Timetabling and Robustness
Computing Good and Delay-Resistant Timetables

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GK MDS, 24 Nov 2008
Overview

- The Periodic Event Scheduling Model (PESP)
  - Basic formulation and examples
- The influence of cycle bases on solving the PESP
  - Why short integral cycle bases help
  - On the complexity of computing them
- Robustness and Recoverability
  - A general paradigm for robustness
  - Computing robust timetables
  - Polyhedral aspects of recoverable robustness
Planning steps in railbound traffic

- Network Planning
- Line Planning
- Timetabling
- Vehicle Scheduling
- Crew Scheduling

PESP model
An informal problem formulation

- Lines are already determined
  - Line = sequence of stations
- Wanted: timetable
  - arrival and departure times
    at stations
  - and technically important points
- subject to temporal constraints of various kinds
  - timetable must be periodic
- Objectives to be minimized
  - total (weighted) changeover time
  - number of used vehicles/trains (rolling stock)
PESP: The core of the model

Consider geographic points $i, j$ along a line

\begin{align*}
\ell_{ij} & \leq \pi_j - \pi_i + p_{ij} \cdot T \leq u_{ij} \\
x_{ij} & \text{ integer notation } x_{ij} \in [\ell_{ij}, u_{ij}]_T
\end{align*}

[Serafini & Ukovich '89]
From the line plan to the graph model

Französische Str.

Stadtmitte

arrival
departure

driving activity
stopping activity
changeover activity
Example: a single line

stopping condition in $B$

\[ \pi_{\text{dep}} \in B - \pi_{\text{arr}} in B \in [2, 4]T \]

xx:59 arr in $B$

xx:02 dep in $B$

travel condition

\[ \pi_{\text{arr}} in A - \pi_{\text{dep}} in B \in [11, 11]T \]

xx:48 dep in $B$

xx:59 arr in $A$

turning condition

\[ \pi_{\text{arr}} in A - \pi_{\text{dep}} in A \in [5, 10]T \]

xx:02 dep in $A$

xx:10 arr in $A$
More complex conditions

- Single tracks used in both directions
- Safety distance between successive trains (headway)
- Coupling to other traffic systems
- Coordinated servicing of central transfer points
- ...

all can be modeled by a digraph with conditions

\[ x_{ij} \in [l_{ij}, u_{ij}]_T \] at the arcs \((i, j)\)

call \(x = (x_{ij})\) a feasible periodic tension
Natural MIP formulation

Linear objective \( \sum w_{ij} x_{ij} \)

- Sum of changeover times
- Time spent at turning points (rolling stock)
- Routing of rolling stock (line changes at endpoints)
- Penalties for weak side constraints

Have side constraints for every arc \( a = (i, j) \)

\[
\begin{align*}
x_a &= \pi_j - \pi_i \\
\ell_a &\leq x_a + p_a \cdot T \leq u_a \\
0 &\leq \pi_j \leq T - 1 \\
p_a &\text{ integer}
\end{align*}
\]

Integer LP in variables \( x_a \) and \( p_a \)
Complexity of the PESP

- Decision problem is NP-complete [Odijk ‘94]
- Generalizes MAXIMUM k-COLORABLE SUBGRAPH, so MAX-PESP is MAXSNP-complete [Liebchen ‘05]
- Natural Mixed Integer Programming (MIP) formulations are extremely hard to solve
- Small instances have entered MIPLIB
  - timetab1: 56 nodes, 226 arcs (after contraction)
  - timetab2: 88 nodes, 381 arcs (after contraction)
Long distance trains in Germany

- Extremely difficult for CPLEX without “enhancements”
- 10 pairs of ICE/IC trains, 40 most important connections
- Led to addition of 2 new instances of 2 new instances to MIPLIB
  - timetab1
  - timetab2

<table>
<thead>
<tr>
<th>Quantity</th>
<th>timetab1</th>
<th>timetab2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Original Digraph</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nodes</td>
<td>4604</td>
<td>5344</td>
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<tr>
<td>Arcs</td>
<td>5053</td>
<td>5859</td>
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<tr>
<td>Run/stop arcs</td>
<td>4582</td>
<td>5318</td>
</tr>
<tr>
<td>safety arcs</td>
<td>225</td>
<td>265</td>
</tr>
<tr>
<td><strong>Contracted Digraph</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nodes</td>
<td>56</td>
<td>88</td>
</tr>
<tr>
<td>Arcs</td>
<td>226</td>
<td>381</td>
</tr>
<tr>
<td>– with $d_{ij} = T - 1$</td>
<td>72</td>
<td>164</td>
</tr>
<tr>
<td>– with $d_{ij} \geq 0.9 \cdot T$</td>
<td>153</td>
<td>253</td>
</tr>
<tr>
<td>– with $d_{ij} \leq 0.1 \cdot T$</td>
<td>41</td>
<td>70</td>
</tr>
<tr>
<td>average span</td>
<td>77.76%</td>
<td>79.19%</td>
</tr>
</tbody>
</table>
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Characterizing periodic tension

[Serafini & Ukovich 89]

Explains the role of (undirected) cycles $C$ by looking at the sum of potential differences $\sum_C x_{ij}$ along $C$

$x$ is a periodic tension for period length $T$

$\Leftrightarrow \sum_C x_{ij}$ is an integer multiple of $T$ for every cycle $C$ of $G$

$\Leftrightarrow \sum_C x_{ij}$ is an integer multiple of $T$ for every fundamental cycle $C$ of $G$ w.r.t. to an arbitrary spanning tree of $G$
Proof that tree condition suffices

\[ \sum_{C} x_{ij} \] is integer multiple \( kT \) of \( T = 4 \) for every fundamental cycle of the blue tree

Use the tree to define a periodic potential

define potential \( \pi \) such that \( \pi_j - \pi_i = x_{ij} \) on tree arcs

\[ \pi_j - \pi_i \] sum up to 0 along cycles

\[ 0 = \pi_v - \pi_u + \sum_{p_+} x_{ij} - \sum_{p_-} x_{ij} \] \( \Rightarrow \) \( x \) is a periodic tension
Characterization is not true for arbitrary basis

Outer 3-cycles and inner 5-cycle form a basis

Matrix $\Gamma^T$ of incidence vectors is

$\begin{align*}
(1,2) & (2,3) & (3,4) & (4,5) & (5,1) & (3,1) & (4,2) & (5,3) & (1,4) & (2,5) \\
C_1 & 1 & 1 & 1 & 1 \\
C_2 & 1 & 1 & 1 & 1 \\
C_3 & 1 & 1 & 1 & 1 \\
C_4 & 1 & 1 & 1 & 1 \\
C_5 & 1 & 1 & 1 & 1 \\
C_6 & 1 & 1 & 1 & 1 & 1
\end{align*}$

$= \frac{1}{2} C_1 + \frac{1}{2} C_2 + \frac{1}{2} C_3 + \frac{1}{2} C_4 + \frac{1}{2} C_5 - \frac{1}{2} C_6$
Better MIP formulations based on cycle bases

- Along every cycle $C$, the period offsets $p_a$ must sum up to an integer multiple $q_C$ of the period length $T$
- Suffices to require this property for the cycles of an integral cycle basis of the underlying graph
- Gives a different IP formulation in $m-n+1$ cycle variables $q_C$

$$\Gamma^T \cdot x = q \cdot T$$

- The formulation is “good” if the cycle basis is “short”

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Integral cycle bases

- Integral basis of the cycle space of a graph
  - basis such that every cycle is obtained by a linear combination of basis cycles with integer coefficients
  - e.g. fundamental cycles of a spanning tree
  - e.g. interior faces of a planar graph

- Cycle matrix $\Gamma$ and determinant $\det(B)$ of a basis $B$
  - columns of $\Gamma$ are the incidence vectors of the cycles $C \in B$
  - $\det(B) := |\text{determinant of a maximal non-singular submatrix of } \Gamma|$ (this is invariant of the choice)

Cycle basis $B$ is integral $\iff |\det(B)| = 1$
Bounds on the cycle multiples $q_C$

- Bounds $\underline{q}_C \leq q_C \leq \overline{q}_C$ yield $q \in \prod_C \text{in basis}[\underline{q}_C, \overline{q}_C]$
  - size of search space $\leq \prod_C \text{in basis}(\overline{q}_C - \underline{q}_C + 1)$

- Cycle inequalities [Odijk 94]
  - $x$ is a **feasible** periodic potential with cycle multiples $q_C$
    $$q_C \leq \frac{1}{T} \left( \sum_{a \in C^+} u_a - \sum_{a \in C^-} \ell_a \right)$$
    for all cycles $C$
  - this gives lower an upper bounds on the $q_C$
    $$\left[ \frac{1}{T} \left( \sum_{a \in C^+} \ell_a - \sum_{a \in C^-} u_a \right) \right] \leq q_C \leq \left[ \frac{1}{T} \left( \sum_{a \in C^+} u_a - \sum_{a \in C^-} \ell_a \right) \right]$$

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Short cycle bases

- Lower and upper bounds from Odijk suggest
  - find an integer cycle basis \( \mathcal{B} \) that is short w.r.t.
    \[
    \sum_{C \in \mathcal{B}} \log(1 + \sum_{a \in C} (u_a - \ell_a))
    \]

- This resembles closely the shortest cycle basis problem:
  - find a basis with minimum length

- Length of basis \( \mathcal{B} \) w.r.t. arc weights \( w(a) \)
  - \( \sum_{C \in \mathcal{B}} w(C) \) with \( w(C) := \sum_{e \in C} w(a) \)
Numerical evidence for good cycle bases

Runtime and memory drop by a factor of $2$ to $10$
How to compute a suitable short bases

- Undirected graphs
  - $O(m^3n)$ [Horton ‘87]
  - $O(m^2n + mn^2\log n)$ [Kavitha, Mehlhorn, Michal & Paluch ‘04]

- Directed graphs
  - $\tilde{O}(m^{\omega+1}n)$ [Liebchen & Rizzi ‘04]
  - $O(m^3n + m^2n^2\log n)$ [Hariharan, Kavitha, Mehlhorn ’06,07]

- Hardness
  - NP-hard for tree bases [Deo, Prabhu, Krishnamoorthy ‘82]
  - MAX-SNP hard for tree bases [Amaldi & Galbiati ’03]

- Complexity open for integral bases
- Cycle exchange approaches [Berger & Gritzmann ’04, others]
The complexity of computing a short cycle basis

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Known algorithms use matroid structure

- Horton type algorithms
  - compute a polynomially sized set of short cycles
  - starting from the empty set, add them greedily (shortest first) until an independent set of \( v = m-n+1 \) cycles is formed

- De Piña type algorithms
  - iteratively compute shortest cycle with some edges in the orthogonal complement of the subspace spanned by the previously added cycles

- Horton can be simulated by de Piña
  - by choosing the next subspace such that the same sequence \( C_1, \ldots, C_v \) of cycles is constructed
Questions about integral cycles bases

- Do the cycles of an integral cycle basis induce a matroid?  
  \text{NO}

- Can a shortest integral cycle basis be computed with the greedy algorithm?  
  \text{NO}

- Do every two shortest integer cycle bases have the same sequence of cycle weights?  
  \text{NO}

- Does the Horton family contain an integral cycle basis?  
  \text{open}

- Can one compute shorter integral cycle bases by using \text{circulations} instead of cycles?  
  \text{open}
Greedy does not work for integral bases (1)

- Need cycle bases with large determinant
  - take circulant graphs $\mathbb{Z}_{n,k}$

- enhance them by bridges to generalized Peterson graphs

- choose special cycles and add chords
Greedy does not work for integral bases (2)

- choose special weights for cycles

- can be realized by arc weights
  5 on inner and outer cycle, 19 on bridges and chords
Greedy does not work for integral bases (3)

- Directed: not integer
- Undirected: not integer
- Greedy: integer
- Integer:
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