

§14 Performance guarantees of simple policies for the expected weighted sum of completion time $\sum w_j C_j$

need more complicated methods: LP-guided construction of a policy

Model: no precedence constraints, m machines

can be generalized to release dates $r_j \geq 0$

Consider LP in completion time variables C_j^{LP}

$$(LP) \left\{ \begin{array}{l} \min \sum_j w_j C_j^{\text{LP}} \quad \text{such that} \\ (1) \quad \sum_{j \in A} p_j \cdot C_j^{\text{LP}} \geq \frac{1}{2m} \left(\sum_{j \in A} p_j \right)^2 + \frac{1}{2} \sum_{j \in A} p_j^2 \quad \forall A \subseteq V \\ (2) \quad C_j^{\text{LP}} \geq p_j \end{array} \right.$$

\uparrow
decentralistic case, processing times = p_j

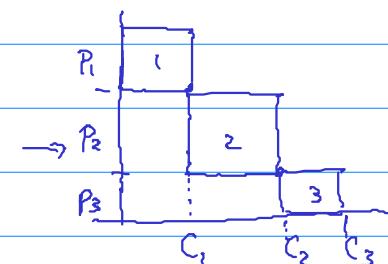
14.1 LEMMA: The completion times of every schedule for $p = (p_1 \dots p_n)$ fulfill (1) and (2)

Proof: let S be a feasible schedule for p with completion times C_1, \dots, C_n
Show that the C_j fulfill (1) and (2)

\uparrow
obvious

Need to show (1)

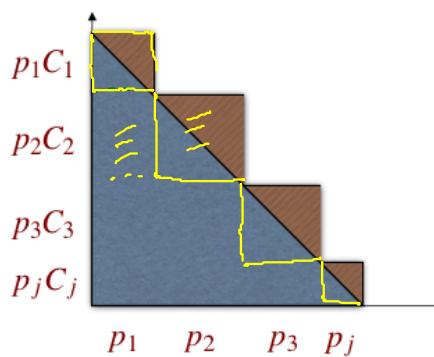
Consider case $m = 1$



Inequalities for 1 machine, deterministic case

[Queyranne 1993, Goemans & Williamson 2000]

$$\sum_{k \in A} p_k C_k \geq \frac{1}{2} \left(\sum_{k \in A} p_k \right)^2 + \frac{1}{2} \sum_{k \in A} p_k^2 \quad A = \{1, \dots, j\}$$



$$\sum_{k \in A} p_k C_k \geq$$

$$\frac{1}{2} \left(\sum_{k \in A} p_k \right)^2 =$$

$$\frac{1}{2} \sum_{k \in A} p_k^2 =$$

$$\forall A \subseteq V$$

lhs is smallest, when
jobs from A are first
in the schedule

Inequalities for m machines, deterministic case

$$\sum_{k \in A} p_k C_k \geq \frac{1}{2m} \left(\sum_{k \in A} p_k \right)^2 + \frac{1}{2} \sum_{k \in A} p_k^2$$

$$\sum_{k \in A} p_k C_k = \sum_{i=1}^m \sum_{k \in A \cap M_i} p_k C_k \geq \sum_{i=1}^m \left(\frac{1}{2} \left(\sum_{k \in A \cap M_i} p_k \right)^2 + \frac{1}{2} \sum_{k \in A \cap M_i} p_k^2 \right)$$

$$= \frac{1}{2} \sum_{i=1}^m \left(\sum_{k \in A \cap M_i} p_k \right)^2 + \frac{1}{2} \sum_{k \in A} p_k^2 \geq \frac{1}{2m} \left(\sum_{k \in A} p_k \right)^2 + \frac{1}{2} \sum_{k \in A} p_k^2$$

$$\sum_{i=1}^m a_i^2 \geq \frac{1}{m} \left(\sum_{i=1}^m a_i \right)^2$$

special case of Cauchy-Schwarz inequality

use inequality
for 1 machine

Cauchy-Schwarz inequality:

$$a, b \in \mathbb{R}^m \Rightarrow \langle a, b \rangle \leq \|a\| \cdot \|b\|$$

scalar product \leq product of norms

$$\Leftrightarrow \left(\sum_i a_i \cdot b_i \right)^2 \leq \left(\sum_i a_i^2 \right) \cdot \left(\sum_i b_i^2 \right)$$

special case $b = (1, 1, \dots, 1) \in \mathbb{R}^m$

$$\Rightarrow \left(\sum_i a_i \right)^2 \leq \sum_i a_i^2 \cdot m \Rightarrow \frac{1}{m} \left(\sum_i a_i \right)^2 \leq \sum_i a_i^2$$

$$\Rightarrow \text{statement with } a_i = \sum_{j \in A \cap M_i} p_j \quad \square$$

14.2 Lemma: If numbers $C_1 \leq C_2 \leq \dots \leq C_n$ fulfill (1), then
 $2 \cdot C_j \geq \frac{1}{m} \sum_{k \in J} p_k$ with $J = \{1, 2, \dots, j\}$

This will be the form how we will use inequality (1)

$$\begin{aligned} \text{Proof: } C_j \cdot \sum_{k \in J} p_k &= \sum_{k=1}^j p_k C_j \geq \sum_{k=1}^j p_k C_k \\ &\stackrel{(1)}{\geq} \frac{1}{2m} \left(\sum_{k \in J} p_k \right)^2 + \frac{1}{2} \sum_{k \in J} p_k^2 \\ &\geq \frac{1}{2m} \left(\sum_{k \in J} p_k \right)^2 \end{aligned}$$

$$\Rightarrow 2 \cdot C_j \geq \frac{1}{m} \sum_{k \in J} p_k \quad \square$$

Idea for approximation algorithm:

A: Solve (LP) \rightarrow optimal solution C_j^{LP}

[can be done in polynomial time although we have exponentially many inequalities]

B: Use the ordering $C_{j_1}^{LP} \leq C_{j_2}^{LP} \leq \dots \leq C_{j_n}^{LP}$ as

job-based priority list $j_1 < j_2 < \dots < j_n$

← call the list L

[different from list scheduling as considered before:

may start j_k only after all j_1, \dots, j_{k-1} have been started,

i.e. list scheduling with condition $S_{j_1} \leq S_{j_2} \leq \dots \leq S_{j_n}$]

define a planning rule Π_L

C: Use Lemma 14.2 to prove a performance guarantee.

More on B

14.3 Theorem:

- (1) Π_L is a policy, called a job-based list-scheduling policy
- (2) Every such policy is dominated by a preselective policy

Proof:

(1) Π_L uses only info from the past at every decision time

(2) The condition $S_{j_1} \leq S_{j_2} \leq \dots \leq S_{j_m}$ implies that, for every forbidden set \bar{F} , the last job in \bar{F} from the list L is a waiting job

since: it is started last in \bar{F} and thus must wait for the completion of any other job from \bar{F}

\Rightarrow the preselective policy Π^* with these waiting jobs dominates Π_L , i.e. $\Pi^* \leq \Pi_L$

It may start jobs earlier than Π_L \square

More on C

14.4 Lemma

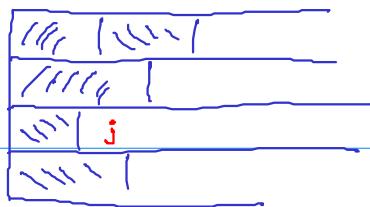
Consider Π_L with list $L: 1 < 2 < 3 < \dots < n$

Let $C_j^{\Pi_L}(x) = \Pi_L[x](j) + x_j$ denote the completion time of job j

Then: $C_j^{\Pi_L}(x) \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k + x_j$ w.r.t Π_L b' x

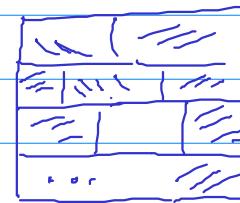
and thus $E[C_j^{\Pi}] \leq \frac{1}{m} \sum_{k=1}^{j-1} E[x_k] + E[x_j]$ in the stochastic case

Proof: L implies that jobs $1, 2, \dots, j-1$ are started before j is started



\uparrow
jobs before j

worst case:
all machines take equally long



The latest time at which a machine becomes available for j

$$\text{is } \frac{1}{m} \sum_{k=1}^{j-1} x_k \Rightarrow S_j \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k \\ \Rightarrow C_j \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k + x_j$$

Taking expectations gives the second inequality

14.5 THEOREM

let $C_1^{\text{LP}} \leq C_2^{\text{LP}} \leq \dots \leq C_n^{\text{LP}}$ be an optimal solution of the (LP)

and let, for fixed x , C_1, \dots, C_n be the completion times

obtained from Π_L with $l : 1 < 2 < 3 < \dots < n$ (LP-ordering)

Then

$$\sum_j w_j C_j \leq \left(3 - \frac{1}{m}\right) \text{OPT}$$

\uparrow optimal value over all schedules for x

So the algorithm "LP-guided job-based priority scheduling"

is a $(3 - \frac{1}{m})$ -approximation algorithm for the deterministic case

Proof:

$$C_j \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k + x_j = \frac{1}{m} \sum_{k=1}^j x_k + \frac{m-1}{m} x_j$$

La 14.4

$$\leq 2 \cdot C_j^{\text{LP}} + \frac{m-1}{m} x_j \leq C_j^{\text{LP}} \leq \left(3 - \frac{1}{m}\right) C_j^{\text{LP}}$$

14.2 LEMMA: If numbers $C_1 \leq C_2 \leq \dots \leq C_n$ fulfill (1), then

Lemma 14.2

$$2C_j \geq \frac{1}{m} \sum_{k \in J} p_k \text{ with } J = \{1, 2, \dots, j\}$$

$$\Rightarrow \sum_j w_j C_j \leq \left(3 - \frac{1}{m}\right) \underbrace{\sum_j w_j C_j^{\text{LP}}}_{\text{LP-objective}} \leq \left(3 - \frac{1}{m}\right) \text{OPT}$$

LP-objective

LP is a relaxation

More on A

can solve the (LP) in polynomial time if the separation problem for (1), (2) can be solved in polynomial time

Given $(C_1, \dots, C_m) \in \mathbb{R}_+^n$, define the violation for $A \subseteq V$

$v(A)$: rhs of (1) - rhs of (1)

$$= \underbrace{\frac{1}{2m} \left(\sum_{j \in A} p_j \right)^2 + \frac{1}{2} \sum_{j \in A} p_j^2}_{\text{rhs}} - \underbrace{\sum_{j \in A} p_j \cdot C_j}_{\text{rhs}}$$

so A violates (1) $\Leftrightarrow v(A) > 0$

14.6 Lemma: let A maximize the violation. Then

$$k \in A \Leftrightarrow C_k - \frac{1}{2} p_k < \frac{1}{m} \sum_{j \in A} p_j$$

Proof: Let $k \notin A$ Then

$$\begin{aligned} v(A \cup \{k\}) &= v(A) + \frac{1}{m} p_k \sum_{j \in A} p_j + \frac{1}{2m} p_k^2 + \frac{1}{2} p_k^2 - p_k C_k \\ &= v(A) + p_k \left[\frac{1}{m} \sum_{j \in A} p_j + \frac{m+1}{2m} p_k - C_k \right] \end{aligned} \tag{1}$$

Let $k \in A$ Then

$$v(A \setminus \{k\}) = v(A) - p_k \left[\frac{1}{m} \sum_{j \in A} p_j + \frac{m-1}{2m} p_k - C_k \right] \tag{2}$$

Let A maximize the violation. Then

$$\underline{k \in A} \Rightarrow v(A \setminus \{k\}) \leq v(A) \Rightarrow \frac{1}{m} \sum_{j \in A} p_j + \frac{m-1}{2m} p_k - C_k \geq 0$$

(2)

$$\Rightarrow C_k \leq \frac{1}{m} \sum_{j \in A} p_j + \frac{m-1}{2m} p_k$$

$$\Rightarrow C_k < \frac{1}{m} \sum_{j \in A} p_j + \frac{1}{2} p_k$$

$$\Rightarrow C_k - \frac{1}{2} p_k < \frac{1}{m} \sum_{j \in A} p_j$$

$$k \notin A \Rightarrow v(A \cup \{k\}) \leq v(A)$$

$$\stackrel{(1)}{\Rightarrow} \frac{1}{m} \sum_{j \in A} p_j + \frac{m+1}{2m} p_k \leq C_k$$

$$\Rightarrow \dots + \frac{1}{2} p_k < C_k$$

$$\Rightarrow C_k - \frac{1}{2} p_k > \frac{1}{m} \sum_{j \in A} p_j$$

14.7 Separation algorithm

(1) Sort jobs w.r.t. increasing $C_j - \frac{1}{2} p_j$ values

Let $1, 2, 3, \dots, n$ be this ordering

(2) The set A with maximum violation is an initial segment

$J = \{1, \dots, j\}$ of this ordering

(3) Check initial segments of this ordering for violation

Proof (2): Let A maximize the violation and $i \in A$

Show $k \in A$ for every $k < i$

$$i \in A \Rightarrow C_i - \frac{1}{2} p_i < \frac{1}{m} \sum_{j \in A} p_j$$

La 14.6

$$k < i \Rightarrow C_k - \frac{1}{2} p_k \leq C_i - \frac{1}{2} p_i < \frac{1}{m} \sum_{j \in A} p_j \stackrel{\text{Lq 14.6}}{\Rightarrow} k \in A \quad \square$$

ordering

Consider now the stochastic counterpart of this problem:

in machines

no precedence constraints

random independent processing times X_j

← important assumption

minimize $\sum_j w_j E(C_j)$ over all policies Π

The LP-based approach

Consider the **achievable region**

$\{ (E[C_1^\Pi], \dots, E[C_n^\Pi]) \in \mathbb{R}^n \mid \Pi \text{ policy} \}$

similar in queuing theory:
Bertsimas, Glazebrook, Nino-Mora

Find a polyhedral relaxation P

Solve the linear program

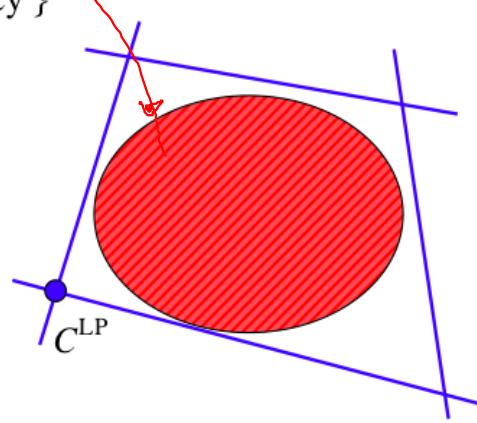
$$(LP) \min \left\{ \sum_j w_j C_j^{\text{LP}} \mid C^{\text{LP}} \in P \right\}$$

Use the list $L: i_1 \leq i_2 \leq \dots \leq i_n$
defined by $C_{i_1}^{\text{LP}} \leq C_{i_2}^{\text{LP}} \leq \dots \leq C_{i_n}^{\text{LP}}$

as list for priority/lin. pres./other policy

↓

job based priority policy as in the deterministic case



Bounds

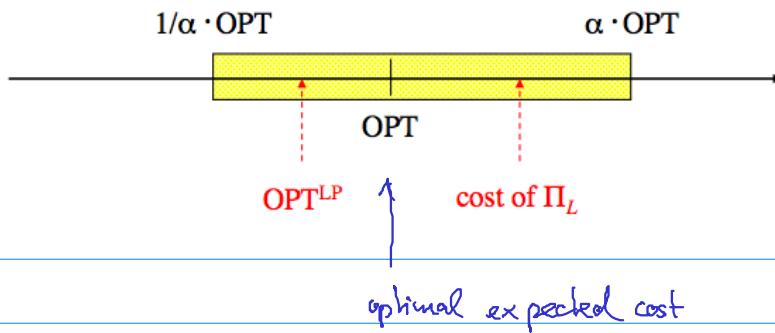
we do not know
the properties of
the achievable
region, only
Boundedness if
 $E(X_j)$ exist

Performance guarantees from the LP

Let Π_L be the policy induced by $L: i_1 \leq i_2 \leq \dots \leq i_n$

← job based priority policy

Hope that $E[\kappa^{\Pi_L}] \leq \alpha \cdot \text{OPT}^{\text{LP}}, \alpha \geq 1$



The tasks

- Find the relaxation P
- Solve the LP optimally in polynomial time to obtain list L
- Prove $E[\kappa^{\Pi_L}] \leq \alpha \cdot \text{OPT}^{\text{LP}}, \alpha \geq 1$

The polyhedral relaxation

14.7 Lemma (\approx Lemma 14.1 in the deterministic case)

Every policy Π for our problem fulfills

$$\sum_{j \in A} E[x_j] \cdot E[c_j^\Pi] \geq \frac{1}{2m} \left(\sum_{j \in A} E(x_j) \right)^2 + \frac{1}{2} \sum_{j \in A} E[x_j]^2 \quad \leftarrow \text{as in the determ. case}$$
$$- \frac{m-1}{2m} \sum_{j \in A} \text{Var}(x_j) \quad \forall A \subseteq V$$

↑ new

Proof: Consider fixed (x_1, \dots, x_n) . Lemma 14.1 gives

$$\sum_{j \in A} x_j C_j^T(x) \geq \frac{1}{2m} \left(\sum_{j \in A} x_j \right)^2 + \frac{1}{2} \sum_{j \in A} x_j^2 \quad \forall A \subseteq V$$

Rewriting in terms of start times $S_j^T(x) = C_j^T(x) - x_j$

$$\sum_{j \in A} x_j S_j^T(x) \geq \frac{1}{2m} \sum_{\substack{i, j \in A \\ i \neq j}} x_i \cdot x_j - \frac{m-1}{2m} \sum_{j \in A} x_j^2 \quad (1)$$

S_j^T and X_j are stoch. independent $\Rightarrow E[X_j S_j^T] = E[X_j] E[S_j^T]$ $(*)$
 will also use $VAR[X_j] = E[X_j^2] - E[X_j]^2$ $(**)$

Take expectations in (1) \Rightarrow

$$\sum_{j \in A} E[X_j \cdot S_j^T] \geq \frac{1}{2m} \sum_{\substack{i, j \in A \\ i \neq j}} E[X_i \cdot X_j] - \frac{m-1}{2m} \sum_j E[X_j^2]$$

$E[X_j] \cdot E[S_j^T]$ $E[X_i] \cdot E[X_j]$
 indep.

$$\Rightarrow \sum_{j \in A} E[X_j] \cdot E[S_j^T] \geq \frac{1}{2m} \left(\sum_{j \in A} E[X_j] \right)^2$$

↑
 add $\frac{1}{2m} \sum_{j \in A} E[X_j]^2$ and subtract here

reformulate this part by adding to it a "nice" 0

$$0 = \frac{m-1}{2m} \sum_{j \in A} E[X_j]^2 - \frac{m-1}{2m} \sum_{j \in A} E[X_j]^2$$

add green to green and red to red

$$\underbrace{\quad}_{= -\frac{m-1}{2m} \sum_{j \in A} VAR[X_j]} = \underbrace{-\frac{1}{2} \sum_{j \in A} E[X_j]^2}_{}$$

$$\Rightarrow \sum_{j \in A} \mathbb{E}[x_j] \cdot \mathbb{E}[s_j^{\pi}] \geq \frac{1}{2m} \left(\sum_{j \in A} \mathbb{E}[x_j] \right)^2 - \frac{1}{2} \sum_{j \in A} \mathbb{E}[x_j]^2$$

$$- \frac{m-1}{2m} \sum_{j \in A} \text{VAR}[x_j]$$

Adding $\sum_{j \in A} \mathbb{E}[x_j]^2$ on both sides gives

$$\sum_{j \in A} \mathbb{E}[x_j] \cdot \mathbb{E}[c_j^{\pi}] \geq \frac{1}{2m} \left(\sum_{j \in A} \mathbb{E}[x_j] \right)^2 + \frac{1}{2} \sum_{j \in A} \mathbb{E}[x_j]^2$$

$$- \frac{m-1}{2m} \sum_{j \in A} \text{VAR}[x_j]$$

□

Similar result as in the deterministic case, except for the variances

Coefficient of variation $\text{CV}[X_j]^2 := \frac{\text{VAR}[X_j]}{\mathbb{E}[X_j]^2}$

≤ 1 for all random variables that are NBUE

New Better than Used in Expectation

$\mathbb{E}[X_j - t \mid X_j \geq t] \leq \mathbb{E}[X_j]$ for all $t > 0$

NBUE: exponential, Erlang, uniform, geometric distributions

NOT: discrete, multi-modal distributions

The final stochastic inequalities

Assume $\text{CV}[X_j]^2 \leq \Delta$

$$\sum_{j \in A} \mathbb{E}[x_j] \mathbb{E}[C_j^{\Pi}] \geq \frac{1}{2m} \left(\left(\sum_{j \in A} \mathbb{E}[x_j] \right)^2 + \sum_{j \in A} \mathbb{E}[x_j]^2 \right)$$

$$- \frac{(m-1)(\Delta-1)}{2m} \left(\sum_{j \in A} \mathbb{E}[x_j]^2 \right) \text{ for all } A \subseteq V$$

$$\geq \frac{1}{2m} \left(\left(\sum_{j \in A} \mathbb{E}[x_j] \right)^2 + \sum_{j \in A} \mathbb{E}[x_j]^2 \right)$$

for NBUE processing times

rewrite the
inequalities

in terms of Δ

4.8 LEMMA

(4)

basis for LP relaxation

Proof Lemma 14.8 : just calculation

14.9 Lemma: let $C = (C_1, \dots, C_n) \in \mathbb{R}^n$, satisfy (4) and $C_j \geq \mathbb{E}[X_j]$

Assume $C_1 \leq C_2 \leq \dots \leq C_n$. Then

$$\frac{1}{m} \sum_{k=1}^j \mathbb{E}[X_k] \leq \left(1 + \max \left\{ 1, \frac{m-1}{m} \Delta \right\} \right) C_j \quad \forall j$$

[Stochastic counterpart of Lemma 14.2: $\frac{1}{m} \sum_{k=1}^j p_k \leq 2 \cdot C_j$

Proof: $C_j \sum_{k=1}^j \mathbb{E}[X_k] \geq \sum_{k=1}^j \mathbb{E}[X_k] \cdot C_k \geq \text{rhs of (4)}$
 $C_1 \leq C_2 \leq \dots$ (4)

$$= \left(\frac{1}{2m} \sum_{k=1}^j \mathbb{E}[X_k] \right)^2 + \frac{m - \Delta(m-1)}{2m} \sum_{k=1}^j \mathbb{E}[X_k]^2$$

rewriting

$$\stackrel{=} \quad C_j \geq \frac{1}{2m} \sum_{k=1}^j \mathbb{E}[X_k] + \frac{m - \Delta(m-1)}{2m} \cdot \underbrace{\frac{\sum_k \mathbb{E}[X_k]^2}{\sum_k \mathbb{E}[X_k]}}$$

\uparrow
division

CASE 1: $\Delta \leq \frac{m}{m-1} \Rightarrow$ positive

$$\Rightarrow 2C_j \geq \frac{1}{m} \sum_{k=1}^j \mathbb{E}[X_k] \stackrel{?}{=} \text{La. 14.2}$$

CASE 2: $\Delta > \frac{m}{m-1} \Rightarrow$ negative

use $C_j \geq C_k \geq \mathbb{E}[X_k]$ for $k = 1, \dots, j$

$$\Rightarrow C_j \geq \max_{k=1, \dots, j} \mathbb{E}[X_k] \geq \left(\sum_k \mathbb{E}[X_k]^2 \right) / \sum_k \mathbb{E}[X_k]$$

let $\mathbb{E}[X_i] = \max_k \mathbb{E}[X_k]$

$$\Rightarrow \sum_k \mathbb{E}[X_k]^2 \leq \mathbb{E}[X_i] \cdot \sum_k \mathbb{E}[X_k]$$

$$\Rightarrow \mathbb{E}[X_j] \geq \frac{\sum_k \mathbb{E}[X_k]^2}{\sum \mathbb{E}[X_k]}$$

$$\therefore = \sum_{k=1}^m \sum_{j=1}^d \mathbb{E}[X_k] \leq \left(1 + \max\{1, \frac{m-1}{m} \Delta\}\right) C_j \quad \square$$

Case 1, 2

14.10 THEOREM Let Π the job-based priority policy induced by the

$$(LP) \quad \begin{cases} \min \sum_j w_j C_j^{LP} & \text{s.t.} \\ (4) \\ C_j^{LP} \geq \mathbb{E}[X_j] \end{cases}$$

Then Π is a $(2 + \max\{1, \frac{m-1}{m} \Delta\})$ -approximation

Proof: Let $C_1^{LP} \leq \dots \leq C_m^{LP}$ be an optimal solution of (LP) and let $L: 1 < 2 < 3 \dots < n$ the priority list for policy Π

$$\text{Then } \mathbb{E}[C_j^\Pi] \leq \frac{1}{m} \sum_{k=1}^j \mathbb{E}[X_k] + \mathbb{E}[X_j]$$

La 14.4

$$= \frac{1}{m} \sum_{k=1}^j \mathbb{E}[X_k] + \underbrace{\frac{m-1}{m} \mathbb{E}[X_j]}_{\leq (1 + \max\{\dots\}) C_j^{LP}} \leq \frac{m-1}{m} C_j^{LP}$$

La 14.9

$$\leq \underbrace{\left(2 - \frac{1}{m} + \max\{\dots\}\right)}_{=: \alpha} C_j^{LP}$$

$$= \sum w_j \mathbb{E}[C_j^\Pi] \leq \alpha \cdot \sum w_j C_j^{LP} \leq \alpha \cdot OPT$$

↑ expected cost of an
optimal policy

Remarks

- (1) LP can be solved in polynomial time (without proof)
- (2) WSEPT (weighted shortest expected processing time first)
leads to guarantee $1 + \frac{(\Delta+1)(m-1)}{2m} \approx \alpha - 1$ different from our job-based priority policy
- (3) can be generalized to release dates $r_j \geq 0$
 \Rightarrow guarantee $\alpha + 1$
need job-based priority policies for that

More on Remark (1)

14.11 Theorem: Assume $w_1/E[X_1] \geq \dots \geq w_n/E[X_n]$

Then (LP) has the optimal solution

$$C_j^{LP} = \frac{1}{m} \sum_{k=1}^j E[X_k] - \frac{(\Delta-1)(m-1)}{2m} E[X_j] \quad j=1, \dots, n$$

Proof idea: rhs of (4) seen as function $f(A)$
is a supermodular function

- \Rightarrow the polyhedron defined by (4) is a supermodular polyhedron
- \Rightarrow optimal solution can be obtained by Edmonds' greedy algorithm for polyhedra

Exercises

14.1 Show that WSEPT leads to a $1 + \frac{(\Delta+1)(m-1)}{2m}$ guarantee (m identical machine, no prec) \Rightarrow WSEPT is optimal for 1 machine

14.2 Generalize the results of this paragraph to the case with release dates (Use job-based priority rules)

14.3 Show that WSEPT may be arbitrarily bad for release dates

Role of the coefficient of variation

Assumed $\text{CV}[X_j]^2 \leq \Delta$ Δ enters performance guarantee

Is this avoidable?

Not for some classes of policies

- ▶ [Skutella, Sviridenko, Uetz 2014]
(in the context of unrelated machines)

The performance ratio of any fixed-assignment policy can be as large as $(1-\delta)\Delta$ for any $\delta > 0$, for large enough m

- ▶ [Labonté, M., Megow 2014]
for every k , there is an instance I_k with $\Delta \leq k$, such that the performance ratio of $WSEPT$ is as large as $\Omega(\sqrt[4]{k})$