

§15 Stochastic online scheduling for  $\sum w_j C_j$   
[Megow, Uetz, Vredeveld 2006]

## Stochastic online scheduling

[Megow, Uetz, Vredeveld 2004<sup>6</sup>]

2 phase model for  $\sum w_j C_j$

- ▶ jobs arrive online and must be assigned to machines now
  - unknown number, have random processing time
- ▶ “next day”, jobs are scheduled on the assigned machines in the “expected performance” model
  - number is now known
- ▶ view this as a scheduling policy, analyze w.r.t. expected performance

optimally on every machine with  
WSEPT

assignment policy

## Algorithm MinIncrease

- ▶ assign job to machine such that  $\sum w_j C_j$  based on  $\mathbb{E}[X_j]$ 
  - has minimum increase
  - can be done in polynomial time
- ▶ MinIncrease matches best known bounds of previous model
  - even better for NBUE processing times and release dates
- ▶ needs LP-based lower bounds in analysis, but not for defining the policy
- ▶ first combinatorial approximation algorithm for release dates

Notation:  $j \rightarrow i$  if job  $j$  is assigned to machine  $i$   
 priority order from WSEPT, i.e.  $w_j / \mathbb{E}[X_j]$  decreasing  
 $H(j) = \{k \in V \mid \text{higher or equal priority as } j\} \Rightarrow j \in H(j)$   
 $L(j) = V \setminus H(j)$  lower priority jobs  
 $k < j \hat{=} k$  arrives before  $j$   
 tie breaking: according to incoming order

### 15.1 Algorithm MinIncrease MI

(1) Upon arrival of job  $j$ , assign it to machine  $i$  that minimizes

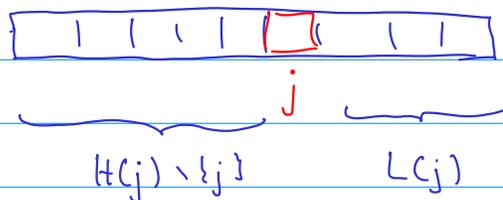
$$w_j \sum_{\substack{k \in H(j) \\ k < j \\ k \rightarrow i}} \mathbb{E}[X_k] + \mathbb{E}[X_j] \sum_{\substack{k \in L(j) \\ k \rightarrow i \\ k < j}} w_k + w_j \mathbb{E}[X_j] =: z(j, i)$$

(2) In the scheduling phase, schedule the jobs on every machine with WSEPT (optimal per machine)

### 15.2 Lemma

$z(j, i)$  is the increase of  $\sum_e w_e \mathbb{E}(C_e)$  on machine  $i$  when  $j$  is assigned to that machine and jobs are scheduled by WSEPT

Proof:



$$z(j, i) = w_j \sum_{\substack{k \in H(j) \\ k < j \\ k \rightarrow i}} \mathbb{E}[X_k] + \mathbb{E}[X_j] \sum_{\substack{k \in L(j) \\ k \rightarrow i \\ k < j}} w_k + w_j \mathbb{E}[X_j]$$

$= w_j \mathbb{E}(C_j)$ 
with lower priority
increase of  $w_k \mathbb{E}[C_k]$  of jobs
□

15.3 LEMMA 
$$\mathbb{E} \left[ \sum_j w_j C_j^{MI} \right] = \sum_j \min_i z(j, i)$$

Proof: Let  $C_j := C_j^{MI}$

$$\mathbb{E} \left[ \sum_j w_j C_j \right] = \sum_j w_j \sum_{\substack{k \in H(j) \\ k \rightarrow i_j \\ \text{same priority jobs } k \\ \text{have } k < j}} \mathbb{E}[X_k]$$



jobs are partitioned into jobs that are before  $j$  on the machine  $i_j$  that  $j$  is assigned to

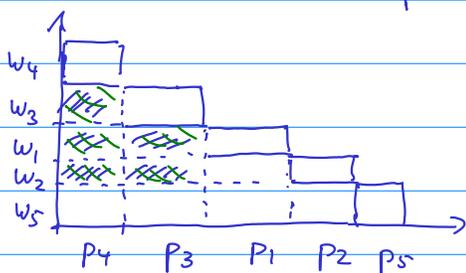
$$= \sum_j w_j \sum_{\substack{k \in H(j) \\ k \rightarrow i_j, k < j}} \mathbb{E}[X_k] + \underbrace{\sum_j w_j \sum_{\substack{k \in H(j) \\ k \rightarrow i_j, k > j}} \mathbb{E}[X_k]}_{=: A} + \sum_j w_j \mathbb{E}[X_j]$$

↑  
partition into jobs that arrive before/after  $j$

Claim:  $A = B := \sum_j \mathbb{E}[X_j] \cdot \sum_{\substack{k \in L(j) \\ k < j \\ k \rightarrow i_j}} w_k$

Proof Claim: use different counting (rowwise and column-wise in 2D-Gantt chart)

Expl: 5 jobs on one machine in arrival order 1 2 3 4 5  
with priority order 4 3 1 2 5  
↑ highest priority



$$p_i = \mathbb{E}[X_i]$$

$$A = \sum_j w_j \sum_{\substack{k \in H(j) \\ k > j}} E[X_k] \quad : \quad \begin{aligned} &w_5 \cdot 0 \\ &w_2 \cdot (p_3 + p_4) \quad // // \\ &w_1 \cdot (p_3 + p_4) \\ &w_3 \cdot p_4 \\ &w_4 \cdot 0 \end{aligned}$$

$$B = \sum_j E[X_j] \sum_{\substack{k \in L(j) \\ k < j}} w_k \quad : \quad \begin{aligned} &p_4 \cdot (w_1 + w_2 + w_3) \quad \approx \approx \approx \\ &p_3 \cdot (w_1 + w_2) \\ &\dots \end{aligned}$$

$$\Rightarrow E[w_j C_j^{ME}] = \sum_j w_j \sum_{\substack{k \in H(j) \\ k > i_j, k < j}} E[X_k] + \underbrace{\sum_j w_j \sum_{\substack{k \in H(j) \\ k \rightarrow i_j \\ k > j}} E[X_k]}_{=: A} + \sum_j w_j E[X_j]$$

$=: A$

" ← Claim

$$\sum_j E[X_j] \sum_{\substack{k \in L(j) \\ k \rightarrow i_j \\ k < j}} w_k$$

La 15.2

$$= \sum_j \min_i z(j, i)$$

#### 15.4 THEOREM

Let  $CV[X_j] \leq \Delta$ . Then ME is a  $g$ -approximation algorithm with  $g = 1 + \frac{(m-1)(\Delta+1)}{2m}$

Proof:  $E[\sum_j w_j C_j^{ME}] = \sum_j \min_i z(j, i)$

La 15.3

$$L_{15.2} = \sum_j w_j \min_i \left\{ w_j \sum_{\substack{k \in H(j) \\ k < j \\ k \rightarrow i}} \mathbb{E}[X_k] + \mathbb{E}[X_j] \sum_{\substack{k \in L(j) \\ k \rightarrow i \\ k < j}} w_k + w_j \mathbb{E}[X_j] \right\}$$

$$= \sum_j w_j \min_i \left\{ \dots \right\} + \sum_j w_j \mathbb{E}[X_j]$$

$$\leq \sum_j \frac{1}{m} \sum_i \left\{ \dots \right\} + \sum_j w_j \mathbb{E}[X_j]$$

↑

min ≤ arithmetic mean

$$= \sum_i \frac{1}{m} \left\{ \sum_j w_j \sum_{\substack{k \in H(j) \\ k < j \\ k \rightarrow i}} \mathbb{E}[X_k] + \mathbb{E}[X_j] \sum_{\substack{k \in L(j) \\ k \rightarrow i \\ k < j}} w_k \right\} + \sum_j w_j \mathbb{E}[X_j]$$

A = B

interchange summation

$$= \sum_i \frac{1}{m} \sum_j w_j \sum_{\substack{k \in H(j) \\ k < j \\ k \rightarrow i}} \mathbb{E}[X_k] + \sum_i \frac{1}{m} \sum_j w_j \sum_{\substack{k \in L(j) \\ k \rightarrow i \\ k > j}} \mathbb{E}[X_k] + \sum_j w_j \mathbb{E}[X_j]$$

$$= \sum_j \frac{1}{m} w_j \sum_{\substack{k \in H(j) \\ k \neq j \\ k \rightarrow i_j}} \mathbb{E}[X_k] + \sum_j w_j \mathbb{E}[X_j]$$

↓

$$\mathbb{E} \left[ \sum_j w_j C_j^{MI} \right] = \sum_j \frac{1}{m} w_j \sum_{\substack{k \in H(j) \\ k \rightarrow i_j}} \mathbb{E}[X_k] + \frac{m-1}{m} \sum_j w_j \mathbb{E}[X_j]$$

↑  
MI

include j in first sum

↑  
A

↑  
B

Now use (see next Lemma)

$$(1) \quad \underset{\substack{\uparrow \\ \text{optimal policy} \\ \uparrow \\ \text{OPT}}}{\mathbb{E}[\text{OPT}]} \geq \sum_j w_j \frac{1}{m} \sum_{\substack{k \in H(j) \\ k \rightarrow i_j}} \mathbb{E}[X_k] - \frac{(m-1)(\Delta-1)}{2m} \sum_j w_j \mathbb{E}[X_j]$$

$$\Rightarrow \text{MI} = A + B \quad \downarrow$$

$$\text{OPT} \geq A - C \geq \text{MI} - B - C \quad \Rightarrow \quad \text{MI} \leq \text{OPT} + B + C$$

$$\Rightarrow \mathbb{E}\left[\sum_j w_j C_j^{\text{MI}}\right] \leq \underbrace{\mathbb{E}[\text{OPT}] + \left[\frac{(m-1)(\Delta-1)}{2m} + \frac{m-1}{m}\right]}_{\frac{(m-1)(\Delta+1)}{2m}} \underbrace{\sum_j w_j \mathbb{E}[X_j]}_{\leq \mathbb{E}[\text{OPT}]}$$

$$= \rho \cdot \mathbb{E}[\text{OPT}]$$

15.5 Lemma: Consider priorities w.r.t. WSEPT.

Then (1) holds.

Proof: Recall Theorem 14.11.

14.11 Theorem: Assume  $\frac{w_1}{\mathbb{E}[X_1]} \geq \dots \geq \frac{w_n}{\mathbb{E}[X_n]}$

Then (LP) has the optimal solution

$$C_j^{\text{LP}} = \frac{1}{m} \sum_{k=1}^j \mathbb{E}[X_k] - \frac{(\Delta-1)(m-1)}{2m} \mathbb{E}[X_j] \quad j=1, \dots, n$$

$$\Rightarrow \sum_j w_j C_j^{\text{LP}} = \frac{1}{m} \sum_j w_j \sum_{\substack{k \in H(j) \\ k \rightarrow i_j}} \mathbb{E}[X_k] - \sum_j w_j \frac{(\Delta-1)(m-1)}{2m} \mathbb{E}[X_j]$$

$$\mathbb{E}[\text{OPT}] \geq \sum_j w_j C_j^{\text{LP}} \quad \square$$

Note: lemma 15.5 is the only part in the proof, where the (LP) is needed.

Remark

(1) WSEPT and MI produce different schedules in general

(3) The lower bound of Thm 14.2 applies also to MI

15.1 Show by example that WSEPT and MI generate different schedules