

## § 18 Bounds for dependent processing times and $C_{\max}$

Previous Bounds require independent processing times!

But dependencies are frequent in practice



difficult to specify

Therefore we follow a worst case approach

### Worst case approach for stochastic dependencies

[Meilijson & Nadas '79, Klein-Haneveld '86]

Consider **expected tardiness**

$$\mathbb{E}_Q[(C_{\max} - t)^+] = \mathbb{E}_Q[\max\{0, C_{\max} - t\}]$$

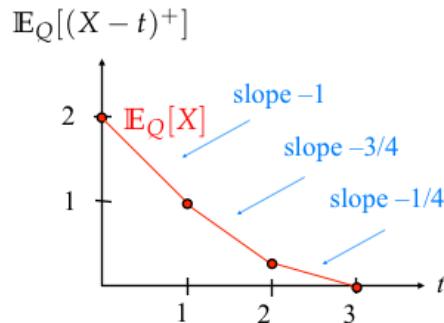
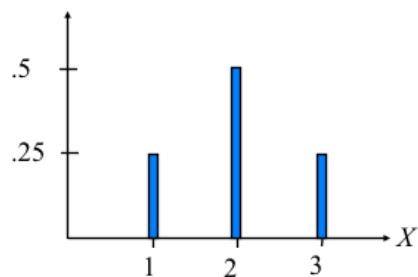
of makespan  $C_{\max}$  in the worst case, i.e.

$$\psi(t) = \sup_Q \mathbb{E}_Q[(C_{\max} - t)^+]$$



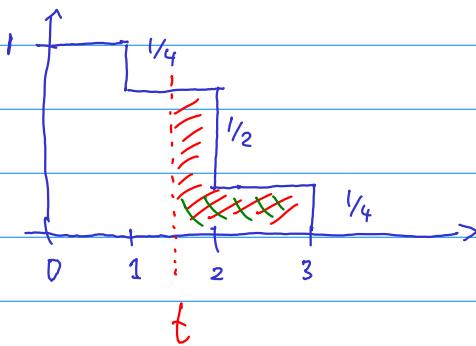
ranges over all joint distributions with the given job processing time distributions as marginals

### Properties of expected tardiness



$\mathbb{E}_Q[(X - t)^+]$  is piecewise linear and convex  
for discrete random variables  $X$

To see this, consider  $1 - \bar{F}_X(t)$  :



$$\mathbb{E}[(X-t)^+] = (3-t) \cdot \frac{1}{4} + (2-t) \cdot \frac{1}{2}$$

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$$\mathbb{E}[(X-t)^+] = \int_t^{\infty} (1 - \bar{F}_X(s)) d\mathbb{P}$$

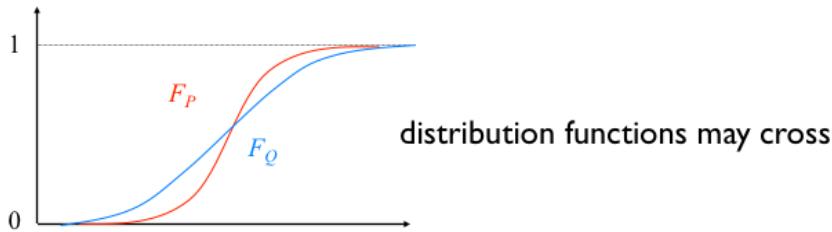
### The stochastic ordering in the convex sense

$P$  is stochastically smaller than  $Q$  in the convex sense if

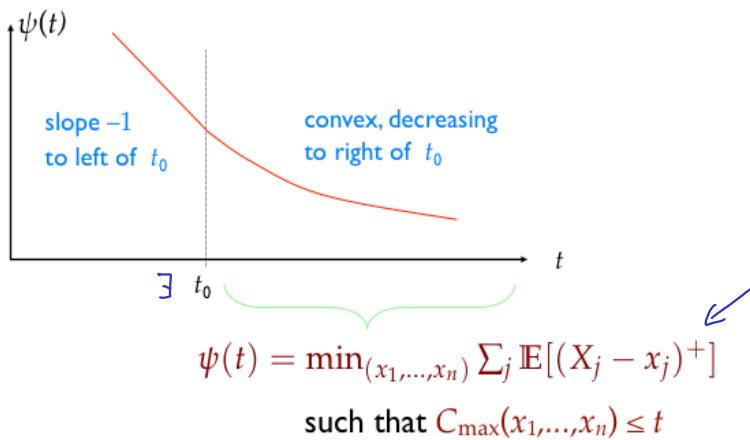
$$P \leq_c Q \Leftrightarrow \int f dP \leq \int f dQ$$

for all monotone convex functions  $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$

$\Leftrightarrow \mathbb{E}[(X-t)^+] \leq \mathbb{E}[(Y-t)^+]$  for all  $t$  if  $X, Y$  are real-valued random variables with distributions  $P, Q$ , respectively



Properties of  $\psi(t) = \sup_Q \mathbb{E}_Q[(C_{\max} - t)^+]$



18.1 THEOREM

independent of  $Q$

special convex separable optimization problem

## Proof of bounding property

Consider chain  $C$  of  $N$  and processing time vector  $x = (x_1, \dots, x_n)$

$$\sum_{j \in C} X_j - t = \underbrace{\sum_{j \in C} x_j - t}_{\leq C_{\max}(x)} + \underbrace{\sum_{j \in C} (X_j - x_j)}_{\leq \sum_{j=1}^n (X_j - x_j) \leq \sum_{j=1}^n (X_j - x_j)^+}$$

$$\sum_{j \in C} X_j - t \leq C_{\max}(x) - t + \sum_j (X_j - x_j)^+$$

$$\Downarrow \quad \leq (C_{\max}(x) - t)^+ + \sum_j (X_j - x_j)^+ \quad \text{all } C, \text{ all } x$$

$$\max_C \underbrace{\sum_{j \in C} X_j - t}_{C_{\max}(X), \quad X = (X_1, \dots, X_n)} \leq \dots$$

$$C_{\max}(X) - t \leq (C_{\max}(x) - t)^+ + \underbrace{\sum_j (X_j - x_j)^+}_{\geq 0}$$

$$(C_{\max}(X) - t)^+ \leq \dots \quad \text{for all } x$$

$$E_Q[\dots] \leq E_Q[\dots] = (C_{\max}(x) - t)^+ + \sum_j E_Q[(X_j - x_j)^+] \quad \text{for all } x, Q$$

$$E_Q[(C_{\max}(X) - t)^+] \leq \inf_x \left\{ (C_{\max}(x) - t)^+ + \underbrace{\sum_j E_Q[(X_j - x_j)^+]}_{\text{optimization problem}} \right\}$$

$$\sup_Q E_Q[(C_{\max}(X) - t)^+] \leq \dots$$

independent of  $Q$

$$\psi(t) \leq \min \sum_j \mathbb{E}_Q[(X_j - x_j)^+] \quad C_{\max}(x) \leq t$$

In more detail:

## 18.2. Theorem (Melijon & Naoros (1971))

Let  $\mathcal{P}$  be the class of joint distributions whose marginal distributions are equal to the distribution of the processing times  $X_j$ . Then:

- $E_Q[(C_{\max} - t)^+] \leq \psi(t)$  for all  $Q \in \mathcal{P}$
- $\exists$  random variable  $Z$  with  $\psi(t) = E[(Z - t)^+]$
- If  $P_Z$  is the distribution of  $Z$ , then  $Q_{C_{\max}} \leq_c P_Z \forall Q \in \mathcal{P}$
- If  $G$  is series-parallel, then  $P_Z = Q_{C_{\max}}$  for some  $Q \in \mathcal{P}$
- $\psi(t)$  is a tight upper bound for  $E_Q[(C_{\max} - t)^+]$  in the sense that, for every  $t$ , there is a  $Q_t \in \mathcal{P}$  s.t  

$$\psi(t) = E_{Q_t}[(C_{\max} - t)^+]$$

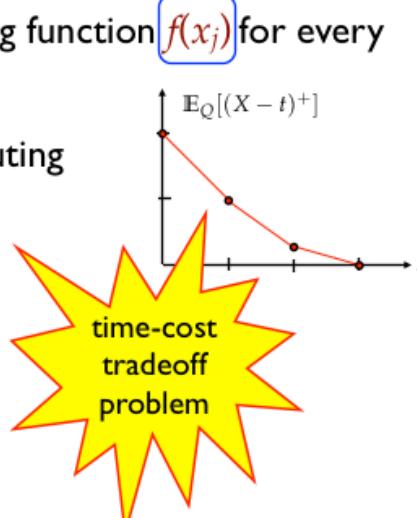
This shows equality in Theorem 18.1

(without proof)

## Solving the convex optimization problem

$$\psi(t) = \min_{(x_1, \dots, x_n)} \sum_j E[(X_j - x_j)^+] \text{ such that } C_{\max}(x_1, \dots, x_n) \leq t$$

- ▶ piecewise linear, convex, decreasing function  $f(x_j)$  for every job  $j$  in objective
- ▶ Interpret  $f(x_j)$  as cost for executing job  $j$  with processing time  $x_j$
- ▶ Side constraints:  
Find processing times  $x_j$  that
  - minimize the total cost
  - do not exceed the deadline  $t$  on the makespan



this interpretation  
leads to a  
time cost tradeoff  
problem

see § 19

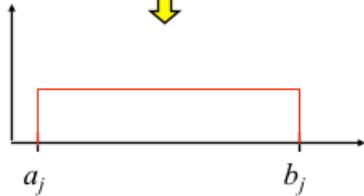
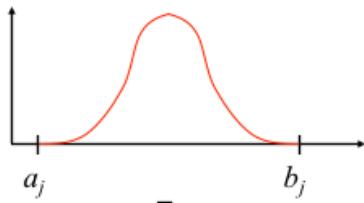
## Time-cost tradeoff problems

- ▶ classical network problem
- ▶ is the dual of a min-cost-flow problem for fixed  $t$   
[Fulkerson '61]
- ▶ can be solved parametrically in  $t$  by a sequence of max-flow problems  
[Kelley '61]
- ▶ very efficient in practice

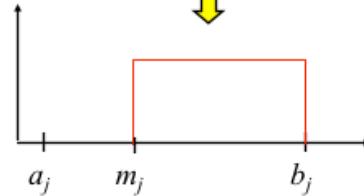
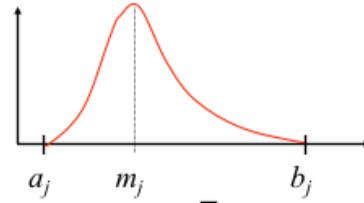
## Compatibility with incomplete information

Incomplete information about  $X_j$  [Cipra '78, Zackova '66]

unimodal & symmetric



only unimodal



These uniform distributions are larger in the convex sense

⇒ we get upper bounds if we replace the original distributions by these

## Exercises

18.1  $\psi(t)$  for series-parallel networks

a) Consider a 2-element chain  $\xrightarrow{X_1 \ X_2}$  and let  $F_1, F_2$  be the distribution functions of  $X_1, X_2$ .

Show that the distribution of the 2-dimensional random variable

$(F_1^{-1}(u), F_2^{-1}(u))$ ,  $u$  uniformly distributed on  $[0,1]$ , is the worst case distribution in the convex sense.

b) Consider two jobs in parallel  $\circlearrowleft_{X_1 \ X_2} \circlearrowright$

With the notation of a), show that the distribution of

$(F_1^{-1}(u), F_2^{-1}(1-u))$  is the worst case distribution for  $\circlearrowleft \circlearrowright$  in the convex sense

c) Use a) and b) to compute the worst case expected tardiness of a series-parallel network