbb{Z}^d} \sup_{t\le s\le t(1+c)} \Big| \frac{u(t,z)}{\sum_{x\in\mathbb{Z}^d} u(t,x)} - \frac{u(s,z)}{\sum_{x\in\mathbb{Z}^d} u(s,x)} \Big| < \varepsilon \Big\} = I(c)\,,$$

for a constant $I(c) \in (0,1)$ *given explicitly by*

$$I(c) = \frac{1}{B(\alpha-d+1,d)} \int_0^1 v^{\alpha-d}(1-v)^{d-1}\varphi_c(v)\mathrm{d}v\,,$$

where the weight $\varphi_c(v)$ *is defined as*

$$\varphi_c(v) = \Big(1 - \tilde{B}(v,\alpha-d,d) + (1+c)^\alpha(\frac{c}{v}+1)^{d-\alpha}\tilde{B}\Big(\frac{v+c}{1+c},\alpha-d,d\Big)\Big)^{-1}.$$

Here B is the Beta function and $\tilde{B}$ the incomplete Beta function

$$\tilde{B}(x,a,b) = \frac{1}{B(a,b)}\int_0^x v^{a-1}(1-v)^{b-1}\mathrm{d}t\,.$$

The two cities theorem

For Pareto-distributed potentials, [König, Lacoin, Mörters, Sidorova'09] show that almost surely for all large t, the total mass $\sum_{z\in\mathbb{Z}^d} u(t,z)$ is asymptotically localized in (at most) two time-dependent points.

Theorem 3 (KLMS'09)

There exist two processes $X_t^{(1)}$ and $X_t^{(2)}$ taking values in $\mathbb{Z}^d$ such that as $t\to\infty$

$$\frac{u(t,X_t^{(1)})}{\sum_{z\in\mathbb{Z}^d} u(t,z)} \to 1 \quad \textit{in probability, and} \quad \frac{u(t,X_t^{(1)})+u(t,X_t^{(2)})}{\sum_{z\in\mathbb{Z}^d} u(t,z)} \to 1 \quad \textit{a.s.},$$

Can define the (right-continuous) processes $X_t^{(i)}$ as

$$X_t^{(1)} = \operatorname{argmax}\{u(t,z) \,:\, z\in\mathbb{Z}^d\}$$
$$X_t^{(2)} = \operatorname{argmax}\{u(t,z) \,:\, z\in\mathbb{Z}^d\setminus\{X_t^{(1)}\}\}.$$

The process $X_t^{(2)}$ is only important for the relative short times, when the total mass relocates from one point to another, so there is a certain freedom in choosing it.

The Feynman-Kac representation

By the Feynman-Kac representation,

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(Y_s)ds\Big\}\mathbf{1}_{\{Y_t=z\}}\Big],$$

where $(Y_s, s \geq 0)$ (under $\mathbb{P}_0$) is a continuous-time simple random walk on $\mathbb{Z}^d$ with generator Δ starting at 0. Therefore, the total mass can be written as

$$\sum_{z\in\mathbb{Z}^d} u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(Y_s)ds\Big\}\Big],$$

Heuristically, for fixed $t > 0$, the paths $(Y_s : 0 \leq s \leq t)$ that have the biggest influence are those that spend most of their time at a site z

- that has a large potential $\xi(z)$ and
- which is not too far away from the origin.

The variational problem

[KLMS'09] show that the total mass can be well-approximated by a variational problem. Let $A_t^{z,\rho}$, $\rho \in (0,1)$ be the strategy for the random walk Y to wander to z during $[0, \rho t)$ and to stay there until time t. Then, for $|z| \gg t$

$$\mathbb{P}_0(A_t^{z,\rho}) \approx \exp\{-|z| \log \frac{|z|}{e\rho t} + \eta(z)\},$$

where $|z|$ is the ℓ^1-norm on $\mathbb{R}^d$and

$$\eta(z) = \log \#\{\text{paths of length } |z| \text{ starting at 0 and ending at } z\}.$$

Hence, we can find a lower bound on the total mass of u

$$\frac{1}{t} \log \sum_{x \in \mathbb{Z}^d} u(t,x) = \frac{1}{t} \log \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(Y_s)ds\Big\}\Big]$$

$$\gtrapprox \sup_{z \in \mathbb{Z}^d} \sup_{\rho \in (0,1)} \Big[(1-\rho)\xi(z) - \frac{|z|}{t} \log \frac{|z|}{e\rho t} + \frac{\eta(z)}{t}\Big] = \max_{z \in \mathbb{Z}^d} \Phi_t(z),$$

where

$$\Phi_t(z) = \Big[\xi(z) - \frac{|z|}{t} \log \xi(z) + \frac{\eta(z)}{t}\Big].$$

So it is plausible that u takes the largest values at the first and second maximizers of Φ.

Ageing for the maximizer process

Since u is localized in $Z_t^{(1)}$ (at most times), ageing for the solution u should be detected also for $Z_t^{(1)}$.

Proposition 4

For any $c > 0$, as $t \to \infty$,

$$\mathrm{Prob}\{Z_t^{(1)} = Z_{t(1+c)}^{(1)}\} \to I(c)\,.$$

The proof uses Poisson point processes. Namely, define scaling functions

$$r_t = \left(\frac{t}{\log t}\right)^{\frac{\alpha}{\alpha-d}} \quad \text{and} \quad a_t = \left(\frac{t}{\log t}\right)^{\frac{d}{\alpha-d}}\,,$$

Then, in [v.d.Hofstad,Mörters,Sidorova '08] it is shown (ignoring technicalities) that $\Pi_t = \sum_{z\in\mathbb{Z}^d} \delta_{\left(\frac{z}{r_t},\frac{\Phi_t(z)}{a_t}\right)}$ converges to a Poisson point process Π with intensity measure

$$\nu(\mathrm{d}x,\mathrm{d}y) = \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y + \frac{d}{\alpha-d}|x|)^{\alpha+1}}\,.$$

Proof (1)

We want to compute

$$\lim_{t\to\infty} \mathrm{Prob}\{Z_t^{(1)} = Z_{t(1+c)}^{(1)}\}$$

$$= \lim_{t\to\infty} \int_{x\in\mathbb{R}^d} \int_{y\geq 0} \mathrm{Prob}\Big\{ \frac{Z_t^{(1)}}{r_t} \in \mathrm{d}x,\ \frac{\Phi_t(Z_t^{(1)})}{a_t} \in \mathrm{d}y,\ Z_t^{(1)} = Z_{t(1+c)}^{(1)} \Big\}.$$

The conditions $\frac{Z_t^{(1)}}{r_t} \in \mathrm{d}x$, $\frac{\Phi_t(Z_t^{(1)})}{a_t} \in \mathrm{d}y$ mean that $\Pi_t = \sum_{z\in\mathbb{Z}^d} \delta_{(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t})}$ should have one point in (x, y) and no points in $\mathbb{R}^d \times (y, \infty)$. Note that for t large

$$\frac{\Phi_t(z)}{a_t} \approx \frac{\xi(z)}{a_t} - \frac{d}{\alpha - d} \frac{|z|}{r_t}.$$

Now, $Z_t^{(1)} = Z_{t(1+c)}^{(1)}$ means that for all $z \in \mathbb{Z}^d$

$$\Phi_{t(1+c)}(z) \leq \Phi_{t(1+c)}(Z_t^{(1)}). \qquad (1)$$

But we know that

$$\frac{\Phi_{t(1+c)}(z)}{a_t} \approx \frac{\xi(z)}{a_t} - \frac{d}{\alpha - d} \frac{|z|}{r_t(1+c)} = \frac{\Phi_t(z)}{a_t} + \frac{d}{\alpha - d} \frac{c}{1+c} \frac{|z|}{r_t}.$$

Proof (2)

Hence, (1) is equivalent to the condition that all the points $(\bar{x}, \bar{y})$ of Π satisfy

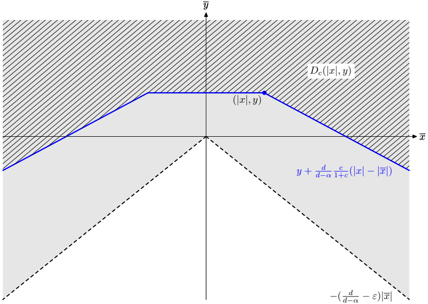
$$\bar{y} + \frac{d}{\alpha - d}\,\frac{c}{1+c}|\bar{x}| \le y + \frac{d}{\alpha - d}\,\frac{c}{1+c}|x|\,.$$

The requirement that $Z_t = Z_{t(1+c)}$ is equivalent to the condition to all the points $(\bar{x}, \bar{y})$ of Π_t satisfy

$$\bar{y} + \frac{d}{\alpha - d}\,\frac{c}{1+c}|\bar{x}| \le y + \frac{d}{\alpha - d}\,\frac{c}{1+c}|x|\,.$$

Hence, let $D_c(|x|, y)$ be the set of all points $(\bar{x}, \bar{y})$ such that either $\bar{y} > y$ or

$$|\bar{x}| > |x| \quad \text{and} \quad \bar{y} > y - \frac{d}{\alpha - d}\,\frac{c}{1+c}(|\bar{x}| - |x|)\,.$$



Proof (3)

Then,

$$\lim_{t\to\infty} \mathrm{Prob}\Big\{ \frac{Z_t^{(1)}}{r_t} \in \mathrm{d}x,\ \frac{\Phi_t(Z_t^{(1)})}{a_t} \in \mathrm{d}y,\ Z_t^{(1)} = Z_{t(1+c)}^{(1)} \Big\}$$
$$= \mathrm{Prob}\{\Pi(\mathrm{d}x \times \mathrm{d}y) = 1, \Pi(D_c(|x|, y)) = 0\}$$
$$= e^{-\nu(D_c(|x|,y))}\nu(\mathrm{d}x, \mathrm{d}y)\,.$$

Integrating, over $x \in \mathbb{R}^d$ and $y \geq 0$,

$$\lim_{t\to\infty} \mathrm{Prob}\{Z_t^{(1)} = Z_{t(1+c)}^{(1)}\} = \int_{x\in\mathbb{R}^d} \int_{y\geq 0} e^{-\nu(D_c(|x|,y))}\nu(\mathrm{d}x, \mathrm{d}y) = I(c)\,,$$

where the last equality follows by a simplification of the integral. □

Ageing for u

In order to prove Theorem 1, for fixed $0 < \varepsilon < \frac{1}{2}$ we have to show that

$$\lim_{t\to\infty} \mathrm{Prob}\Big\{ \sup_{z\in\mathbb{Z}^d} \sup_{t\le s\le t(1+c)} \Big| \frac{u(t,z)}{\sum_{x\in\mathbb{Z}^d} u(t,x)} - \frac{u(s,z)}{\sum_{x\in\mathbb{Z}^d} u(s,x)} \Big| < \varepsilon \Big\}$$
$$= \lim_{t\to\infty} \mathrm{Prob}\big\{ Z_t^{(1)} = Z_{t(1+c)}^{(1)} \big\},$$

Idea of proof: for most times, the total mass $\sum_{z\in\mathbb{Z}^d} u(t,z)$ is localised in only $Z_t^{(1)}$. Moreover, one can show that $Z_t^{(1)}$ does not return to the same point, so that

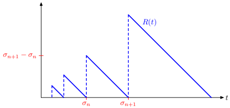
$$Z_t^{(1)} = Z_{t(1+c)}^{(1)} \iff Z_s^{(1)} = Z_t^{(1)} \quad \forall s \in [t, t(1+c)].$$

Finally, one only has to take care of the (relatively short) transition times when $Z_t^{(1)}$ relocates.

Almost sure ageing

Recall that $X_t^{(1)} = \mathrm{argmax}\{u(t,z) : z \in \mathbb{Z}^d\}$. Then, define the residual lifetime function as

$$R(t) = \sup\left\{s \geq 0 : X_t^{(1)} = X_{t+s}^{(1)}\right\}.$$



Clearly, as $t \to \infty$,

$$\mathrm{Prob}\left\{\frac{R(t)}{t} \geq c\right\} = \mathrm{Prob}\{X_t^{(1)} = X_{t(1+c)}^{(1)}\} \to I(c).$$

Hence $R(t)/t$ converges weakly to a random variable with law given by $1 - I$. What about almost sure asymptotics?

Theorem 5 (Mörters,O.,Sidorova 09)

For any nondecreasing function $h : (0,\infty) \to (0,\infty)$ *we have, almost surely,*

$$\limsup_{t\to\infty} \frac{R(t)}{th(t)} = \begin{cases} 0 & \text{if } \int_1^\infty \frac{\mathrm{d}t}{th(t)^d} < \infty \\ \infty & \text{if } \int_1^\infty \frac{\mathrm{d}t}{th(t)^d} = \infty \end{cases}$$

Proof of the almost sure result [upper bound]

- Let h be such that $\int_{t>1} \frac{\mathrm{d}t}{th(t)^d} < \infty$.
- For $\varepsilon > 0$,

$$\mathrm{Prob}\Big\{ \frac{R(e^n)}{e^n h(e^n)} \geq \varepsilon \Big\} = \mathrm{Prob}\{ Z_{e^n} = Z_{e^n(1+\varepsilon h(e^n))} \} \sim C\varepsilon^{-d} h(e^n)^{-d}$$

- Recall for c large

$$\lim_{t\to\infty} \mathrm{Prob}\{ X_t = X_{t(1+c)} \} = I(c) \sim c^{-d} \, .$$

  Replace by a moderate deviations result, for h increasing not too quickly,

$$\lim_{t\to\infty} h(t)^d \mathrm{Prob}\{ X_t = X_{t(1+h(t))} \} = C \, ,$$

  for some explicit constant C.
- But, by the integral condition, we find that $\sum_{n\geq 1} h(e^n)^{-d} < \infty$, so that by Borel-Cantelli,

$$\limsup_{n\to\infty} \frac{R(e^n)}{e^n h(e^n)} \leq \varepsilon \, .$$

- Finally, use "linear" structure of the function $t \mapsto R(t)$ to show that the same holds for any sequence $t_n \to \infty$.

Proof of almost sure result [sharpness of upper bound]

- Let $h(t)$ be such that $\int_{t>1} \frac{\mathrm{d}t}{th(t)^d} = \infty$. Define the event, for $\kappa > 0$,

$$E_n = \left\{ \frac{R(e^n)}{e^n h(e^n)} \geq \kappa \right\}$$

Then, $\mathrm{Prob}(E_n) \sim C\kappa^{-d} h(e^n)^{-d}$, so

$$\sum_{n \geq 1} \mathrm{Prob}(E_n) = \infty \,.$$

- Need to replace Borel-Cantelli by Kochen-Stone lemma:

$$\mathrm{Prob}\{E_n \text{ i.o. }\} \geq \limsup_{k \to \infty} \frac{\left(\sum_{n=1}^k \mathrm{Prob}(E_n) \right)^2}{\sum_{m=1}^k \sum_{n=1}^k \mathrm{Prob}(E_n \cap E_m)} \,.$$

Therefore, have to control correlations and show

$$\mathrm{Prob}(E_n \cap E_m) \sim C^2 \kappa^{-2d} h(e^n)^{-d} h(e^m)^{-d}$$

It follows that

$$\limsup_{t \to \infty} \frac{R(t)}{th(t)} \geq \kappa \,.$$

A functional scaling limit theorem

Ageing only looks at temporal structure, but what can be said about the spatial limit of the maximizer process?

[van der Hofstad, Mörters, Sidorova 2009] show that

$$\Big(\frac{\log t}{t}\Big)^{\frac{d}{\alpha-d}} \frac{\log \sum_{z\in\mathbb{Z}^d} u(t,z)}{t} \Rightarrow Y\,,$$

where Y is a random variable with density on $\mathbb{R}^+$.

Theorem 6 (Mörters, O., Sidorova 09)

As $T\to\infty$, we have the following

$$\Big(\Big(\big(\tfrac{\log T}{T}\big)^{\frac{\alpha}{\alpha-d}} X_{tT}, \big(\tfrac{\log T}{T}\big)^{\frac{d}{\alpha-d}} \frac{\log \sum_{x\in\mathbb{Z}^d} u(tT,x)}{tT}\Big) \,:\, t>0\Big) \\ \Rightarrow \Big(\big(Y_t^{(1)}, Y_t^{(2)} + \tfrac{d}{\alpha-d}\big(1-\tfrac{1}{t}\big)|Y_t^{(1)}|\big) \,:\, t>0\Big)\,,$$

in distribution on the space of càdlàg functions $f:(0,\infty)\to\mathbb{R}^d\times\mathbb{R}$ with respect to the Skorokhod topology on compact intervals.

Definition of the limit process Y

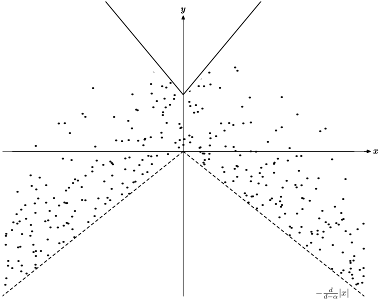
Let Π be a Poisson point process on $H = \{(x,y) : y > -\frac{d}{\alpha-d}|x|\}$ with intensity measure

$$\nu(\mathrm{d}x, \mathrm{d}y) = \frac{\alpha\,\mathrm{d}x\,\mathrm{d}y}{(y + \frac{d}{\alpha-d}|x|)^{\alpha+1}}\,.$$

Consider a cone with tip in $(0,z)$, $z>0$, given by all (x,y) such that

$$y \geq z - \frac{d}{\alpha-d}(1-\frac{1}{t})|x|\,.$$

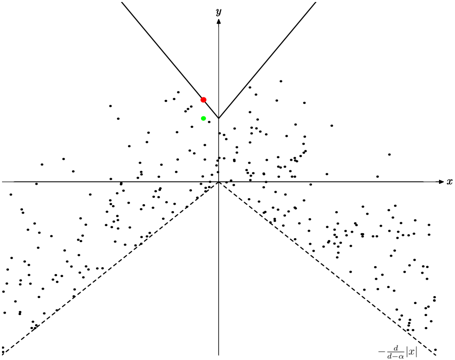
Let $Y_t = (Y_t^{(1)}, Y_t^{(2)})$ be the first point of Π that is hit by the cone as we decrease z (take the one with largest second component if unclear).



Interpretation of theorem

$$\Big(\big(\tfrac{\log T}{T}\big)^{\frac{\alpha}{\alpha-d}} X_{tT}, \big(\tfrac{\log T}{T}\big)^{\frac{d}{\alpha-d}} \frac{\log \sum_{x\in\mathbb{Z}^d} u(tT,x)}{tT} \,:\, t\in[\varepsilon,M] \Big)$$
$$\Rightarrow \big(Y_t^{(1)}, Y_t^{(2)} + \tfrac{d}{\alpha-d}\big(1-\tfrac{1}{t}\big)|Y_t^{(1)}| \,:\, t\in[\varepsilon,M]\big)\,,$$

Note the second component corresponds to the second component of the tip of the cone that defines Y_t.



Consequences of the construction

- $((Y_t^{(1)}, Y_t^{(2)}) : t > 0)$ is a (time-inhomogeneous) Markov process and so is the process $((Y_t^{(1)}, Y_t^{(2)} + \frac{d}{\alpha - d}(1 - \frac{1}{t})|Y_t^{(1)}|) : t > 0)$ is Markov.
- Projecting on the first component gives

$$\Big(\frac{X_{tT}}{r_T} \,:\, t \in [\varepsilon, M]\Big) \Rightarrow (Y_t^{(1)}\,,\, t \in [\varepsilon, M])\,,$$

  but the latter is not Markov.
- Similarly, the time-reversed version of $(Y_t^{(1)}, Y_t^{(2)})$ is Markov.
- Not surprisingly, convergence breaks down at $t = 0$ and limiting process jumps faster and faster as $t \downarrow 0$.

Proof of functional scaling limit result (1)

Aim: show for $a_t = (\frac{t}{\log t})^{\frac{d}{\alpha-d}}$, $r_t = (\frac{t}{\log t})^{\frac{\alpha}{\alpha-d}}$

$$\Big(\Big(\frac{Z_{tT}}{r_T}, \frac{\Phi_{tT}(Z_{tT})}{a_T}\Big) \,:\, t>0\Big) \Rightarrow ((Y_t^{(1)}, Y_t^{(2)} + \tfrac{d}{\alpha-d}(1-1/t)|Y_t^{(1)}|) \,:\, t>0)\,.$$

Want to express

$$\mathrm{Prob}\{\tfrac{Z_{tT}}{r_T} \in A; \tfrac{\Phi_{tT}(Z_{tT})}{a_T} \in B\}\,.$$

in terms of the Point process $\Pi_T = \{(z/r_T, \Phi_T(z)/a_T) : z \in \mathbb{Z}^d\}$.Fix $t>0$, then for large T we have

$$\frac{\Phi_{tT}(z)}{a_T} \approx \frac{\xi(z)}{a_T} - \tfrac{d}{\alpha-d}\frac{|z|}{tr_T} \approx \frac{\Phi_T(z)}{a_T} + \tfrac{d}{\alpha-d}\big(1-\tfrac{1}{t}\big)\frac{|z|}{r_T}\,.$$

Therefore, $(Z_{tT}/r_T, \Phi_T(Z_{tT})/a_T) = (x,y)$ if for all $z \in \mathbb{Z}^d$,

$$y + \tfrac{d}{\alpha-d}(1-\tfrac{1}{t})|x| \geq \frac{\Phi_T(z)}{a_T} + \tfrac{d}{\alpha-d}(1-\tfrac{1}{t})\frac{|z|}{r_T}\,.$$

Define the cone

$$\mathcal{C}_t(x,y) = \{(\overline{x},\overline{y}) : \overline{y} + \tfrac{d}{\alpha-d}(1-\tfrac{1}{t})|\overline{x}| > y + \tfrac{d}{\alpha-d}(1-\tfrac{1}{t})|x|\}$$

Proof of functional scaling limit result (2)

Then,

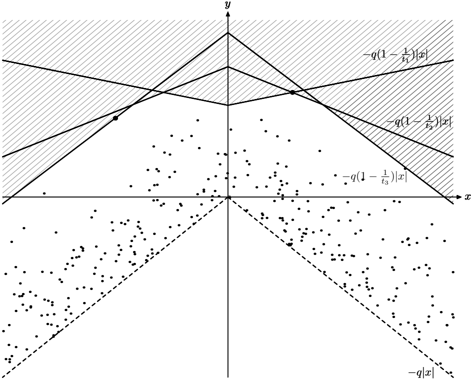
$$\mathrm{Prob}\{\tfrac{Z_{tT}}{r_T} \in A; \tfrac{\Phi_{tT}(Z_{tT})}{a_T} \in B\} \approx \iint\limits_{\substack{x\in A,\\ y+\frac{d}{\alpha-d}(1-\frac{1}{t})|x|\in B}} \mathrm{Prob}\left\{ \begin{array}{l} \Pi_T(\mathrm{d}(x,y)) = 1; \\ \Pi_T(\mathcal{C}_t(x,y)) = 0 \end{array} \right\}$$

$$\xrightarrow{T\to\infty} \iint\limits_{\substack{x\in A,\\ y+\frac{d}{\alpha-d}(1-\frac{1}{t})|x|\in B}} \mathrm{Prob}\left\{ \begin{array}{l} \Pi(\mathrm{d}(x,y)) = 1; \\ \Pi(\mathcal{C}_t(x,y)) = 0 \end{array} \right\}$$

$$= \mathrm{Prob}\{Y_t^{(1)} \in A, Y_t^{(2)} + \tfrac{d}{\alpha-d}|Y_t^{(1)}| \in B\}\,.$$

Proof of scaling limit theorem (2)

- To make it more rigorous, one has to look at finite-dimensional distributions



  and show tightness.
- To transfer to the setting of $X_t = \mathrm{argmax}\{u(t,x)\}$, one has to use that
  (a) X_t and Z_t jump at approximately the same time so that the differences cannot be seen in the scaling limit.
  (b) The value of the variational problem $\Phi_t(Z_t)$ approximates total mass $\frac{1}{t}\log\sum_{x\in\mathbb{Z}^d} u(t,x)$ well enough.

Outlook

What about potentials with less heavy-tailed distribution?

- If potentials are less heavy-tailed, "relevant islands" are no longer points (borderline: double exponential distributions). So need different techniques.
- What is the right formulation for ageing?
- [DEMBO, DEUSCHEL 07] study the correlations (for $(X_t(i) : i \in \mathbb{Z}^d)$ a system of diffusion processes given by some stochastic differential system) for some test functions f, g

$$\mathrm{corr}(f(X_t), g(X_{t+s}))\,.$$

  and deduce ageing from the behaviour for different scalings for $s, t \to \infty$.