

§ 16 Evaluating the distribution of κ^Π for a fixed policy Π

Policies and detailed project analysis

- ◆ First step:
 - determine best/good policy Π w.r.t. expected cost $E(\kappa^\Pi)$
- ◆ Second step:
 - use Π for detailed analysis
 - » cost distribution
 - » time-cost tradeoff

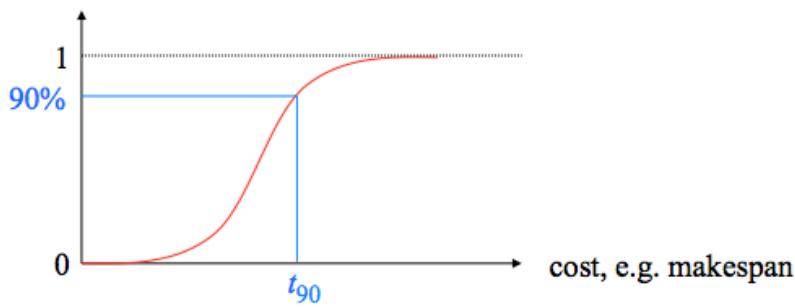
Early Start policies \Rightarrow methods from this part applicable

ES-policies just do early start scheduling w.r.t. precedence constraints
 \Rightarrow many algorithms from combinatorial optimization available

Consider mostly $\kappa = C_{\max}$

Detailed analysis of cost distribution

Ideal: distribution function F of κ^Π



Modest: percentiles

$$t_{90} = \inf\{t \mid \Pr\{\kappa^\Pi \leq t\} \geq 0.90\}$$

$\underbrace{F(t)}_{F(t)}$

Obtaining stochastic information is hard

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MEAN

Given: Stochastic project network with discrete independent processing times

Wanted: Expected makespan

DF

Given: Stochastic project network with discrete independent processing times, time t

Wanted: $\Pr\{\text{makespan} \leq t\}$

← Problems to solve



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- ◆ MEAN and DF are polynomially equivalent for 2-point distributions
- ◆ DF and the 2-point versions of DF and MEAN are #P-complete
- ◆ Unless P = NP, MEAN and DF cannot be solved in time polynomial in the number of values that the makespan attains

16.1 Theorem

Reduction from computing network reliability

$\#P$ = complexity class for counting problems (called "sharpP" or "numberP")
A problem is in $\#P$ if one can compute in non-deterministic polynomial time
the number of "yes"-answers for every instance of that problem.
i.e.

$P \in \#P : \Leftrightarrow \exists \text{ polynomial } p(n), \exists \text{ non-deterministic algorithm A such that}$
 $\forall I \in P, A(I)$ gives the number of "yes" instances
to instance I in time $p(|I|)$

$|I|$ encoding length of I

Examples:

(1) $SAT \in NP$: does a CNF formula have a fulfilling assignment?

$\#SAT \in \#P$: how many fulfilling assignments does a CNF formula have?

(2) $HAMILTON CYCLE \in NP$: does graph G have a Hamiltonian cycle

$\#HAMILTON CYCLE \in \#P$: how many ...

(3) $LINEAR EXTENSION$: does partial order P have a linear extension?

$\#LINEAR EXTENSION$: how many ...

Similar to prove NP-completeness, one defines $\#P$ -complete and shows:

$P \in \#P$ is $\#P$ -complete

$\Leftrightarrow \exists$ polynomial Turing reduction from P' to P , where P' is $\#P$ complete.

So counting on P would solve the counting problem on P'

Examples (1)-(3) are $\#P$ -complete

$\Rightarrow \#P$ -complete problems can arise from polynomially solvable problems

Proof of Theorem 16.1 needs a different representation of a partial order G that will also be useful for other algorithm

Transform $G = (V, E)$ into an s, t -dag $D = (N, A)$

D has a unique source s and a unique sink t

Every job $j \in V$ is an arc in D , s.t. precedences are preserved

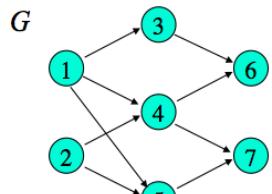
i.e. $(i, j) \in E \Rightarrow \exists$ path from the end of i to the start of $j \in D$

one may use additional arcs (dummy arcs) to properly represent precedence constraints

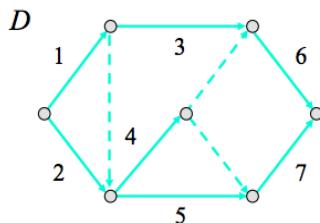
NOTE: The induced suborder of the non-dummy jobs of D must be equal to the original order G

D is called an arc-diagram of G

Arc diagrams



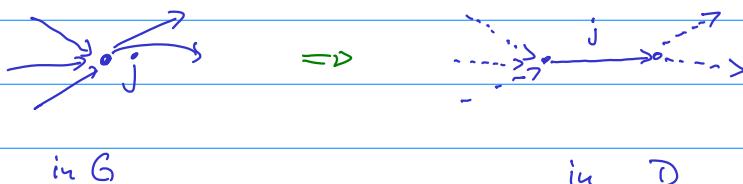
Node diagram
jobs are nodes of digraph G



Arc diagram
jobs are arcs of digraph D
dummy arcs \dashrightarrow may be necessary
standard graph algorithms apply
makespan = longest path

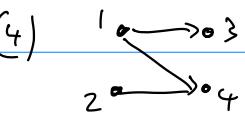
Note:

- (1) arc diagrams are not unique
- (2) a standard (polynomial) construction is as follows:
 - (i) take the edges of G as dummy arcs
 - (ii) blow up every vertex of G to an arc



- (iii) identify sources and sinks
contract "superfluous" dummy arcs

(3) Constructing an arc diagram with the minimum number of dummy arcs is NP-hard

- (4)  shows that dummy arcs are necessary

Proof of Theorem 16.1 (only for the 2-point version of DF)

Use reduction from RELIABILITY (is #P-complete)

Given: s,t-dag D with failure probabilities q_j on arcs j

Wanted: the probability that there is an s,t-path without arc failures

the reliability of D

Let I be an instance of RELIABILITY

Construct instance I' of 2-DF s.t.

$$\text{reliability of } D = 1 - \text{Prob} \{ C_{\max} \leq L-1 \} \text{ for some appropriate } L \in \mathbb{N}$$

$$= 1 - F(L-1)$$

\Rightarrow can compute reliability (D) from $F(L-1)$

Construction of I' :

Take s, t -digraph as arc diagram of G

Every job (arc) j has processing time $X_j = \begin{cases} 0 & \text{with prob. } q_j \\ p_j > 0 & \text{with prob. } 1 - q_j \end{cases}$

with p_j integer such that all s, t -paths have equal length L

(is possible in polynomial time)

start with $p_j=1$ and set $L = \text{length of longest path}$

$\text{early start } ES(j) \quad LF(j) = L - \text{length of longest path from } v \text{ to } t$
 latest finish

if $LFC(j) > ESC(j) + P_j$ then set $P_j := LFC(j) - ESC(j) - P_j$

Then $\text{Prob}(\text{some path has no failure}) \leftarrow \text{reliability}(D)$

$$= 1 - \text{Prob}(\text{ all paths fail })$$

$$= 1 - \text{Prob} \left(\bigcap_k \{ \text{path } k \text{ fails} \} \right)$$

$$= 1 - \text{Prob} \left(\cap_k \{ \text{some arc fails on path } k \} \right) \leftarrow \text{in I}$$

$$= 1 - \text{Prob} \left(\cap_k \{ \text{path } k \text{ has length } < L \} \right) \quad \leftarrow \text{in I}'$$

$$= 1 - \text{Prob} (C_{\max} \leq L-1) \quad \square$$

Exercises:

16.1 Characterization of partial orders that have an arc diagram without dummy arcs

A partial order is **N-free** if its Hasse diagram does not contain **N** as subgraph (so $\forall i, j \text{ Impred}(i) = \text{Impred}(j)$ or $\text{Impred}(i) \cap \text{Impred}(j) = \emptyset$)

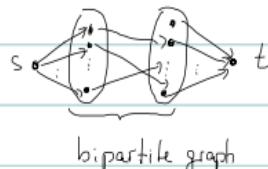
A partial order has the **CAC - property** if every maximal chain and every maximal antichain have a non-empty intersection (also called the chain-antichain property)

Show that the following are equivalent for a partial order $G = (V, E)$:

- (1) G has an arc-diagram without dummy-arcs.
- (2) G is N-free
- (3) G has the CAC-property

16.2 Strengthen the proof of Theorem 16.1 to processing times $X_j \in \{0, 1\}$.

Hint: Use the fact that RELIABILITY is already $\#P$ -complete for s,t-dags of the form



16.3 Show that the 2-point versions of MEAN and DF with $X_j \in \{0, 1\}$ are polynomially equivalent