

### §3 The stochastic project scheduling model

Now random processing times  $X_j$  instead of fixed  $x_j$ .

Assume that we know the joint distribution  $P$  of  $(X_1, \dots, X_n) = X$   
 [mostly, the  $X_j$  will be independent, but not necessarily]

$[G, P]$  is called a stochastic project network

For a (measurable) performance function  $\kappa$

↑  
 always assumed in the future, holds for  $C_{max}$   
 and all the others considered here

the distribution  $P_{\kappa^G}$  of  $\kappa^G(\cdot)$  is well defined

Remember:  $\kappa^G(x) = \kappa \left( \underbrace{ES_G(x)}_{\text{stetig}} + x \right)$   
 $\qquad\qquad\qquad \underbrace{\qquad\qquad}_{\text{measurable}}$

The underestimation error of deterministic planning -

3.1 Theorem: Let  $[G, P]$  be a stochastic project network

Let  $E(X) = (E(X_1), \dots, E(X_n))$  be the vector of average job processing times.

If  $\kappa$  is convex, then  $\kappa^G(E(X)) \leq E(\kappa^G(X))$

i.e. the cost based on average processing times  
 underestimates the expected cost

Proof:  $\kappa^G(x) = \kappa \left( \underbrace{ES_G(x)}_{\text{convex}} + x \right) =: f(x_1, \dots, x_n)$   
 $\qquad\qquad\qquad \underbrace{\qquad\qquad}_{\text{convex}}$

$\Rightarrow$  Theorem follows from Jensen's inequality for convex functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}$

$$f(\mathbb{E}(X_1), \dots, \mathbb{E}(X_n)) \leq \mathbb{E}(f(X_1, \dots, X_n))$$

Expl. for convex  $K$ :  $C_{\max}, \sum w_j C_j, \sum w_j T_j, T_{\max}$

elementary proof for  $K = C_{\max}$

Let  $\mathcal{C} := \{C^1, \dots, C^m\}$  be the set of maximal chains of  $G$

For  $C^i$ , let  $Y_i := \sum_{j \in C^i} X_j$  be the (random) length of  $C^i$

Then  $C_{\max}^G(X) = \max_i Y_i$  and

$$C_{\max}^G(\mathbb{E}(X)) = \max_i \sum_{j \in C^i} \mathbb{E}(X_j) = \max_i \mathbb{E}\left(\underbrace{\sum_{j \in C^i} X_j}_{Y_i}\right) = \max_i \mathbb{E}(Y_i)$$

$$= \mathbb{E}(Y_{i_0}) \quad \leq \quad \mathbb{E}(\max_i Y_i) = \mathbb{E}(C_{\max}^G(X))$$

$\uparrow$  max attained for  $i = i_0$

since  $Y_{i_0} \leq \max_i Y_i$

□

Remarks:

- (1) Equality holds for  $C_{\max}$  iff one chain is critical with probability 1
- (2) Error can get arbitrarily large
  - with growing  $n$
  - with fixed  $n$  and growing variance of the  $X_i$

Expl:  $G_n \stackrel{\text{def}}{=} n\text{-element antichain}$

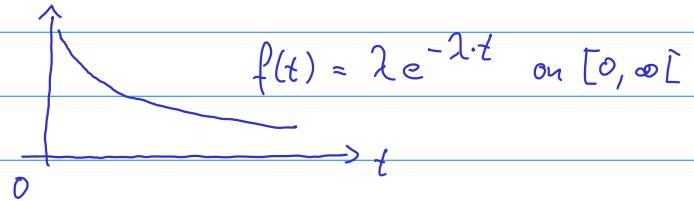
①  
②  
⋮  
④

Every  $X_j \sim \exp(\lambda)$  with  $\lambda = 1$ , independent

↑

exponential distribution  $E(X_j) = \frac{1}{\lambda}$

density function



$$f(t) = \lambda e^{-\lambda t} \text{ on } [0, \infty]$$

these distributions are memoryless!

the distribution of  $X_j$  conditioned on  $X_j \geq t$

is the same as the distribution of  $X_j$

Then  $E(C_{\max}^{G_n}(X)) = E(\text{until first completion}) + E(\text{remaining jobs})$

$$= E(\min X_j) + E(C_{\max}^{G_{n-1}}(\tilde{X}))$$

↑  
one job less  
memoryless property

$$= \frac{1}{\lambda_1 + \dots + \lambda_n} + E(\dots)$$

↑ min of exp. distributions  $X_1 \sim \exp(\lambda_1), \dots, X_n \sim \exp(\lambda_n)$

is exp dist.  $\exp(\lambda_1 + \dots + \lambda_n)$

$$= \frac{1}{n} + E(\dots)$$

$$= \sum_{i=1}^n \frac{1}{i} \quad \text{harmonic series}$$

induction

$\sim \log n + \text{Eulerian constant}$

0.577

On the other hand  $C_{\max}^{G_n}(E(X)) = 1$   
 $\Rightarrow$  absolute error, relative error  $\rightarrow \infty$

### Summary

- stochastic influences are important, need to be considered
- has led to PERT method at NASA in 50ies

↑

bad, considers only distribution of  $Y_{i_0} \leftarrow E(Y_{i_0}) = \max_i E(Y_i)$

### Homework

3. Show: Equality holds for  $C_{\max}$  iff one chain is critical

with probability 1

↑

in the proof of Theorem 3.1