§1 Projects and Partial Orders

What makes a project

- activities or jobs $\quad u, v, i, j \qquad V = \{1, \ldots, n\}$
- job data
  - processing time / duration (deterministic or random)
  - resource requirement
  - ⋮
  - processing cost
  - ⋮
- project data
  - available resources
  - limited budget
- project rules
  - rules for carrying out the project
  - - temporal conditions
  - - resource conditions

simplest case: precedence constraints

binary relation "$<$" on V

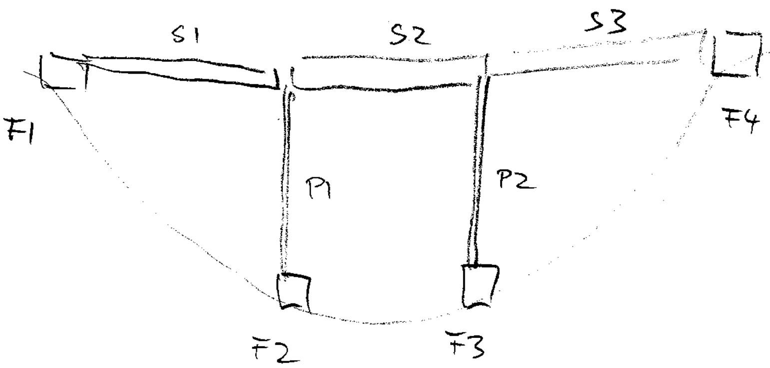
$i < j$: j must wait for i
i "precedes" j

⇓

defines a partial order $\quad i<j \Rightarrow j \not< i$ (asymmetric)

$i<j, j<k \Rightarrow i<k$ (transitive)

Bridge construction project



F : foundations

P : piers

S : superstructure

$V = \{ F1, .., F4, P1, P2, S1, .. S3 \}$

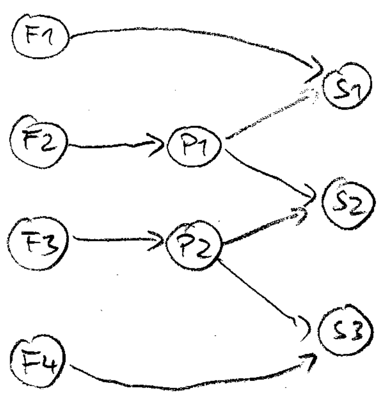
$F1 < S1$

$F2 < P1 < S1 \qquad P1 < S2$

$F3 < P2 < S2 \qquad P2 < S3$

$F4 < S3$

Representation as "activity on node" diagram
= directed acyclic graph (dag) $G = (V, E)$



$V =$ set of jobs

$E = \{ (i,j) \mid i < j, \nexists \text{ no } k \text{ with } i < k < j \}$

transitive reduction of "<"

Notation: $\mathrm{Pred}_G(i) := \{ j \in V \mid j < i \}$ set of predecessors

$\mathrm{Succ}_G(i) := \{ k \in V \mid i < k \}$ set of successors

$\mathrm{ImPred}_G(i) := \{ j \in V \mid (j,i) \in E \}$ immediate predecessors

$\mathrm{ImSucc}_G(i)$ ... ... successors

i <u>maximal</u> in G $:\Leftrightarrow$ $\mathrm{Succ}(i) = \emptyset$

<u>minimal</u>

i <u>greatest</u> $:\Leftrightarrow$ i is the only maximal element of G

<u>smallest</u> minimal

i, j are comparable $(i \sim_G j)$ if $i < j$ or $j < i$

incomparable $(i \parallel_G j)$ if not $i \sim_G j$

<u>1.1. Lemma</u>

Let $(V, <)$ be a (finite) partial order and (V, E) be its transitive reduction. E is the smallest (under $\subseteq$) binary relation on V such that $E^{\text{trans}} = <$

↑

transitive closure

$:=$ smallest transitive relation containing E

$= \{ (i,j) \mid \exists$ finite sequence $i = i_0, i_1, \dots, i_k = j$ with $(i_0, i_1), \dots, (i_{k-1}, i_k) \in E \}$

Proof (Sketch):

We need all pairs $(i,j) \in E$ to obtain all $r < s$ by transitivity

by definition of E not obtainable by transitivity

$\Rightarrow$ E contained in every relation R with $R^{trans} = <$

Show: $\boxed{E^{trans} = <}$

Let $i < j$ and $(i,j) \notin E$

By definition of E, $\exists\ k \in V$ with $i < k < j$

Iterate: $<$ is acyclic and finite $\Rightarrow$ $i = i_0 \to i_1 \to i_2 \to \cdots \to j$ $\square$

$\in E$

Note: Lemma does not hold for
- relations with cycles
- infinite ground sets

Homework:

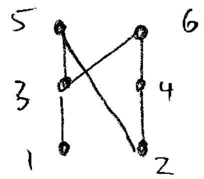
$\boxed{1.1}$ Formulate an algorithm for constructing the transitive closure of a digraph

$\boxed{1.2}$ Formulate an algorithm for constructing the transitive reduction of a partial order (or dag)

$\boxed{1.3}$ Prove an $O(n^3)$ time bound for your algorithms

Use transitive reduction of $<$ for representation

Hassediagramm in mathematics



dag in scheduling, computer science