§15 Stochastic online scheduling for $\sum_j w_j C_j$

Consider online model for $m \,||\, E[\Sigma w_j C_j]$

Jobs are presented to the scheduler sequentially one by one

- each job must directly be assigned to one of the machines (without knowing the number of future jobs)
- after assignment, the jobs are sequenced on the machines

Main result (Megow et al 2006)

$\boxed{\text{a simple assignment policy obtains the same approximation ratio as WSEPT } \left(1+\frac{(\Delta+1)(m-1)}{2m}, \text{ see Exercise 14.1}\right)}$

Notation:

$j \to i$ if job j is assigned to machine i

priority order $w_j / E[X_j]$ (larger is higher)

$H(j) = \{k \in V \mid$ higher priority than $j\} \cup \{j\}$

$L(j) = V - H(j)$ lower priority jobs

tie breaking: according to incoming sequence $1, 2, \dots$
so all priorities are different!

15.1 Algorithm [Algorithm MinIncrease MI]

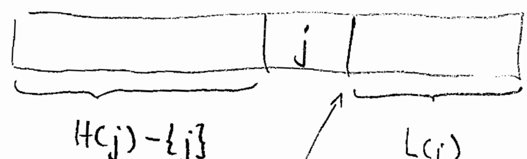
(1) Upon arrival of a job j, assign it to machine i that minimizes $w_j \sum_{k \in H(j),\, k<j,\, k \to i} E[X_k] + E[X_j] \sum_{k \in L(j),\, k \to i} w_k + w_j E[X_j]$

$=: z(j,i)$

(2) Once all jobs have been assigned to machines, schedule those on each machine from higher to lower priority (non-increasing $w_j / \mathbb{E}[X_j]$)

15.2 Lemma: $z(j,i)$ is the increase of $\sum w_j \mathbb{E}[C_j]$ on machine i when job j is added and jobs are processed by non-increasing $w_j / \mathbb{E}[X_j]$.

Proof:



delayed by $\mathbb{E}[X_j]$

$$z(j,i) = w_j\Big(\sum_{\substack{k\in H(j)\\ k<j}} \mathbb{E}[X_k] + \mathbb{E}[X_j]\Big) + \mathbb{E}[X_j]\cdot\Big(\sum_{\substack{k\in L(j)\\ k<j}} w_k\Big)$$ □

15.3 LEMMA:

$$\underbrace{\mathbb{E}\Big[\sum_j w_j C_j^{MI}\Big]}_{\text{expected value of policy MI}} = \sum_j \underbrace{\min_i z(j,i)}_{\text{min increase for assigning } j \text{ at its arrival}}$$

Proof: Let $C_j := C_j^{MI}$

$$\mathbb{E}\Big[\sum_j w_j C_j\Big] = \underbrace{\sum_j w_j \sum_{k\in H(j),\, k\to i_j} \mathbb{E}[X_k]}$$

partition into jobs that arrive before/after j

i_j = machine to which j is assigned

$$= \sum_j w_j \sum_{k \in H(j),\, k \to i,\, \underbrace{k<j}_{\text{arrive before}}} E[X_k] + \underbrace{\sum_j w_j \sum_{\substack{k \in H(j) \\ k \to i \\ k > i}} E[X_k]}_{\text{arrive after}\ =:\ A} + \sum_j w_j E[X_j]$$

job j itself

Claim: $\overbrace{\sum_j w_j \sum_{\substack{k \in H(j) \\ k > j \\ k \to i}} E[X_j]}^{A} = \overbrace{\sum_j E[X_j] \sum_{\substack{k \in L(j) \\ k < j \\ k \to i}} w_k}^{=:B}$ on each machine i

Proof Claim: Use different counting (row and colums in 2d - Gantt chart)

Expl: 5 jobs on a machine, priority order $4 < 3 < 1 < 2 < 5$ arrive in order 1 2 3 4 5 $\quad p_j = E[X_j]$

$\Rightarrow$ left sum:

j:
$w_1 \cdot (p_3 + p_4)$
$w_2 \cdot (p_3 + p_4)$
$w_3 \cdot (p_4)$
$w_4 \cdot (0)$
$w_5 \cdot (0)$

higher priority come after j

right sum:
$p_1 \cdot (0)$
$p_2 \cdot (0)$
$p_3 \cdot (w_1 + w_2)$
$p_4 \cdot (w_1 + w_2 + w_3)$
$p_5 \cdot (0)$

have lower priority come before job j

$$\Rightarrow E\left[\sum w_j C_j\right] \overset{A=B}{=} \sum_j \left(w_j \sum_{\substack{k \in H(j) \\ k \to i_j \\ k < j}} E[X_k] + E[X_j] \sum_{\substack{k \in L(j) \\ k \to i_j \\ k < j}} w_k + w_j E[X_j] \right)$$

$$= \sum_j \min_i z(j, i) \qquad \square$$

15.4 THEOREM: Let $CV[X_j] \leq \Delta$. Then MinIncrease is a ϱ-approximation algorithm with $\varrho = 1 + \frac{(m-1)(\Delta+1)}{2m}$

Proof: lemma 15.3 $\Rightarrow$ $\mathbb{E}[MI(I)] = \sum_j \min_i z(j,i)$

La 15.2
$$= \sum_j \min_i \Big\{ w_j \sum_{\substack{k \in H(j) \\ k < j \\ k \to i}} \mathbb{E}[X_k] + \mathbb{E}[X_j] \sum_{\substack{k \in L(j) \\ k < j \\ k \to i}} w_k + w_j \mathbb{E}[X_j] \Big\}$$

$$= \sum_j \min_i \Big\{ \quad \text{--"--} \quad + \quad \text{--"--} \quad \Big\} + \sum_j w_j \mathbb{E}[X_j]$$

$$\leq \sum_j \frac{1}{m} \sum_i \Big\{ w_j \sum_{\substack{k \in H(j) \\ k < j \\ k \to i}} \mathbb{E}[X_k] + \mathbb{E}[X_j] \sum_{\substack{k \in L(j) \\ k < j \\ k \to i}} w_k \Big\} + \sum_j w_j \mathbb{E}[X_j]$$

min $\leq$ arithmetic mean

$$= \sum_i \frac{1}{m} \Big\{ \sum_j w_j \sum_{\substack{k \in H(j) \\ k < j \\ k \to i}} \mathbb{E}[X_k] + \underbrace{\sum_j \mathbb{E}[X_j] \sum_{\substack{k \in L(j) \\ k < j \\ k \to i}} w_k}_{\overset{\text{Claim}}{\underset{A=B}{=}} \sum_j w_j \sum_{\substack{k \in H(j) \\ k > j \\ k \to i}} \mathbb{E}[X_k]} \Big\} + \sum_j w_j \mathbb{E}[X_j]$$

interchange summation order

$$= \sum_i \frac{1}{m} \sum_j w_j \sum_{\substack{k \in H(j) \\ k \neq j \\ k \to i}} \mathbb{E}[X_k] + \sum_j w_j \mathbb{E}[X_j]$$

interchange summation order

$$= \sum_j \frac{1}{m} w_j \underbrace{\sum_i \sum_{\substack{k\in H(j)\\ k\neq j\\ k\to i}} \mathbb{E}[X_k]}_{} + \sum_j w_j \mathbb{E}[X_j]$$

$$= \sum_{\substack{k\in H(j)\\ k\neq j}} \mathbb{E}[X_k]$$ since machines form a partition of the jobs $k\in H(j)\setminus\{j\}$

$$= \sum_j \frac{1}{m} w_j \sum_{k\in H(j)} \mathbb{E}[X_k] + \frac{m-1}{m}\sum_j w_j \mathbb{E}[X_j]$$

include j in first sum

Now use (see next Lemma)

$$\mathbb{E}[OPT(I)] \geq \sum_j w_j \frac{1}{m} \sum_{k\in H(j)} \mathbb{E}[X_k] - \frac{(m-1)(\Delta-1)}{2m}\sum_j w_j \mathbb{E}[X_j] \quad (1)$$

$$(1) \Rightarrow \mathbb{E}[MI(I)] \leq \mathbb{E}[OPT(I)] + \underbrace{\left[\frac{(m-1)(\Delta-1)}{2m} + \frac{m-1}{m}\right]}_{=\frac{(m-1)(\Delta+1)}{2m}} \underbrace{\sum_j w_j \mathbb{E}[X_j]}_{\leq\ \mathbb{E}[OPT(I)]}$$

$$\Rightarrow \mathbb{E}[MI(I)] \leq \rho\cdot \mathbb{E}[OPT(I)] \quad \square$$

15.5 LEMMA: Consider priorities according to non-increasing $w_j/E[X_j]$ values.

Then (1) holds

Proof: Recall Theorem 14.11, i.e. an optimal solution of the LP defined by inequality (4) and $C_j^{LP} \geq E(X_j)$ in §14 is given by

$$C_j^{LP} = \frac{1}{m} \sum_{k \in H(j)} E[X_k] - \frac{(\Delta-1)(m-1)}{2m} E[X_j]$$

$$\Rightarrow \sum_j w_j C_j^{LP} = \sum_j w_j \frac{1}{m} E[X_k] - \frac{(\Delta-1)(m-1)}{2m} \sum_j w_j E[X_j]$$

and $E[OPT(I)] \geq \sum_j w_j C_j^{LP}$ $\Rightarrow$ Lemma 15.5 $\square$

Remark:

(1) Performance guarantee of MI matches the best known in the offline setting with a fixed number of jobs, which is given by WSEPT

(2) WSEPT requires knowledge of all jobs and their w_j and $E[X_j]$, while MI does not need this data of the jobs still to arrive

(3) WSEPT and MI generate different schedules in general

(4) The lower bound of Theorem 14.12 applies also to MI

Exercises:

15.1 Show by an example that WSEPT and MI generate different schedules, i.e. $\exists\, x$ with $WSEPT(x) \neq MI(x)$

A simple lower bound for MI

15.7 EXAMPLE : $\mathbb{E}[MI(I)] / \mathbb{E}[OPT(I)] \geq 1/2$ asymptotically

Let instance I be given by

$n-1$ deterministic jobs with $x_j = 1$, $w_j = 1$, $j = 1, \dots, n-1$

1 stochastic job with a 2-point distribution:

$$X_n = \begin{cases} 1 & \text{with prob. } 1 - 2/n \\ n^2/4 & \text{"} \quad 2/n \end{cases}$$

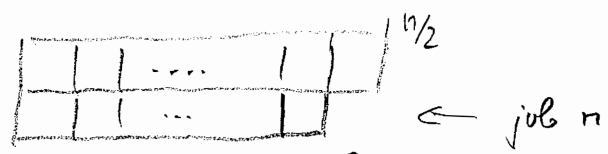
$$w_n = \mathbb{E}[X_n] = 1 - \frac{2}{n} + \frac{n}{2} \Rightarrow \frac{w_n}{\mathbb{E}[X_n]} = 1$$

$m = 2$

jobs come in the order $1, 2, \dots, n$

$\Rightarrow$ MI assigns the $n-1$ deterministic jobs first

$\Rightarrow$ load after assignment



$\Rightarrow$ job n is assigned to the less busy machine

$\Rightarrow \mathbb{E}[MI(I)] = \mathbb{E}[w_j C_j] = 3\frac{n^2}{4} + o(n^2)$

An optimal policy starts only job n and fixes $t = 1$ as next tentative decision time

$X_n = 1 \Rightarrow$ distribute jobs $1, \dots, n-1$ on the 2 machines

$X_n > 1 \Rightarrow$ put jobs $1, \dots, n-1$ on the other machines

$\Rightarrow \mathbb{E}[OPT(I)] = \frac{n^2}{2} + o(n^2)$

$\Rightarrow$ ratio $\to \frac{1}{2}$ for $n \to \infty$