§ 11 CHARACTERIZATION OF ES- AND PRESELECTIVE POLICIES

Have shown

- ES-policies are convex, continuous, monotone
- preselective policies are continuous and monotone

Now: They are already characterized by these properties

11.1 THEOREM: Every monotone policy is dominated by a preselective policy

11.2 COROLLARY: Let Π be an arbitrary policy. Then

Π is monotone $\iff$ Π is preselective (up to dominance, i.e. maybe not earliest start)

Proof: Consider $[G, \mathcal{F}]$. Let A be an antichain of G, and x be a duration vector.

Call job j <u>selected</u> <u>for</u> <u>(A,x)</u> $:\iff$ $\boxed{\text{$j$ waits for any $i \in A$ for all y with $y \geq_A x$}}$

$\geq_A$: only larger on A.

i.e. $y_k \geq x_k$ for all $k \in A$, $y_k = x_k$ otherwise

i.e. for every such y there is $i \in A$ (may depend on y) with $\Pi[y](i) + y_i \leq \Pi[y](j)$

Intuition for this notion:

for x, several jobs might wait

making jobs of A longer reveals the "real" selected job

Let $S(A,x)$ be the set of jobs selected for (A,x)

(1) $\boxed{F \in \mathcal{F} \Rightarrow S(F,x) \neq \emptyset}$ ⇐ for every policy Π

Consider x^m with $x^m_k := \begin{cases} x_k & k \notin F \\ x_k + m & k \in F \end{cases}$ "long on F"

Π policy $\Rightarrow \exists$ job j_m that waits w.r.t. x^m

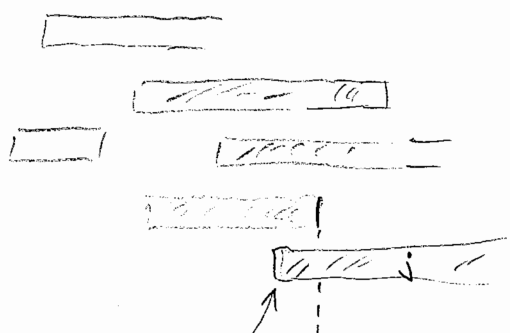
$\Rightarrow$ some job j occurs infinitely often in sequence $(j_m)_m$

Claim: $j \in S(F,x)$

suppose not, say j does not wait w.r.t. y with $y \geq_F x$

Then $\Pi[y](j) < \underbrace{\min_{\substack{i \in F \\ i \neq j}} \{ \Pi[y](i) + y_i \}}_{\text{first completion on } F}$

y



first completion on F

start of j under Π remains the same if we enlarge processing times on F (same history up to start of j)

$\Rightarrow$ j does not wait for all x^m with $x^m/_F \geq x/_F$, contradiction

$\boxed{\text{non-waiting is invariant under } F\text{-monotonicity}}$ (1*)

(2) $\boxed{\begin{array}{l}\left(\forall F, \forall x, \forall y: \; x \leq y \Rightarrow S(F,x) \subseteq S(F,y)\right) \\ \Rightarrow \Pi \text{ preselective (may be not earliest start)}\end{array}}$

Suppose (...) is valid but Π is not preselective

$\Rightarrow \exists F \in \mathfrak{F}$ such that Π is not preselective on F

$\Rightarrow$ for every $j \in F$ there is x^j s.t. j does not wait w.r.t. x^j

$\overset{(1^*)}{\Rightarrow}$ $j \notin S(F, x^j)$ for all $j \in F$

Consider $x := \min_j x^j$ componentwise

Let $j_0 \in S(F,x) \neq \emptyset$ — contradiction

$\Rightarrow \quad x \leq x^{j_0} \quad \underset{(\ldots)}{\Rightarrow} \quad j_0 \in S(F, x^{j_0})$

(proof is a diagonalization argument)

(3) $\boxed{\Pi \text{ monotone} \Rightarrow \left(\forall F, \forall x, \forall y: \; x \leq y \Rightarrow S(F,x) \subseteq S(F,y)\right)}$

Suppose not. $\Rightarrow \exists F, x, y$ with $x \leq y$ but $S(F,x) \not\subseteq S(F,y)$

$\Rightarrow \exists j \in S(F,x) \setminus S(F,y)$

Consider x^m, y^m as above [+m on jobs in F]

$j \in S(F,x) \Rightarrow$ j waits for x and thus for all x^m

$j \notin S(F,y) \Rightarrow$ j does not wait for some $\bar{y} \geq_F y$

$\overset{(*)}{\Rightarrow}$ j does not wait for all $y^m \geq_F \bar{y}$, even same start time

non-waiting is invariant under F-monotonicity

$$\Rightarrow \quad m \leq \Pi[x^m](j) \leq \Pi[y^m](j) = \Pi[\bar{y}](j)$$

$j \in S(F,x)$ $\qquad$ Π monotone, $x^m \leq y^m$

$\Rightarrow$ contradiction. $\square$

Similar techniques:

11.3 THEOREM: Every continuous and elementary policy is preselective.

without proof

11.4 COROLLARY: Let Π be an arbitrary policy. Then

Π is preselective $\Leftrightarrow$ Π is monotone (up to dominance)

$\Leftrightarrow$ Π is continuous and elementary

11.5 CONSEQUENCES (1) Graham anomalies of type a) come in pairs

(2) Preselective policies = natural class fulfilling stability (need continuity, thus preselective)

11.6 THEOREM: Π convex $\Rightarrow$ Π ES-policy

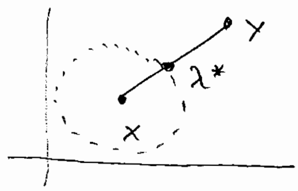
Proof:

(1) $\boxed{\Pi \text{ convex} \Rightarrow \Pi \text{ is dominated by a preselective policy}}$

Show: $\boxed{x \leq y \Rightarrow S(F,x) \subseteq S(F,y) \quad \forall x,y,F}$

Let $x \leq y$ but $j \in S(F,x) \setminus S(F,y)$

Consider line $x^\lambda = (1-\lambda)x + \lambda y \quad 0 \leq \lambda \leq 1$



Let λ^* be the first λ with $j \notin S(F, x^\lambda)$

W.l.o.g. x, y close enough (on the line) to x^{λ^*} s.t. $z := 2x - y > 0$

$\Rightarrow x = \frac{1}{2} y + \frac{1}{2} z$

Consider x^m, y^m as before (add m to jobs on F)

$$\Rightarrow x^m = \underbrace{\frac{1}{2} y^{2m} + \frac{1}{2} z}$$

$$= \frac{1}{2} y^{2m} + x - \frac{1}{2} y = \frac{1}{2} y + \mathbb{1}_F^m + x - \frac{1}{2} y^m = x^m$$

$$\Rightarrow \underset{\substack{\uparrow \\ j \in S(F,x)}}{m \leq} \Pi[x^m](j) \underset{\substack{\uparrow \\ \Pi \text{ convex}}}{\leq} \frac{1}{2} \Pi[y^{2m}](j) + \frac{1}{2} \Pi[z](j)$$

$$\underset{\substack{\uparrow \\ j \notin S(F,y)}}{=} \frac{1}{2} \Pi[y](j) + \frac{1}{2} \Pi[z](j)$$

cf Proof of Thm. 11.1, (1*)

contradiction

(2) $\boxed{\Pi \text{ convex} \Rightarrow \exists \text{ ES-policy } \Pi^* \leq \Pi}$

Π convex $\overset{(1)}{\Rightarrow}$ Π preselective $\Rightarrow$ $\exists$ waiting job j on $F \in \mathcal{F}$

Claim: $\boxed{j \text{ waits always for the same job } i}$

Suppose not $\Rightarrow$ for every $i \in F \setminus \{j\}$ there is x^i with

$$\underbrace{\Pi[x^i](i) + x^i_i > \Pi[x^i](j)}_{j \text{ does not wait for } i \text{ w.r.t. } x^i}$$

Consider $x^{i,m}$ (make jobs in F longer by m)

$\Rightarrow$ $\Pi[x^{i,m}](j)$ does not change

$\uparrow$ monotonicity of non-waiting, (1*)

Let $z^m := \frac{1}{|F|-1} \sum_{i \in F \setminus \{j\}} x^{i,m}$ convex combination

$$m \underset{\uparrow}{\leq} \Pi[z^m](j) \underset{\uparrow \Pi \text{ convex}}{\leq} \frac{1}{|F|-1} \sum_i \Pi[x^{i,m}](j)$$

constant

j is waiting job on F

$\Rightarrow$ start $\geq$ first completion on $F \setminus \{j\}$

every $i \in F \setminus \{j\}$ is at least $\frac{1}{|F|-1} \cdot m$ long w.r.t. x^{im}

$\Rightarrow$ at least m w.r.t. z

$\Rightarrow$ contradiction

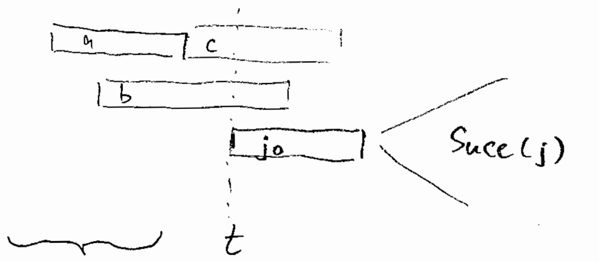
$\Rightarrow$ every F is settled by a waiting pair $i < j$

$\Rightarrow$ $\exists$ ES policy $\Pi^* \leq \Pi$

(3) $\boxed{\Pi \text{ is elementary}}$

suppose not $\Rightarrow$ $\exists\ x^0$ and j_0 such that j_0 starts in $\Pi[x^0]$
at a time t where no job $j \neq j_0$ ends

$\Pi[x^0]$



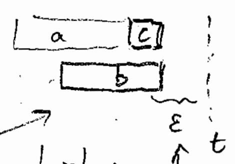
$\left.\begin{array}{l} C(t) \text{ set of completed jobs (before time } t) \\ B(t) \text{ set of busy jobs (at } t) \end{array}\right\}$ in $\Pi[x^0]$

Let $X :=$ set of all $x \in \mathbb{R}^n_{\geq}$ that look the same to Π at t

$= \{ x \in \mathbb{R}^n_{\geq} \mid x_j = x_j^0 \ \forall j \in C(t),\ x_j > t - \Pi[x^0](j) \ \forall j \in B(t) \}$

$\Rightarrow \Pi[x](j_0) = t \quad \forall x \in X$

Let $z \in \mathbb{R}^n_{\geq}$ with $z_j = x_j^0 \quad \forall j \in C(t) \cup Succ_G(j_0) \cup \{j_0\}$
z_j short for all other jobs
↳ s.t. all these jobs end before t, say at $t-\varepsilon$



only job j_0 and its successors are left

$\Pi[z]$ same history as $\Pi[x^0]$

$\Rightarrow z \notin X$

j is the only job that can be started

policies have no "total" idle time $\Rightarrow \Pi[z](j_0) = t - \varepsilon < t$

Choose $y \in X$ with $\frac{1}{2}y + \frac{1}{2}z \in X$ (choose y_j large for $j \in B(t)$ in $\Pi[x^0]$)

Then $t = \Pi[\underbrace{\tfrac{1}{2}y+\tfrac{1}{2}z}_{\in X}](j_0) \underset{\Pi \text{ convex}}{\leq} \tfrac{1}{2}\underbrace{\Pi[y](j_0)}_{=t} + \tfrac{1}{2}\underbrace{\Pi[z](j_0)}_{<t}$

$< t$, contradiction

(4) $\boxed{\Pi \text{ is an ES-policy}}$

consider the most restrictive problem $[G^*, F^*]$ for which Π is still a policy

- add to G all $i<j$ with $\Pi[x](i)+x_i < \Pi[x](j) \quad \forall x$
- let F^* be all antichains of G^* that are not scheduled simultaneously by Π for all x

$\Downarrow$

Π is convex policy for $[G^*, F^*]$

$\overset{(1)}{\Rightarrow} \exists$ ES-policy Π^* for $[G^*, F^*]$ with $\Pi^* \leq \Pi$

Claim: $\boxed{\Pi^* = \Pi}$

proof by induction along decision times t of Π for arbitrary, fixed vector x.

<u>$t=0$:</u>

$S^*(0)$ set of jobs started by Π^* at $t=0$

$S(0)$ ---- Π ----

If $S^*(0) \neq S(0) \overset{\Pi^* \leq \Pi}{\Rightarrow} \exists\ j \in S^*(0) \setminus S(0)$

Π elementary $\Rightarrow$ j waits for another completion

$\Rightarrow$ $S^*(0)$ not simultaneously in $\Pi[x]$

$\underset{[G^*, F^*] \text{ most restrictive}}{\Rightarrow}$ $S^*(0)$ either forbidden or not an antichain

$\Rightarrow$ π^* not a policy for $[G^*, F^*]$ $\Rightarrow$ contradiction

inductive step: $S(t') = S^*(t')$ for all decision times $t' < t$

consider decision time t (= next completion, and thus the same in π^* and π)

same argument as at $t = 0$ $\Rightarrow$ $S(t) = S^*(t)$ $\square$