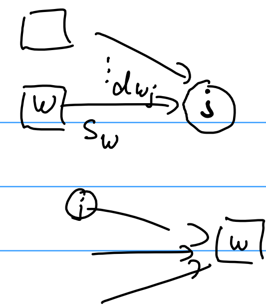


$$j \in V \quad \text{AND node:} \quad S_j \geq \max_{(w,j)\in A} \{S_w + d_{wj}\}$$

$$w \in \mathcal{W} \quad \text{OR node:} \quad S_w \geq \min_{(j,w)\in A} \{S_j + d_{jw}\}$$

min-max system

$(\infty, \infty, \ldots, \infty)$ is a solution

$(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$ solutions $\Rightarrow$ $(\min\{S_1, T_1\}, \ldots, \min\{S_n, T_n\})$ solution

Certificate for SOLVABILITY ∈ NP

Any feasible schedule $S = (S_1, \ldots, S_n)$ is a certificate for SOLVABILITY ∈ NP

Certificate for SOLVABILITY ∈ coNP

Let $S = (S_1, \ldots, S_n)$ be the unique minimal solution (maybe with ∞)

For every AND node j, one can delete all but one incoming arcs without changing S_j

requires a little proof

Then every cycle has non-negative length

requires a little proof

Relaxing in every AND-node
$\Rightarrow$ relaxed problem with only min inequalities (~ OR nodes)
$\Rightarrow$ can check $S_j > K$ by shortest path algorithms in polynomial time
$\Rightarrow$ relaxed problem is certificate for SOLVABILITY ∈ coNP

Complexity of checking solvability

A schedule and a ~~tightened~~ relaxed subproblem are polynomially checkable certificates for membership in NP and coNP

$\Downarrow$

SOLVABILITY $\in$ NP $\cap$ coNP

for arbitrary MIN-MAX systems

no polynomial algorithm known, not known to be NP-complete
or coNP-complete

same complexity status as LINEAR PROGRAMMING and PRIMES

now known to be in P
(Ellipsoid method)

$\in$ P
2002 Agarwal + ...
NP-certificate requires
number theory

$\exists$ simple pseudo-polyn. algorithm for MIN-MAX systems

Exercises:

10.1 Show that Algorithm 10.4. can be implemented to run in $O(|V| + |W| \cdot \log |W| + |A|)$ time

10.2* Derive a polynomial-time algorithm for finding the unique minimal (feasible) solution ≥ 0 of a min-max system with non-negative arc weights

Hint: Try to relate feasibility of the min-max system to structural feasibility in the sense of Lemma 10.6. What changes?

10.3 Derive a pseudopolynomial-time algorithm for finding the unique minimal (feasible) solution ≥ 0 of a min max system with arbitrary arc weights

§ 11 Characterizing ES- and preselective policies

Have shown

- ES policies are convex, continuous, monotone
- preselective policies are continuous, monotone

Now: They are already characterized by these properties

11.1 Thm. Every monotone policy Π is dominated by a preselective policy Π^*, i.e. $\Pi^* \leq \Pi$

(Π also selects waiting jobs on forbidden sets, but need not be earliest start)

Corollary 11.2

Proof: Consider $[G, \mathcal{F}]$, let A be an antichain of G and x be a vector of processing times

Call job j selected for (A, x) $:\Longleftrightarrow$ j waits for any $i \in A$ for all y with $y \geq_A x$

only larger than x on A

i.e. $y_k \geq x_k$ $\forall k \in A$, $y_k = x_k$ otherwise

i.e. for every such y there is some $i \in A$ (may depend on y) with $\Pi[y](i) + y_i \leq \Pi[y](j)$

Intuition for this notion:

for x several jobs might wait for any $i \in A$

making jobs large on A reveals the "right" selected waiting job

Let $S(A,x)$ be the set of jobs selected for (A,x) (could be empty)

(1) $\boxed{F \in \mathcal{F} \Rightarrow S(F,x) \neq \emptyset}$ for our policy Π

Consider x^m with $x^m_k = \begin{cases} x_k & k \notin F \\ x_k + m & k \in F \end{cases}$ "long" on F, $m \in \mathbb{N}$

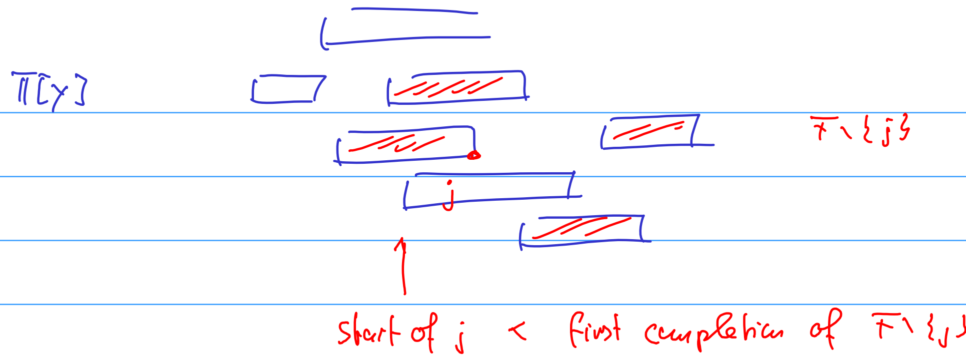
Π policy $\Rightarrow$ $\exists$ job j_m that waits w.r.t. x^m

$\Rightarrow$ some j occurs infinitely often in the sequence $(x^m)_{m \in \mathbb{N}}$

Claim $\boxed{j \in S(F,x)}$

Suppose not, say j does not wait for y with $y \geq_F x$

Then $\Pi[y](j) < \underbrace{\min_{i \in F,\, i \neq j} [\Pi[y](i) + y_i]}$ first completion on $F \setminus \{j\}$



Π non-anticipative $\Rightarrow$ increasing the length on F
does not change the start of j
(history is the same)

$\Rightarrow$ j does not wait for all x^u with $x^u|_F \geq y|_F$, contradiction

non-waiting is invariant under F-monotonicity (1*)

(2) $\left[\ \forall F,\ \forall x,\ \forall y:\quad x \leq y \Rightarrow S(F,x) \subseteq S(F,y)\ \right]$
$\Rightarrow$ Π is preselective (maybe not earliest start)

monotonicity of selected sets

Suppose that [...] is valid but Π is not preselective
$\Rightarrow$ $\exists\, F \in \mathcal{F}$ such that Π is not preselective on F
$\Rightarrow$ $\forall\, j \in F\ \exists\ x^j$ s.t. j does not wait w.r.t. x^j and F
$\overset{(1^*)}{\Rightarrow}$ $j \notin S(F, x^j)$ $\forall j \in F$

Consider $x := \min\limits_j x^j$ componentwise

Let $j_0 \in S(F,x) \neq \emptyset$ because of (1)

$\Rightarrow$ $x \leq x^{j_0}$ $\underset{[...]}{\Rightarrow}$ $j_0 \in S(F, x^{j_0})$

contradiction

(proof is a diagonalization argument)

$\mathcal{F}$ $\quad j: \quad x^{j_1} \quad (x_1^{j} \dots 0 \ x_n^{j})$

$j_0 \longrightarrow x^{j_2} \quad (x_1^{j_2} \dots x_n^{j_2}) \qquad j_i \notin S(\mathcal{F}, x^{j_i})$

$\vdots$

$x^{j_F}: \quad (x_1^{j_F} \ 0 \ x_n^{j_F})$

---

x component wise min

$\underset{(1)}{\Rightarrow} j_0 \in \mathcal{F}$ with $j_0 \in S(\mathcal{F}, x)$

$x \leq x^{j_2}$

(3) $\boxed{\Pi \text{ monotone } \Rightarrow [\dots]}$

Suppose not $\Rightarrow \exists\ \mathcal{F}, x, y : x \leq y$ but $S(\mathcal{F},x) \not\subseteq S(\mathcal{F},y)$

$\Rightarrow \exists\ j \in S(\mathcal{F},x) \setminus S(\mathcal{F},y)$

Consider x^m, y^m as above

$j \in S(\mathcal{F},x) \Rightarrow j$ waits for all x^m and x

$j \notin S(\mathcal{F},y) \Rightarrow j$ does not wait for some $\bar{y} \geq_{\mathcal{F}} y$

$\underset{(1^*)}{\Rightarrow} j$ does not wait for all $y^m \geq_{\mathcal{F}} \bar{y}$

(j even has the same start time as for $\bar{y}$)

$$\Rightarrow m \leq \Pi[x^m](j) \underset{\substack{\uparrow \\ \Pi \text{ monotone} \\ x^m \leq y^m}}{\leq} \Pi[y^m](j) = \underset{\substack{\uparrow \\ \text{fixed value}}}{\Pi[\bar{y}](j)} \quad \forall m$$

contradiction □

Similar techniques

11.3 THEOREM: Every continuous and elementary policy is preselective

11.4 Corollary: Let π be an arbitrary policy. Then

π is preselective $\iff$ π is monotone (up to earliest start)

$\iff$ π is continuous and elementary

11.5 Consequences

(1) Graham anomalies (of type a) come in pairs

(2) preselective policies are a natural class fulfilling stability requirements (need continuity, get thus preselectivity)

11.6 THEOREM: π convex $\Rightarrow$ π ES-policy

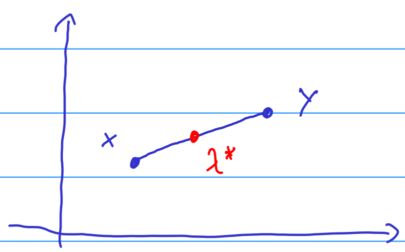
Proof:

(1) $\boxed{\pi \text{ convex } \Rightarrow \pi \text{ is dominated by a preselective policy}}$

Show $x \leq y \Rightarrow S(F,x) \subseteq S(F,y)$ $\quad \forall F, x, y$ with $x \leq y$

By contradiction, let $x \leq y$ but $j \in S(F,x) \setminus S(F,y)$

Consider line from x to y given by $x^\lambda = (1-\lambda)x + \lambda y \quad 0 \leq \lambda \leq 1$



Let λ^* be the first λ with $j \notin S(F, x^\lambda)$

w.l.o.g x, y are close enough to $x^{\lambda*}$

s.t. $z := 2x - y > 0$

$\Rightarrow z$ is a processing time vector

and $x = \frac{1}{2} y + \frac{1}{2} z$

Consider x^m, y^m as before

$\Rightarrow \quad x^m = \underbrace{\frac{1}{2} y^{2m} + \frac{1}{2} z}$

$\frac{1}{2} y^{2m} + x - \frac{1}{2} y = \frac{1}{2} y + \frac{1}{2} \mathbb{1}_F \cdot 2m + x - \frac{1}{2} y = x + \mathbb{1}_F \cdot m = x^m$

$$\Rightarrow \quad m \le \pi[x^m](j) \underset{\pi \text{ convex}}{\le} \frac{1}{2}\underbrace{\pi[y^{2m}](j)}_{\substack{\text{is a fixed}\\ \text{start for}\\ \text{some } \bar{y} \ge_F y\\ \text{because of } (1^*)}} + \frac{1}{2}\underbrace{\pi[z](j)}_{\text{fixed}} \qquad \forall m$$

$\Rightarrow$ contradiction $\square$

(2) $\boxed{\pi \text{ convex} \Rightarrow \exists \text{ ES policy } \pi^* \le \pi}$

π convex $\overset{(1)}{\Rightarrow}$ π is preselective $\Rightarrow$ $\exists$ waiting job j on $F \in \mathcal{F}$

Claim $\boxed{j \text{ waits always for the same job } i \in F}$

Suppose not: $\Rightarrow$ $\forall\, i \in F \setminus \{j\}$ $\exists$ x^i with j does not wait for i

Consider $x^{i,m}$ $\underset{(1^*)}{\Rightarrow}$ $\pi[x^{im}](j)$ does not change

Let $z^m := \frac{1}{|F|-1} \sum_{i \in F\setminus\{j\}} x^{im}$ convex combination

$$m \le \pi[z^m](j) \underset{\pi \text{ convex}}{\le} \frac{1}{|F|-1} \cdot \sum_i \pi[x^{im}](j) \qquad \forall m$$

fixed value

j is a waiting job on F

every $i \in F$ is at least $\frac{1}{|F|-1} \cdot m$ long w.r.t. x^{im}

$\Rightarrow$ .... m long .... z^m

$\Rightarrow$ contradiction

$\Rightarrow$ every F is settled by a waiting pair $i < j$

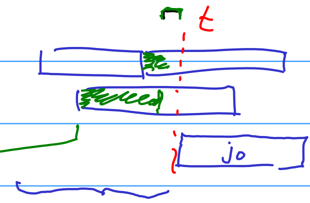
$\Rightarrow$ $\exists$ ES-policy $\pi^* \le \pi$

(3) $\boxed{\pi \text{ is elementary}}$

Suppose not $\Rightarrow$ $\exists$ x^0 and j_0 such that j_0 starts in $\pi[x^0]$ at a time t

where no other job ends

$\Pi[x^\circ]$



z makes busy jobs very short

$C(t)$ set of completed jobs until t } in $\Pi[x^\circ]$
$B(t)$ set of busy jobs at t

and also jobs ≠ j started after t that are not successors of j

let $X :=$ set of all $x \in \mathbb{R}^n_>$ that look the same to Π at time t

$\Rightarrow \Pi[x](j_0) = t \quad \forall x \in X$

↑ non-anticipativity

definition of z => all jobs started before t in $\Pi[z]$ end before t,
and only job j and its successors are left

$\Rightarrow z \notin X$

policies do (w.l.o.g.) not have total idle time (no job processed)

$\Rightarrow \Pi[z](j_0) < t$

Choose $y \in X$ with $\frac{1}{2} y + \frac{1}{2} z \in X$

↑ choose y_j large enough for $j \in B(t)$

Then $t = \Pi[\frac{1}{2}y + \frac{1}{2}z](j_0) \underset{\Pi \text{ convex}}{\leq} \frac{1}{2} \underbrace{\Pi[y](j_0)}_{t} + \frac{1}{2} \underbrace{\Pi[z](j_0)}_{<t} < t$, contradiction

(4) $\boxed{\Pi \text{ is an ES-policy}}$

consider the most restrictive problem $[G^*, \mathcal{F}^*]$ for which Π is still a policy

- add to G all $i <_j$ with $\Pi[x](i) + x_i \leq \Pi[x](j) \quad \forall x$

Note that G^* extends G

- Let $\mathcal{F}^*$ be all antichains of G^* that are not scheduled simultaneously by Π for all x

$\Downarrow$

Π is a convex policy for $[G^*, \mathcal{F}^*]$

$\Downarrow$

$\exists$ ES policy Π^* for $[G^*, \mathcal{F}^*]$ with $\Pi^* \leq \Pi$

Claim $\boxed{\Pi^* = \Pi}$

proof by induction along the decision times of Π for arbitrary fixed x

$\underline{t = 0:}$

$S^*(0)$ set of jobs started by Π^* at $t = 0$

$S(0)$ .... Π .....

if $S^*(0) \neq S(0)$ $\underset{\Pi^* \leq \Pi}{\Rightarrow}$ $\exists\, j \in S^*(0) \setminus S(0)$

Π is elementary $\Rightarrow$ j waits for another completion

$\Rightarrow$ $S^*(0)$ not simultaneously in $\Pi[x]$

$\Rightarrow$ $S^*(0)$ either forbidden in $\mathcal{F}^*$ or not antichain of G^*

$\Rightarrow$ Π^* not a policy for $[G^*, \mathcal{F}^*]$, contradiction

Inductive step: $S(t') = S^*(t)$ for all decision times $t' < t$

consider decision time t = next completion, and thus the same in Π^* and Π

$\Rightarrow$ same argument as at $t = 0$ gives $S(t) = S^*(t)$

So $\Pi = ES_{G^*}$ $\square$

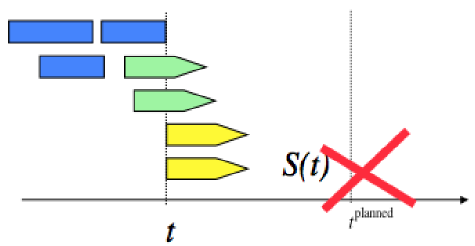
§12 SET-Policies

Set policies: A general class of robust policies

Only exploitable information at time t

- set of completed jobs
- set of busy jobs

Jobs start only at completions of other jobs



Special cases

- priority policies
- preselective policies

only info about the set of completed jobs is used

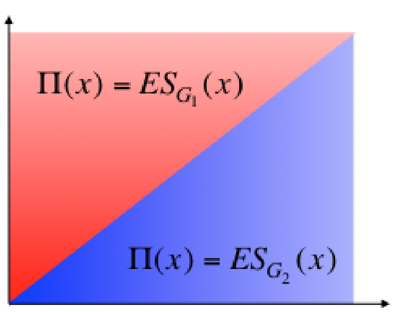
Set policies behave locally like ES-policies

For every set policy Π, there exists

- a partition of $\mathbb{R}^n_>$ into finitely many convex polyhedral cones $Z_1,\dots,Z_k$
- and feasible partial orders $G_1,\dots,G_k$

such that $\Pi(x) = ES_{G_i}(x)$ for $x \in Z_i$

even interval orders



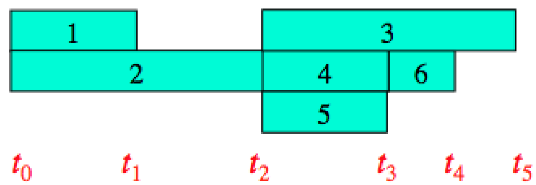
Graham anomalies only at boundaries of cones!

Stability for continuous distributions!

└ boundaries of cones have Lebesgue measure 0

Proof of local behaviour of set policies

Decisions depend only on sets
They determine an "abstract" interval order



Processing time vectors x leading to that interval order
fulfil a system of homogeneous equations and inequalities

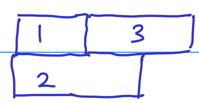
$x_1 < x_2 \quad x_4 + x_6 < x_3 \quad x_4 = x_5$ ⇨ polyhedral cone

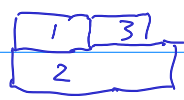
Example

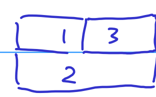
1. Π must make a decision at $t = 0$
2. $m = 2$ say, Π starts $1, 2$ at $t = 0$
3. then 3 as early as possible

} set policy

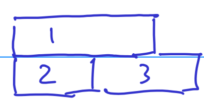
in total get 7 different possible "abstract" schedules defining polyhedral cones

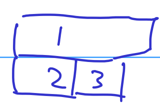
G_1  $x_1 < x_2 \quad x_2 < x_1 + x_3$

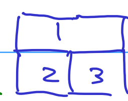
G_1  $x_1 < x_2 \;,\; x_2 > x_1 + x_3$

G_1  $x_1 < x_2 \;,\; x_2 = x_1 + x_3$

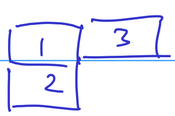
$x_1 < x_2$

G_2 

G_2 

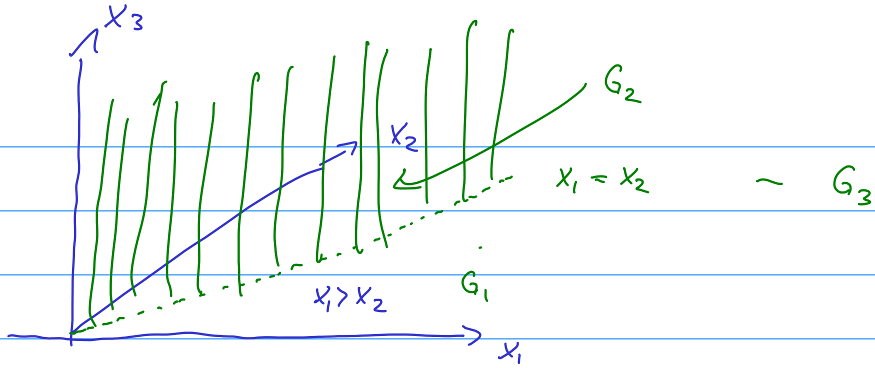
G_2 

$x_1 > x_2$

G_3  $x_1 = x_2$

but only three interval orders are induced

G_1	G_2	G_3
$1 \to 3$, 2	$2 \to 3$, 1	$1 \to 3$, $2 \to 3$

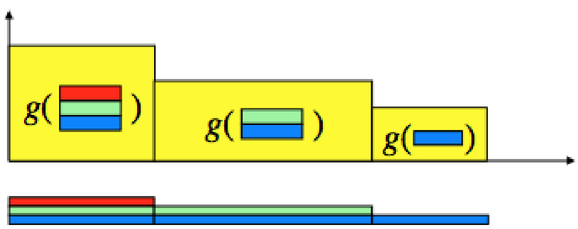


Optimality of set policies

If • all jobs are exponentially distributed and independent
• the cost function κ is *additive*
then there is an optimal set policy Π (among all policies).

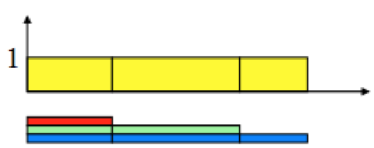
1985
M., Radermacher, Weiss

κ is *additive* if there is a set function $g: 2^V \to \mathbb{R}$ (the *cost rate*)
with $\kappa(C_1, ..., C_n) = \int g(U(t))dt$ $U(t) =$ set of uncompleted jobs at t



Special cases of optimal set policies

$\kappa = C_{\max}$

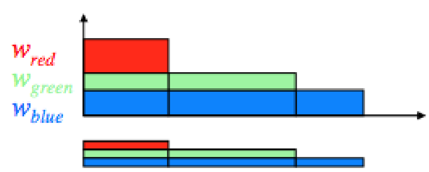


$$g(U) = \begin{cases} 1 \text{ if } U \neq \emptyset \\ 0 \text{ if } U = \emptyset \end{cases}$$

LEPT is optimal for
$P \mid p_j \sim \exp \mid C_{\max}$

Weiss '82

$\kappa = \sum w_j C_j$



$$g(U) = \sum_{j \in U} w_j$$

SEPT is optimal for
$P \mid p_j \sim \exp \mid \sum C_j$

no weights

Weiss & Pinedo '80

Given a set policy Π and independent exponential processing times, and additive cost function κ
its expected cost can be calculated via a decision tree

$$\mathbb{E}[\kappa^\Pi] = \text{cost until first expected completion} + \text{expected cost for remaining problem}$$

same problem with less jobs because exponential distributions are memory-less

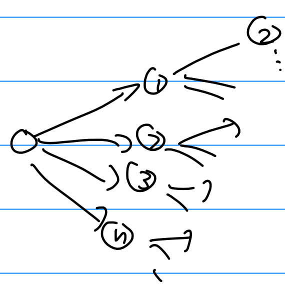
$$= \sum_{\substack{\text{all possible sets } C \\ \text{of jobs that} \\ \text{complete together first}}} Q(\text{completion of } C)\cdot\Big[\text{expected completion}\cdot g(V) + E[\kappa^\Pi_{\text{remaining problem}}]\Big]$$

$$= \sum_{j\in V} Q(j \text{ ends first})\cdot\big[E(\text{completion of } j)\big]\cdot g(V) + E[\kappa^\Pi_{\text{remaining problem}}]\big]$$

double completions have prob. 0

$$= \sum_{j\in V} \frac{\lambda_j}{\lambda_1+\dots+\lambda_n}\left(\frac{1}{\lambda_j}\cdot g(V) + E[\kappa^\Pi_{\text{remaining problem}}]\right)$$

$\Rightarrow$ tree



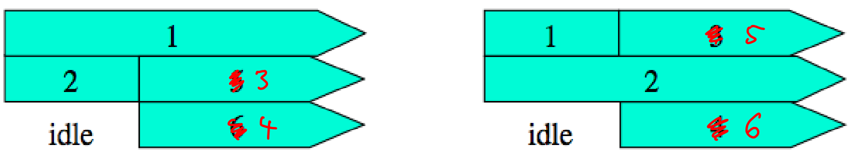
Set policies may be involved

No precedence constraints, $m = 3$ identical machines, 6 jobs

$X_j \sim \exp(a_j)$ with $a_1 = a_2 = a > 1,\ a_3 = a_4 = a_5 = a_6 = 1$

$$g(U) = \begin{cases} w \gg 1 & \text{if } 1 \in U \text{ and } 2 \in U \\ 1 & \text{if } 1 \notin U \text{ and } (2,5 \in U \text{ or } 2,6 \in U) \\ 1 & \text{if } 2 \notin U \text{ and } (1,3 \in U \text{ or } 1,4 \in U) \\ 0 & \text{otherwise} \end{cases}$$

Optimal set policy involves deliberate idleness



Q: what are conditions on $g(U)$ such that the optimal set policy does not have idle time ?

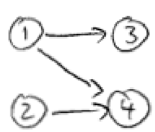
Candidate $\quad \sum w_j C_j \quad$ open

strange $\quad g$ is submodular ??

Exercises $\quad$ Exercise is relevant

12.1 $\quad$ Calculate the expected makespan by the algorithm in Remark 12.6 for the ES-policy $\Pi = ES_H$ with

H:



$X_j \sim \exp(\lambda_j)$ with $\lambda_1 = 1 \quad \lambda_2 = 2$

$\lambda_3 = 2 \quad \lambda_4 = 1$