

## §II Characterizing ES and preselective policies

Have shown:

- ES-policies are convex, continuous, monotone
- preselective policies are continuous, monotone

Now: They are already characterized by these properties

II.1 THEOREM: Every monotone policy is dominated by a preselective policy

II.2. COROLLARY: let  $\pi$  be an arbitrary policy. Then

$\pi$  is monotone  $\Leftrightarrow \pi$  is preselective but not necessarily

↑  
earliest start

chooses a waiting job on every  $f \in F$

Proof: Consider  $[G, F]$ . Let  $A$  be an antichain of  $G$

and  $x$  a processing time vector

Call job  $j$  selected for  $(A, x)$  : $\Leftrightarrow$

$j$  waits for any  $i \in A \setminus \{j\}$   
for all  $y$  with  $y \geq_A x$

$y$  only longer on  $A$

i.e.  $y_k \geq x_k \wedge k \in A, y_k = x_k$  otherwise

i.e. for every such  $y$  there is  $i \in A$  (may depend on  $y$ )  
with  $\pi[y](i) + y_i \leq \pi[y](j)$

Intuition for this notion:

for  $x$ , several jobs might wait

making jobs of  $A$  larger reveal the "real" selected job

let  $S(A, x)$  be the set of jobs selected for  $(A, x)$

$$(1) \boxed{F \in \mathcal{F} \Rightarrow S(F, x) \neq \emptyset} \quad (\text{holds for every policy})$$

Consider  $x^m$  with  $x_k^m := \begin{cases} x_k & k \notin F \\ x_k + m & k \in F \end{cases}$  "long on  $F$ "

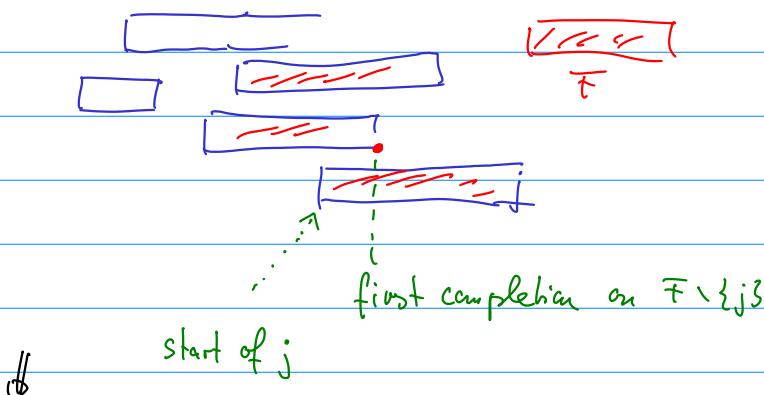
$\Pi$  policy  $\Rightarrow \exists$  job  $j_m$  that waits w.r.t.  $x^m$   
 $\Rightarrow$  some job  $j$  occurs infinitely often in sequence  $(j_m)_{m \in \mathbb{N}}$

Claim  $\boxed{j \in S(F, x)}$

Suppose not, say  $j$  does not wait w.r.t.  $y$  with  $y \geq_F x$

$$\text{Then } \Pi[y](j) < \min_{\substack{i \in F \\ i \neq j}} \{ \Pi[y](i) + x_i \}$$

first completion on  $F \setminus \{j\}$



start of  $j$  under  $\Pi$  remains the same if we enlarge processing times on  $F$  (same history up to the start of  $j$ )

$\hookrightarrow \boxed{\text{non-waiting is invariant under } F\text{-monotonicity}} \quad (1^*)$

↯

$j$  does not wait for all  $x^m$  with  $x^m|_F \geq y|_F$

$\Rightarrow j$  does not occur infinitely often in the sequence  $(j_m)_m$

$\Rightarrow$  contradiction

$$(2) \boxed{(\forall F, \forall x, \forall y : x \leq y \Rightarrow S(F, x) \subseteq S(F, y))}$$

$\Rightarrow \Pi$  preselective (maybe not earliest start)

Suppose (...) is valid, but  $\Pi$  is not preselective

$\Rightarrow \exists F \in \mathcal{F}$  such that  $\Pi$  does not select a fixed waiting job

$\Rightarrow \forall j \in F \exists x^j$  s.t.  $j$  does not wait w.r.t.  $x^j$

$\stackrel{(1*)}{\Rightarrow} j \notin S(F, x^j)$

$$\begin{array}{lll}
j_1 \in F & (x_1^{j_1}, \dots, x_n^{j_1}) & x^{j_1} \\
j_2 \in F & (x_1^{j_2}, \dots, x_n^{j_2}) & x^{j_2} \quad \leftarrow j_0 \\
\vdots & \textcircled{O} & \\
j_k \in F & (x_1^{j_k}, \dots, x_n^{j_k}) & x^{j_k}
\end{array}$$

Consider  $x := \min_{j \in F} x^j$  componentwise  $\Rightarrow x > 0$

contradiction

Let  $j_0 \in S(F, x) \subseteq S(F, x^{j_0})$

$x \leq x^{j_0}$   $\underbrace{\dots}_{\text{...}}$

$$(3) \boxed{\Pi \text{ monotone} \Rightarrow (\forall F, \forall x, \forall y : x \leq y \Rightarrow S(F, x) \subseteq S(F, y))}$$

Suppose not  $\Rightarrow \exists F, \exists x, \exists y$  with  $x \leq y$  but  $S(F, x) \not\subseteq S(F, y)$

$\Rightarrow \exists j \in S(F, x) \setminus S(F, y)$

Consider  $x^m, y^m$  as above

$j \in S(\bar{\pi}, x)$   $\Rightarrow j$  waits for  $x$ , and thus for all  $x^m$

$j \in S(\bar{\pi}, y)$   $\Rightarrow j$  does not wait for some  $\bar{y} \geq_{\bar{\pi}} y$

$\Rightarrow j$  does not wait for all  $y^m \geq_{\bar{\pi}} \bar{y}$   
( $I^x_j$ )

$\Rightarrow m \leq \bar{\pi}[x^m](j) \leq \bar{\pi}[y^m](j) = \bar{\pi}[\bar{y}](j) \quad \forall m$

$j \in S(\bar{\pi}, x^m)$        $\bar{\pi}$  monotone

$$x^m \leq y^m$$

$\Rightarrow$  contradiction       $\square$

Similar techniques:

II.3 THEOREM: Every continuous and elementary policy is preselective

without proof.

II.4 COROLLARY: Let  $\bar{\pi}$  be an arbitrary policy. Then

$\bar{\pi}$  is preselective  $\Leftrightarrow \bar{\pi}$  monotone (up to earliest start)

$\Leftrightarrow \bar{\pi}$  continuous and elementary

II.5 CONSEQUENCES

(1) Graham anomalies of type a) occur in pairs

e.g. non-continuity  $\Rightarrow$  non-monotonicity

(2) Preselative policies = natural class of policies fulfilling stability  
(needs continuity)

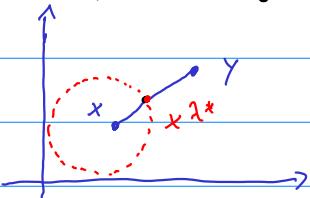
II.6.  $\pi$  convex policy  $\iff \pi$  ES-policy

Proof:

(1)  $\pi$  convex  $\Rightarrow \pi$  is poselective (up to earliest start)

Show:  $x \leq y \Rightarrow S(F, x) \subseteq S(F, y) \quad \forall x, y, F$

Let  $x \leq y$  but  $j \in S(F, x) \setminus S(F, y)$



Consider line  $x^2 = (1-\lambda)x + \lambda y \quad 0 \leq \lambda \leq 1$

Let  $\lambda^*$  be the first  $\lambda$  with  
 $j \notin S(F, x^2)$

w.l.o.g.  $x, y$  close enough to  $x^{*\lambda}$  such that  $z := 2x - y > 0$

$$\Rightarrow x = \frac{1}{2}y + \frac{1}{2}z$$

Consider  $x^m, y^m$  as before (add  $m$  to all jobs in  $F$ )

$$\Rightarrow x^m = \frac{1}{2}y^{2m} + \frac{1}{2}z$$

$$= \frac{1}{2}y^{2m} + x - \frac{1}{2}y = \frac{1}{2}y + \pi_F^m + x - \frac{1}{2}y$$

$$= x + \pi_F^m = x^m$$

$$m \leq \pi[x^m](j) \leq \frac{1}{2}\pi[y^{2m}](j) + \frac{1}{2}\pi[z](j)$$

$j \in S(F, x)$

$\pi$  convex

$$= \frac{1}{2}\pi[y](j) + \frac{1}{2}\pi[z](j) \quad \forall m$$

$\uparrow$

$j \notin S(F, y)$

invariance of non-waiting under  $F$ -monotonicity

= constant

(2)  $\bar{\pi}$  convex  $\Rightarrow$   $\exists$  ES-policy  $\pi^* \leq \bar{\pi}$

$\bar{\pi}$  convex  $\stackrel{(r)}{=} \bar{\pi}$  preselective  $\Rightarrow \exists$  waiting job  $j$  for every  $F \in \mathcal{F}$

Claim:  $j$  waits always for the same job (independent of  $x$ )

Suppose not:  $\Rightarrow$  for every  $i \in F \setminus \{j\}$  there is  $x^i$  with  
 j. does not wait for  $i$  w.r.t.  $x^i$   
 i.e.  $\bar{\pi}[x^i](i) + x_i^i > \bar{\pi}[x^i](j)$

Consider  $x^{i,m}$  (make jobs in  $F$  longer by  $m$ )

$\Rightarrow \bar{\pi}[x^{i,m}](j)$  does not change

↳ monotonicity of non waiting ( $\pi^*$ )

Let  $z^m := \frac{1}{|F|-1} \sum_{i \in F \setminus \{j\}} x^{i,m}$  convex combination

$$m \leq \bar{\pi}[z^m](j) \leq \frac{1}{|F|-1} \sum_{i \in F \setminus \{j\}} \bar{\pi}[x^{i,m}](j)$$

$\uparrow$   $\bar{\pi}$  convex

$j$  is waiting job on  $F$

$\underbrace{\hspace{10em}}$

constant

$\Rightarrow$  contradiction

$\Rightarrow$  every  $F$  is settled by a waiting pair  $i < j$

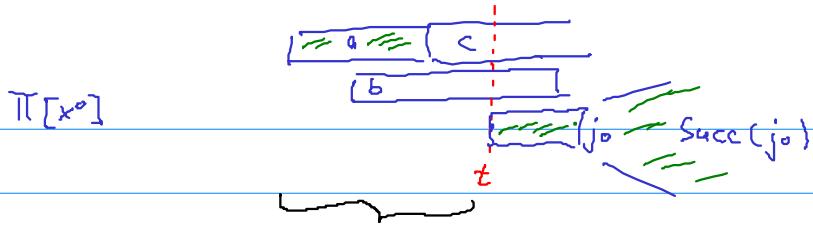
$\Rightarrow \exists$  ES-policy  $\pi^* \leq \bar{\pi}$

$\uparrow$  has the same waiting pairs and is Earliest Start

(3)  $\bar{\pi}$  is elementary

Suppose not:  $\Rightarrow \exists x^o$  and  $j_o$  s.t.  $j_o$  starts in  $\bar{\pi}[x^o]$

at a time  $t$  at which no  $j \neq j_o$  ends



$C(t) = \text{set of completed jobs until } t \quad \left\{ \begin{array}{l} \text{in } \Pi[x^0] \\ \text{in } \Pi[x] \end{array} \right.$

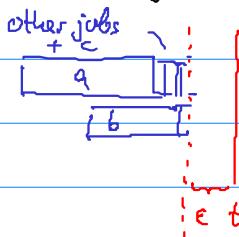
$B(t) = \text{set of busy jobs at } t$

Let  $X := \text{set of all } x \text{ that look the same to } \Pi \text{ at } t$   
 $= \{x \in \mathbb{R}^n \mid x_j = x_j^0 \ \forall j \in C(t), \bar{x}_j = \bar{x}_j^0 \ \forall j \in B(t)\}$

$\Rightarrow \Pi[x](j_0) = t \quad \forall x \in X$

Let  $z \in \mathbb{R}^n$  with  $z_j = x_j^0 \ \forall j \in C(t) \cup \text{Succ}_G(j_0) \cup \{j_0\}$   
in the picture  
 $z_j$  short for all other jobs

s.t. all these jobs end before  $t$ , say at  $t - \varepsilon$



only jobs  $j_0$  and its successors are left after  $t - \varepsilon$

$z \notin X$ , policies have no total idle time (no job is busy)  
 $\Rightarrow \Pi[z](j_0) = t - \varepsilon < t$

Choose  $y \in X$  with  $\frac{1}{2}y + \frac{1}{2}z \in X$  (choose  $y_j$  large for  $j \in B(t)$  in  $\Pi[x^0]$ )

$$\text{Then } t = \Pi\left[\frac{1}{2}y + \frac{1}{2}z\right](j_0) \leq \underbrace{\frac{1}{2}\Pi[y](j_0)}_{\Pi \text{ convex}} + \underbrace{\frac{1}{2}\Pi[z](j_0)}_{< t} < t$$

contradiction

(4)  $\bar{\pi}$  is an ES-policy

consider the most restrictive problem  $[G^*, F^*]$  for which  $\bar{\pi}$  is still a policy

- add to  $G$  all  $i < j$  with  $\bar{\pi}[x](i) + x_i \leq \bar{\pi}[x](j)$   $\forall x$
- let  $F^*$  be the set of all antichains of  $G^*$  that are not scheduled simultaneously by  $\bar{\pi}$   $\forall x$

iff

$\bar{\pi}$  is a convex policy for  $[G^*, F^*]$

$\stackrel{(1)}{\Rightarrow}$   $\exists$  ES-policy  $\pi^*$  for  $[G^*, F^*]$  with  $\pi^* \leq \bar{\pi}$

Claim

$$\pi^* = \bar{\pi}$$

proof by induction along the decision times of  $\bar{\pi}$  for arbitrary fixed vector  $x$

$t=0$ :

$S^*(0)$  = set of jobs started by  $\pi^*$  at  $t=0$

$S(0)$  =  $\dots \bar{\pi} \dots$

If  $S^*(0) \neq S(0)$   $\stackrel{\pi^* \leq \bar{\pi}}{=} \exists j \in S^*(0) \setminus S(0)$

$\bar{\pi}$  elementary  $\Rightarrow j$  waits for another completion

$\Rightarrow S^*(0)$  not simultaneously in  $\bar{\pi}[x]$

$\Rightarrow S^*(0)$  either forbidden or not an antichain

Def ↑  
most restrictive

$\Rightarrow$  contradiction to the fact that  $S^*(0)$  is scheduled simultaneously in  $\pi^*[x]$

inductive step:  $S(t') = S^*(t')$   $\forall$  decision times  $t' < t$

consider now  $t =$  next completion time, and thus the same in  $\bar{\pi}$  and  $\pi^*$   
 same argument as for  $t=0 \Rightarrow S(t) = S^*(t)$   $\square$