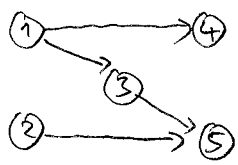
§7 EARLY START (ES) POLICIES

Recall the ES-function induced by a partial order H on V $\quad (H = (V, E_H))$

$$ES_H[\cdot] : \mathbb{R}^n_{\geq} \to \mathbb{R}^n$$

$ES_H[x]$ = ES-schedule w.r.t. H and x

Expl: H



$$ES_H[x] = (0, 0, x_1, x_1, \max\{x_1 + x_3, x_2\})$$

$\Downarrow$

$ES_H[\cdot]$ is a function with the same domain and range as a policy Π for $[G, \mathcal{F}]$

$ES_H[\cdot]$ has nice properties: - continuous
- monotone ...

Question: Given $[G, \mathcal{F}]$, when is ES_H a policy for $[G, \mathcal{F}]$?

7.1 THEOREM: $ES_H[\cdot]$ is a planning rule for $[G, \mathcal{F}]$ iff

(1) H extends G, i.e. $i <_G j \Rightarrow i <_H j$

(2) No antichain of H is forbidden, i.e.

$$F \in \mathcal{F} \Rightarrow F \text{ is not an antichain of } H$$

Proof: "$\Rightarrow$"

(1) Assume $i <_G j$ but $i \not<_H j$. Define $x \in \mathbb{R}^n_{\geq}$ by

$$x_k := \begin{cases} \varepsilon & k \neq i \\ 1 & k = i \end{cases} \qquad \varepsilon < \frac{1}{n^2}$$

Then $ES_H[x](j) \underset{\substack{\uparrow \\ i \not<_H j}}{\leq} (n-2)\varepsilon < 1$

$x_i = 1$
$\Rightarrow$ j does not wait for i in schedule $ES_H[x]$

$\Rightarrow$ $ES_H[x]$ does not respect G

$\Rightarrow$ ES_H is not a planning rule for $[G, \mathcal{F}]$

(2) Suppose $F \in \mathcal{F}$ is an antichain of H. Define $x \in \mathbb{R}^n_>$ by

$$x_k = \begin{cases} \varepsilon & k \notin F \\ 2 & k \in F \end{cases} \qquad \varepsilon < \frac{1}{n^2}$$

Then, for $j \in F$, $ES_H[x](j) \underset{\substack{\uparrow \\ \text{j has only short predecessors}}}{\leq} (n-2)\varepsilon < 1$

$\Rightarrow$ at $t = 1$, all jobs in F are busy

$\Rightarrow$ ES_H is not a planning rule for $[G, \mathcal{F}]$

"$\Leftarrow$": H extends G $\Rightarrow$ ES_H respects G

No antichain of H is forbidden $\underset{\substack{\uparrow \\ \text{only antichains can be processed} \\ \text{simultaneously in } H}}{\Rightarrow}$ ES_H respects $\mathcal{F}$ $\square$

Definition: Call a partial order H on V <u>feasible</u> for $[G, \mathcal{F}]$ if

(1) it <u>respects</u> G, i.e. it extends G

(2) it <u>respects</u> $\mathcal{F}$, i.e. no antichain of H is in $\mathcal{F}$

7.2 THEOREM: $ES_H[\cdot]$ is a planning rule for $[G, \mathcal{F}]$ iff H is feasible for $[G, \mathcal{F}]$. In this case, ES_H is already a policy, i.e. non-anticipative

Proof: First part follows from Thm. 7.1. Show non-anticipativity:
Let x, y look the same to ES_H at time t and $ES_H[x](j) = t$.
$\Rightarrow$ x, y have same history up to t
$\Rightarrow$ all $i \in Pred_H(j)$ have $x_i = y_i$
$\Rightarrow$ (ES_j = longest path length in $Pred_H(j)$) $ES_H[y](j) = ES_H[x](j) = t$ $\square$

Let $\mathcal{E}(G)$ denote set of all extensions of G (including G itself).
$\mathcal{E}(G)$ is a partial order under the ordering relation

$$H_1 \prec H_2 \quad :\Leftrightarrow \quad H_2 \text{ extends } H_1$$
$$(\text{i.e. } i <_{H_1} j \Rightarrow i <_{H_2} j)$$

We call $\mathcal{E}(G)$ the <u>extension order</u>. of G

$\mathcal{E}(G)$ has many nice properties (Exercises)

(1) $H_1 = (V, E_1)$, $H_2 = (V, E_2)$

$$H_1 \in ImPred(H_2) \Leftrightarrow \left| E_2^{trans} \setminus E_1^{trans} \right| = 1$$

partial order constraints (including trans. arcs) differ exactly by 1

$$H_1 \prec H_2 \Leftrightarrow E_1^{trans} \subseteq E_2^{trans}$$

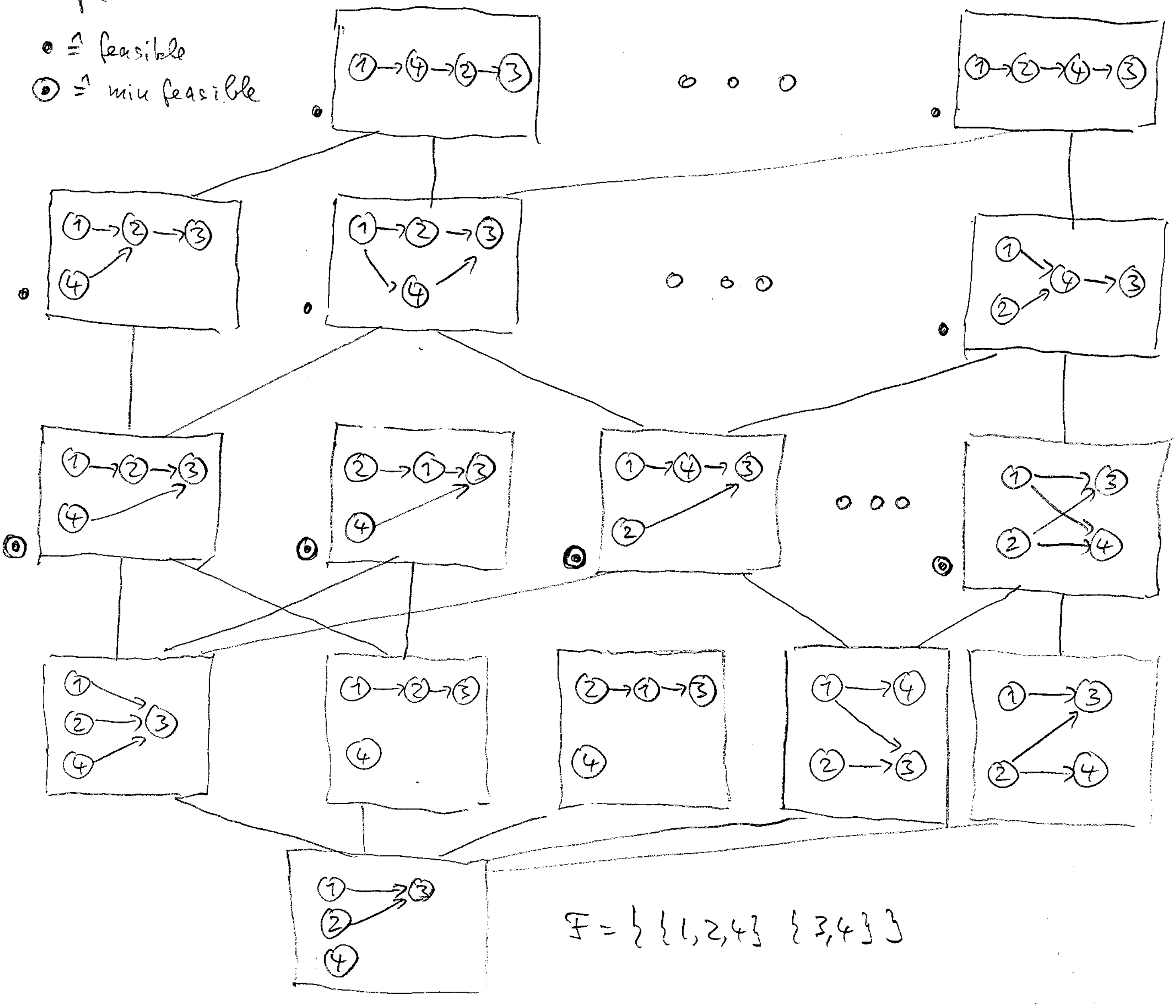
(2) All maximal chains from H_1 to H_2 have the same length

(3) H_1, H_2 have a largest common predecessor $H = (V, E)$ with $E^{trans} = E_1^{trans} \cap E_2^{trans}$

(4) H_1, H_2 have a smallest common successor $H = (V, E)$ iff $E_1 \cup E_2$ is acyclic. Then $E^{trans} = (E_1 \cup E_2)^{trans}$

Note: (3) and (4) say that $\mathcal{E}(G)$ equipped with an artificial greatest element, is a <u>semi lattice</u>.

Expl:



$\mathcal{F} = \{\{1,2,4\}\ \{3,4\}\}$

7.3 CONSEQUENCES

(1) The set of feasible orders is upwardly closed in $\mathcal{E}(G)$, i.e.

$$H_1 \text{ feasible}, \quad H_1 \prec H_2 \quad \Rightarrow \quad H_2 \text{ feasible}$$

(2) $H_1 \prec H_2 \iff ES_{H_1} \leq ES_{H_2}$

(3) ES_H is a minimal ES-policy $\iff$ H is minimal feasible

(4) The set of ES policies may be identified with the set of feasible orders

Proof: Obvious

ES-policies lead to optimal schedules in the deterministic case

7.4. THEOREM: Consider $[G, \mathcal{F}]$. For every x, there is an ES-policy $\Pi = ES_H$ such that

$$OPT(\kappa, x) = \kappa^{\Pi}(x) = \kappa^{H}(x)$$

[Note that Π and thus H depends on x]

In particular:

$$\begin{aligned} OPT(\kappa, x) &= \min\{\kappa^H(x) \mid H \text{ feasible}\} \\ &= \min\{\kappa^H(x) \mid H \text{ minimal feasible}\} \end{aligned}$$

Proof: "$\leq$" Let H be such that the min of rhs is attained

Then $ES_H[x]$ is a feasible schedule for $G, \mathcal{F}, x$

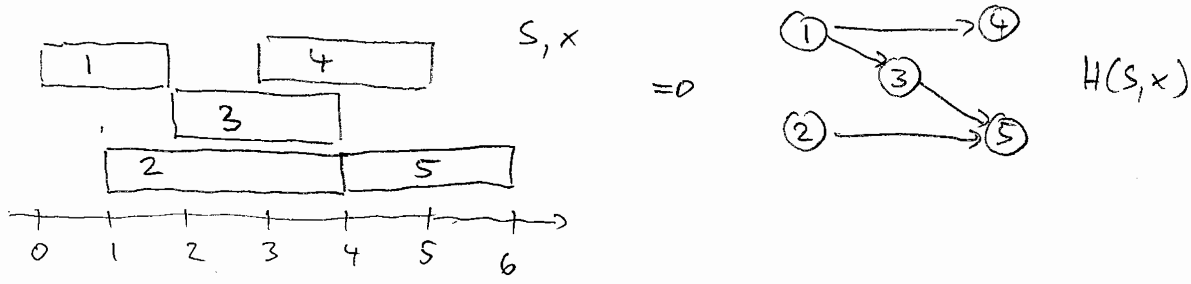
$$\Rightarrow \quad \kappa(ES_H[x], x) = \kappa^H(x) \geq OPT(\kappa, x)$$

"$\geq$" Let S be an optimal schedule for $G, \mathcal{F}, x, k$

Consider the partial order $H = H(S,x)$ induced by S and x:

$$i <_H j \quad \text{if} \quad S_i + x_i \le S_j$$

Expl:



Note:

(1) Every schedule S and vector x induce such an order
[Important that $x_i > 0$, i.e. $x \in \mathbb{R}^n_{>}$]

(2) $H(S,x)$ is an <u>interval order</u>

i.e. an order H whose elements j can be represented by intervals $I_j \subseteq \mathbb{R}^1$ such that

$i <_H j$ if I_i is entirely to the left of I_j (except for endpoints)

For $H(S,x)$, $I_j = [S_j, S_j + x_j]$

(3) Interval orders have nice properties (see exercises)

Claim 1 $\boxed{\text{$A$ is an antichain of $H(S,x)$ $\iff$ there is a time t such that all $j \in A$ are busy, i.e. $S_j < t < S_j + x_j$ for all $j \in A$}}$

proof by induction on $|A|$

Claim 1 $\hat{=}$ Helly property of a set of intervalls
i.e. $\boxed{\text{any two intersect} \Rightarrow \text{all intersect}}$

Claim 2: $\boxed{H(S,x) \text{ extends } G}$

$$i <_G j \underset{\substack{\uparrow \\ S \text{ feasible}}}{\Rightarrow} S_i + x_i \le S_j \underset{\substack{\uparrow \\ \text{Def } H(S,x)}}{\Rightarrow} i <_{H(S,x)} j$$

Claim 3: $\boxed{\text{No antichain of } H(S,x) \text{ is forbidden}}$

A antichain of $H(S,x) \overset{\text{Claim 1}}{\Rightarrow}$ busy at same time in schedule S

$\underset{\substack{\uparrow \\ S \text{ feasible}}}{\Rightarrow}$ A is not a forbidden set

Claim 2,3 $\Rightarrow$ $H(S,x)$ is a feasible order

Claim 4 $\boxed{ES_{H(S,x)}[x] \le S}$

S respects $H(S,x)$ by construction of $H(S,x)$

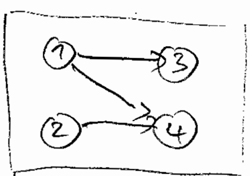
$ES_{H(S,x)}[x]$ is componentwise best schedule for $H(S,x)$ $\Rightarrow$ "$\le$"

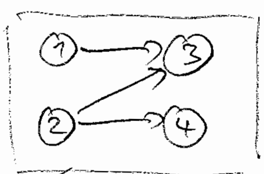
$\Rightarrow \kappa^{H(S,x)}(x) \le \kappa(S,x)$ $\square$

Consequence: In the deterministic case, we have

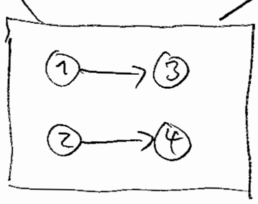
$$OPT(\kappa,x) = \min\{\kappa^H(x) \mid H \text{ is a (minimal) feasible interval order}\}$$

This need not be true in the stochastic case:

H_1



H_2

interval orders

H


not an interval order

For every x, $ES_H[x] = \min\{ES_{H_1}[x], ES_{H_2}[x]\}$

$\Rightarrow$ one of H_1, H_2 is as good as H

For random processing times, when we minimize expected makespan, H may be better, e.g. every job has proc. times 1 and 3 with probability $1/2$ each, and independent from each other

$\Rightarrow E[C_{max}^{H_1}] = E[C_{max}^{H_2}] = 4.875$, $E[C_{max}^{H}] = 4.75$

7.5 THEOREM: Let Π be a policy for $[G, \mathcal{F}]$. Then

Π is an ES-policy $\Leftrightarrow$ Π is convex

Proof: Later, need a more general class of policies first $\square$

Exercises

7.1 Prove the properties of $\mathcal{E}(G)$

7.2 <u>Characterization of interval orders</u>

Show that the following conditions are equivalent:

(1) H is an interval order

(2) H does not contain $\begin{array}{c} \circ \longrightarrow \circ \\ \circ \longrightarrow \circ \end{array}$ (2+2)

as induced suborder (may be transitive arcs)

(3) There is an ordering $j_1, j_2, \dots, j_n$ of V with

$$\mathrm{Pred}(j_1) \subseteq \mathrm{Pred}(j_2) \subseteq \dots \subseteq \mathrm{Pred}(j_n)$$

(4) There is a numbering $A_1, A_2, \dots, A_m$ of the maximal antichains of H such that, for every $j \in V$, all A_k containing j occur consecutively in the numbering

[consecutiveness property of maximal antichains]

7.3 The m-machine probleme can be solved in polynomial time on interval orders

7.4 For every minimal feasible order H, there are κ and Q such that H is the only optimal (minimal) feasible order

$$E_Q[\kappa^H] < E_Q[\kappa^{H'}] \quad \text{for all (minimal) feasible } H' \neq H$$

Does this also hold for $\kappa = C_{max}$?