# Moment explosions and long-term properties of stochastic volatility models

Martin Keller-Ressel
TU Wien, FAM Research Group
mkeller@fam.tuwien.ac.at

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# Moment Explosions in Stochastic Volatility models

$X_t$ ... (discounted, risk-neutral) log-price-process
$V_t$ ... a latent factor (or factors), such as stochastic variance, stochastic volatility or stochastic arrival rate of jumps.
$S_t := \exp(X_t)$ ... (discounted, risk-neutral) price-process

As a minimal assumption in this talk, $(X_t, V_t)_{t \geq 0}$ is time-homogeneous Markovian.

## Moment Explosion

Let $u \in \mathbb{R}$. We speak of a moment explosion at time $T_*(u)$, if

$$T_*(u) := \sup \{ t \geq 0 : \mathbb{E}[S_t^u] < \infty \} < \infty .$$

and say that no moment explosion occurs, if $T_*(u) = \infty$.

# Applications of moment explosions

- **Equity and FX Models** $S_t \ldots$ stock price/ exchange rate. Moment explosions can be used to obtain approximations for the implied volatility of deep in-the-money and out-of-the-money options. Lee [2004], Benaim and Friz [2008].
- **Interest rate modeling** $S_t \ldots$ forward LIBOR rate/swap rate. Andersen and Piterbarg [2007] give examples of derivatives with super-linear payoff, where moment explosions lead to infinite prices of derivatives.
- **numerical procedures** E.g. discretization schemes for SDEs. Moment explosions can lead to break-down of error estimates depending on higher-order moments (see e.g. Alfonsi [2008]).
- **utility maximization** Moment explosions may lead to infinite expected utility, cf. Kallsen and Muhle-Karbe [2008].

# Critical Moments

## Critical Moments

Define the **upper/lower critical moments** by

$$u_+(T) = \sup\{u \geq 1 : \mathbb{E}[S_T^u] < \infty\} = \sup\{u \geq 1 : T_*(u) < T\} \;,$$

$$u_-(T) = \inf\{u \leq 0 : \mathbb{E}[S_T^u] < \infty\} = \inf\{u \leq 0 : T_*(u) < T\} \;.$$

- $u_\pm(T)$ are the piecewise inverse functions of $T_*(u)$ on $(1, \infty)$ and $(-\infty, 0)$ respectively.
- $T_*(u) = \infty$ for all $u \in [0, 1]$ because of the martingale property of $(S_t)_{t\geq 0}$.
- In many models moment explosion times are more tractable than critical moments.

# Lee's moment formula

By a result of Lee [2004], $u_\pm(T)$ are related to the large-strike behavior of the volatility smile:

## Lee's moment formula

Let $V(T,\xi)$ be the implied Black-Scholes-Variance of a European call with time-to-maturity $T$ and log-moneyness $\xi = \log\left(\frac{e^{-rT}}{S_0}\right)$. Then

$$\limsup_{\xi \to -\infty} \frac{V(T,\xi)}{|\xi|} = \frac{\varsigma(-u_-(T))}{T}$$

and

$$\limsup_{\xi \to \infty} \frac{V(T,\xi)}{|\xi|} = \frac{\varsigma(u_+(T)-1)}{T}$$

where $\varsigma(x) = 2 - 4\left(\sqrt{x^2+x} - x\right)$ and $u_\pm(T)$ are the critical moment functions.

# The basic models

- In the **Black-Scholes** model

$$\mathbb{E}\left[S_T^u\right] = S_0 \exp\left(T\frac{\sigma^2}{2}(u^2 - u)\right) ,$$

which means that **no moment explosions occur for any order** $u \in \mathbb{R}$.

- In an **exponential-Lévy** model

$$\mathbb{E}\left[S_T^u\right] = S_0 \exp\left(T\kappa(u)\right) ,$$

which means that

$$T_*(u) = \begin{cases} +\infty & \kappa(u) < \infty , \\ 0 & \kappa(u) = \infty . \end{cases}$$

or equivalently $u_\pm(T) = \kappa_\pm$, i.e. the critical moment functions are constant.

# The Heston model (1)

It gets more interesting in a true stochastic volatility model...

## Heston Model

$$
\begin{aligned}
dS_t &= S_t \sqrt{V_t}\, dW_t^1, \quad S_0 = s\,, \\
dV_t &= -\lambda(V_t - \theta)\, dt + \eta \sqrt{V_t}\, dW_t^2\,, \quad V_0 = v, \\
&\langle dW_t^1, dW_t^2 \rangle = \rho\, dt.
\end{aligned}
$$

$(X_t, V_t)_{t \geq 0}$ is a (time-homogenous) diffusion with generator

$$
\mathcal{L} = \frac{v}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) + \frac{v}{2}\eta^2 \frac{\partial^2}{\partial v^2} + \lambda(\theta - v)\frac{\partial}{\partial v} + \rho\eta v \frac{\partial^2}{\partial x \partial v}\,.
$$

# The Heston model (2)

We have the homogeneity condition

$$\mathbb{E}\left[e^{uX_T}|X_t = x, V_t = v\right] = e^{ux}\mathbb{E}\left[e^{uX_T}|X_t = 0, V_t = v\right]$$

such that $f = f(t, v; u) = \mathbb{E}\left[e^{uX_t}|X_0 = 0, V_0 = v\right]$, satisfies (up to localization) the parabolic partial differential equation

$$\partial_t f = \mathcal{A}f := \left(\frac{v}{2}\eta^2\frac{\partial^2}{\partial v^2} + \left[\lambda(\theta - v) + \rho\eta u v\right]\frac{\partial}{\partial v} + \frac{v}{2}\left(u^2 - u\right)\right) f$$

with initial condition $f(0, \cdot; u) = 1$.

# The Heston model (3)

The exponentially-affine ansatz

$$f(t, v; u) = \exp\left(\phi(t, u) + v\psi(t, u)\right)$$

reduces the PDE to the ODE system

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(t, u) &= F(u, \psi(t, u)), \qquad & \phi(0, u) = 0, \\
\frac{\partial}{\partial t}\psi(t, u) &= R(u, \psi(t, u)), \qquad & \psi(0, u) = 0,
\end{aligned}$$

where

$$\begin{aligned}
F(u, w) &= \lambda\theta w \quad , \\
R(u, w) &= \frac{w^2}{2}\eta^2 + (\rho\eta u - \lambda)w + \frac{1}{2}(u^2 - u) \; .
\end{aligned}$$

# The Heston model (4)

The second ODE is a **Riccati differential equation**, whose solution blows up at finite time, corresponding to the moment explosion of $S_t$. Explicit calculations (cf. Andersen and Piterbarg [2007]) yield

## Moment Explosion in the Heston Model

$$T_*^{\text{Heston}}(u) = \begin{cases} +\infty & \Delta(u) \geq 0, \chi(u) < 0 \\ \frac{1}{\sqrt{\Delta}} \log\left(\frac{\chi(u)+\sqrt{\Delta}}{\chi(u)-\sqrt{\Delta}}\right) & \Delta(u) \geq 0, \chi(u) > 0 \\ \frac{2}{\sqrt{-\Delta}}\left(\arctan\frac{\sqrt{-\Delta}}{\chi} + \pi \mathbf{1}_{\{\chi<0\}}\right) & \Delta(u) < 0, \end{cases} \tag{1}$$

where $\chi(u) = \rho\eta u - \lambda$ and $\Delta(u) = \chi(u)^2 - \eta^2(u^2 - u)$.

# Extensions to other models

- **SABR-type models**
  (Andersen and Piterbarg [2007], Lions and Musiela [2007]):

$$
\begin{aligned}
dS_t &= V_t^\delta S_t^\beta \, dW_t^1, \qquad S_0 = s \\
dV_t &= \eta V_t^\gamma \, dW_t^2 + b(V_t)\, dt, \qquad V_0 = v, \quad \langle dW_t^1, dW_t^2 \rangle = \rho \, dt.
\end{aligned}
$$

  where $\delta, \gamma > 0$, $\beta \in [0,1]$.

- **Affine stochastic volatility models** (Keller-Ressel [2008]):
  $(X_t, V_t)_{t \geq 0}$ is a time-homogeneous Markov process, such that

$$
\mathbb{E}\left[ e^{uX_t} | X_0 = x, V_0 = v \right] = e^{ux} \exp\left( \phi(t,u) + v\psi(t,u) \right)
$$

  Note that $(X_t, V_t)_{t \geq 0}$ is not necessarily a diffusion, but may exhibit jumps.

1. Moment Explosions
   - Introduction
   - Moment Explosions in SABR-type Models
   - Moment Explosions in affine SV Models

2. Long-time Asymptotics for affine SV Models
   - Long-term behaviour of the price process

# SABR-type models (1)

We discuss the case $u \geq 0, \beta = 1$:

The function

$$f = f(t, v; u) := \mathbb{E}\left[e^{uX_t} | X_0 = 0, V_0 = v\right]$$

is (up to a localization) the unique viscosity solution of the parabolic PDE

$$\frac{\partial}{\partial t} f = \mathcal{A} f := \frac{v^{2\gamma}}{2} \eta^2 \frac{\partial^2 f}{\partial v^2} + \left[b(v) + \eta\rho u v^{\delta+\gamma}\right] \frac{\partial f}{\partial v} + \frac{v^{2\delta}}{2} \left(u^2 - u\right) f$$

with initial condition $f(0, \cdot; u) \equiv 1$.

# SABR-type models (2)

Lions and Musiela use the exponentially-affine (in $v^q$) ansatz

$$f(t, v; u) = \exp\left(\phi(t, u) + v^q \psi(t, u)\right) .$$

Suitable choice of $q$, $\phi$ and $\psi$ allows to construct

- supersolutions ($\mathcal{A}\overline{f} - \frac{\partial \overline{f}}{\partial t} \leq 0$), leading to lower bounds for $T_*(u)$.
- subsolutions ($\mathcal{A}\underline{f} - \frac{\partial \underline{f}}{\partial t} \geq 0$), leading to (matching) upper bounds.

The Heston model is recovered as the special case $\beta = 1$, $\delta = \gamma = 1/2$, $b(v) = -\lambda(v - \theta)$, for which supersolution = subsolution!

# Results of Lions & Musiela (1)

(i) $\beta < 1$: no moment explosion occurs, i.e. $\mathbb{E}\left(S_t^u\right) < \infty$ for all $u \geq 1, t \geq 0$;

(ii) $\beta = 1, \gamma + \delta < 1$: as in (i), no moment explosion occurs;

(iii) $\beta = 1, \gamma + \delta = 1$: If $\gamma = \delta = \frac{1}{2}$, a Heston-type model is obtained. Formula (1) remains valid with $\lambda$ replaced by $-\lim_{v\to\infty} b(v)/v$.

If $\gamma \neq \delta$, the model can be transformed into a Heston-like model by the change of variables $\widetilde{V_t} := V_t^{2\delta}$. The time of moment explosion $T_*(u)$ can be related to formula (1), by

$$T_*(u) = \frac{1}{2\delta} T_*^{\text{Heston}}(u).$$

# Results of Lions & Musiela (2)

(iv) $\beta = 1, \gamma + \delta > 1$: Let

$$b_\infty = \lim_{v \to \infty} b(v)/v^{\delta+\gamma}$$
$$\rho^*(u) = -\sqrt{(u-1)/u} - b_\infty/(\eta u) \ .$$

The moment explosion time is given by

$$T_*(u) = \begin{cases} +\infty & \rho < \rho^*(u) \ , \\ 0 & \rho > \rho^*(u) \ . \end{cases}$$

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   - Moment Explosions in SABR-type Models
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2. Long-time Asymptotics for affine SV Models
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# Affine Stochastic Volatility Models (1)

We consider time homogeneous Markov processes $(X_t, V_t)_{t \geq 0}$, such that

$$\mathbb{E}\left[e^{uX_t} | X_0 = x, V_0 = v\right] = e^{ux} \exp\left(\phi(t,u) + v\psi(t,u)\right)$$

We can prove that $\phi$ and $\psi$ are differentiable in $t$, and thus that $(X_t, V_t)_{t \geq 0}$ is a regular affine process in the sense of Duffie et al. [2003].

In particular this implies that $(X_t, V_t)_{t \geq 0}$ is a semi-martingale.

# Affine Stochastic Volatility Models (2)

Define

$$F(u,w) = \frac{\partial}{\partial t}\phi(t,u,w)\Big|_{t=0}, \qquad R(u,w) = \frac{\partial}{\partial t}\psi(t,u,w)\Big|_{t=0}.$$

General results on affine processes yield that

## $F$ and $R$ are of Levy-Khintchine form:

$$\begin{aligned} F(u,w) &= (u,w)\cdot\frac{a}{2}\cdot\begin{pmatrix}u\\w\end{pmatrix} + b\cdot\begin{pmatrix}u\\w\end{pmatrix} + \\ &+ \int_{D\setminus\{0\}}\left(e^{xu+yw} - 1 - h_F(x,y)\cdot\begin{pmatrix}u\\w\end{pmatrix}\right) m(dx,dy)\,, \\ R(u,w) &= (u,w)\cdot\frac{\alpha}{2}\cdot\begin{pmatrix}u\\w\end{pmatrix} + \beta\cdot\begin{pmatrix}u\\w\end{pmatrix} + \\ &+ \int_{D\setminus\{0\}}\left(e^{xu+yw} - 1 - h_R(x,y)\cdot\begin{pmatrix}u\\w\end{pmatrix}\right) \mu(dx,dy) \end{aligned}$$

# Affine Stochastic Volatility Models (3)

The functions $\phi$ and $\psi$ satisfy the 'generalized Riccati equations':

## Generalized Riccati Equations

$$\begin{aligned} \partial_t \phi(t,u,w) &= F(u, \psi(t,u,w)), \quad & \phi(0,u,w) = 0 \\ \partial_t \psi(t,u,w) &= R(u, \psi(t,u,w)), \quad & \psi(0,u,w) = w \ . \end{aligned}$$

- scalar, autonomous ODE; $u$ enters as parameter, $w$ as initial condition.
- The martingale condition on $\exp(X_t)$ implies that

$$F(0,0) = R(0,0) = F(1,0) = R(1,0) = 0 \ .$$

- Our results will follow from a qualitative analysis of the Riccati equations & convexity properties of $F$ and $R$.

# Affine Stochastic Volatility Models (4)

The class of ASVMs such defined, includes the Heston model with and without added jumps, the models of Bates [1996, 2000] and the BNS model.

The following function appears in several conditions and can be interpreted as the exponential rate at which a given moment of the price process' transitional distribution converges to its asymptotic state.

## Definition

For each $u \in \mathbb{R}$ where $R(u, 0) < \infty$, define $\chi(u)$ as

$$\chi(u) := \frac{\partial R}{\partial w}(u, w)\bigg|_{w \uparrow 0} .$$

# Moment Explosions in ASVMs (1)

**Lemma**

Suppose that $\chi(0) < 0$ and $\chi(1) < 0$. Then there exist an interval $I$, such that $[0,1] \subseteq I$, and a unique function $w \in C^1(I^\circ) \cap C(I)$, that cannot be $C^1$-extended beyond $I^\circ$, such that

$$R(u, w(u)) = 0 \qquad \text{for all} \quad u \in I$$

and $w(0) = w(1) = 0$.
Moreover $w(u) < 0$ for all $u \in (0,1)$ and

$$\frac{\partial R}{\partial w}(u, w(u)) < 0, \quad \text{for all } u \in I.$$

$w(u)$ are exactly the asymptotically stable equilibria of the second generalized Riccati equation.

# Moment Explosions in ASVMs (2)

- **Solution converges to equilibrium point:** No moment explosion occurs.
- **Solution does not converge to equilibrium point:** Use extension theorems for autonomous ODEs – Each solution can be extended to come arbitrarily close to the boundary of the right hand side's domain. Reaching the boundary in finite time $\Rightarrow$ Moment explosion.

# Moment Explosions in ASVMs (3)

## Theorem (Moment Explosions in ASVMs)

*Suppose that* $\chi(0) < 0$ *and* $\chi(1) < 0$*, and define* $J = \{u : F(u, w(u)) < \infty\}$ *and*

$$f_+(u) := \sup\{w \geq 0 : F(u, w) < \infty\}$$
$$r_+(u) := \sup\{w \geq 0 : R(u, w) < \infty\} \ .$$

*Suppose that* $F(u, 0) < \infty$*,* $R(u, 0) < \infty$ *and* $\chi(u) < \infty$*.*

- *If* $u \in J$ *then*
$$T_*(u) = +\infty \ .$$
- *If* $u \in \mathbb{R} \setminus \overline{J}$*, then*
$$T_*(u) = \int_0^{\min(f_+(u), r_+(u))} \frac{1}{R(u, \eta)} \, d\eta \ .$$

# Moment Explosions in ASVMs (4)

**Theorem (continued)**

*If* $F(u,0) = \infty$, $R(u,0) = \infty$, *or* $\chi(u) = \infty$, *then*

- $$T_*(u) = 0 \, .$$

- Heston model

$$T_*(u) = \begin{cases} +\infty & \Delta(u) \geq 0 \\ \frac{2}{\sqrt{-\Delta(u)}} \left( \arctan \frac{\sqrt{-\Delta(u)}}{\chi(u)} + \pi \mathbf{1}_{\{\chi(u)<0\}} \right) & \Delta(u) < 0 \,. \end{cases}$$

where $\chi(u) = \rho\zeta u - \lambda$ and $\Delta(u) = \chi(u)^2 - \zeta^2(u^2 - u)$.

- Bates [2000] model (with state-dependent jumps)

$$T_*(u) = \begin{cases} +\infty & \Delta(u) > 0 \\ \frac{2}{\sqrt{-\Delta(u)}} \left( \arctan \frac{\sqrt{-\Delta(u)}}{\chi(u)} + \pi \mathbf{1}_{\{\chi(u)<0\}} \right) & -\infty < \Delta(u) < 0 \\ 0 & \Delta(u) = -\infty \,. \end{cases}$$

where $\chi(u) = \rho\zeta u - \lambda$, $\Delta(u) = \chi(u)^2 - \zeta^2(u^2 - u + 2\widetilde{\kappa}(u))$, and $\widetilde{\kappa}(u)$ is the cumulant generating function of the compensated jump measure.

- BNS model

$$T_*(u) = -\frac{1}{\lambda} \log \max \left(1 - \frac{2\lambda(\max(\kappa_+ - \rho u, 0))}{u(u-1)}, 0\right) \ ,$$

where $\kappa_+ = \sup\{u \in \mathbb{R} : \kappa(u) < \infty\}$.

Extensions:

- The times of moment explosion can also be calculated for the stochastic volatility model 'in the stationary variance regime', i.e. when $V_0$ follows the stationary distribution of $(V_t)_{t \geq 0}$.

# Long-time behavior of affine SV models

Relation to Moment Explosions in affine SV models:

- Both types of results are obtained through the qualitative analysis of the generalized Riccati equations.
- Both types of results have applications to the asymptotics of the implied volatility surface.
- Moment Explosions $\Longrightarrow$ Large-Strike Asymptotics<br>Long-Term Behaviour $\Longrightarrow$ Large-Time-to-Maturity Asymptotics.

# Long-time behavior of the Log-price process (1)

**Equilibrium Points revisited:**

**Lemma**

Suppose that $\chi(0) < 0$ and $\chi(1) < 0$. Then there exist an interval $I$, such that $[0,1] \subseteq I$, and a unique function $w \in C^1(I^\circ) \cap C(I)$, that cannot be $C^1$-extended beyond $I^\circ$, such that

$$R(u, w(u)) = 0 \qquad \text{for all} \quad u \in I$$

and $w(0) = w(1) = 0$.
Moreover $w(u) < 0$ for all $u \in (0,1)$ and

$$\frac{\partial R}{\partial w}(u, w(u)) < 0, \quad \text{for all } u \in I.$$

$w(u)$ are exactly the asymptotically stable equilibria of the second generalized Riccati equation.

# Long-time behavior of the Log-price process (2)

**Theorem (Long-term behavior of $(X_t)_{t\geq 0}$)**

*Suppose that* $\chi(0) < 0$ *and* $\chi(1) < 0$ *and define*

$$m(u) = F(u, w(u)), \quad J = \{u \in I : m(u) < \infty\} \ .$$

*Then* $w(u)$ *and* $m(u)$ *are cumulant generating functions of infinitely divisible random variables and*

$$\lim_{t\to\infty} \psi(t, u, 0) = w(u) \quad \textit{for all} \quad u \in I \ ;$$
$$\lim_{t\to\infty} \frac{1}{t}\phi(t, u, 0) = m(u) \quad \textit{for all} \quad u \in J \ .$$

The convergence to $w(u)$ and $m(u)$ is exponential with rate $\chi(u)$.
Interpretation: for large $t$, $(X_t)_{t\geq 0}$ 'looks like' the Levy process with char. exponent $m(u)$.

# Example: The Heston Model (1)

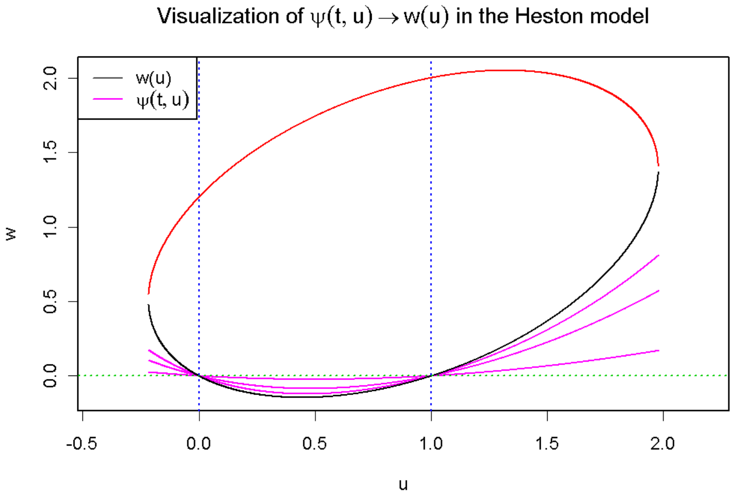
## Heston in SDE form

$$dX_t = -\frac{1}{2} V_t \, dt + \sqrt{V_t} \, dW_t^1$$
$$dV_t = -\lambda(V_t - \theta) \, dt + \gamma \sqrt{V_t} \, dW_t^2$$
$$\langle dW_t^1, dW_t^2 \rangle = \rho \, dt$$

## Heston in affine form

$$F(u, w) = \lambda \theta w$$
$$R(u, w) = -\frac{1}{2} u - \lambda w + \frac{u^2}{2} + \frac{\gamma^2 w^2}{2} + \rho \gamma w u$$

# Example: The Heston Model (2)



# Example: The Heston Model (3)

In the Heston model

$$w(u) = \frac{(\lambda - u\rho\gamma) - \sqrt{(\lambda - u\rho\gamma)^2 - \gamma^2(u^2 - u)}}{\gamma^2}$$

$$m(u) = \lambda\theta \frac{(\lambda - u\rho\gamma) - \sqrt{(\lambda - u\rho\gamma)^2 - \gamma^2(u^2 - u)}}{\gamma^2}\ .$$

This is the cumulant generating function of a **Normal-Inverse-Gaussian** distribution.

# Application to the implied volatility smile (1)

Write the price of a European call with time-to-maturity $T$ and log-moneyness $\xi = \log(e^{(-rT)}K/S_0)$ as Fourier Integral, and use a saddlepoint approximation:

$$\frac{1}{S_0}C(T,\xi) = 1 - \frac{e^{(1-u_*)\xi}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iz\xi}\exp\left(\Phi_T(u_* + iz)\right)}{(z + i(1-u_*))(z - iu_*)}\,dz =$$

$$= 1 - \frac{\exp\left((1-u_*)\xi + Tm(u_*) + C\right)}{2u_*(1-u_*)\pi\sqrt{T}}\sqrt{\frac{1}{2\pi m''(u_*)}} + \mathcal{O}\left(\frac{1}{T}\right)$$

under the condition $m'(u_*) = 0$.

# Application to the implied volatility smile (2)

Comparing with a Black-Scholes price yields:

## Long-term Asymptotics for the volatility smile

Let $u_*$ be the solution of

$$m'(u_*) = 0 \; .$$

Then

$$\sigma^2_{\text{imp}}(T,\xi)\Big|_{\xi=0} = -8m(u_*) + \mathcal{O}(T^{-1})$$

$$\frac{\partial}{\partial \xi}\sigma^2_{\text{imp}}(T,\xi)\Big|_{\xi=0} = \frac{1}{T}(8u_* - 4) + \mathcal{O}(T^{-2})$$

for $T \to \infty$.

# Summary

- In SABR-type models, the time of moment explosion can be analyzed by constructing super- and subsolutions to the Kolmogorov PDE.
- In affine models, the time of moment explosions can be analyzed by determining the blow-up time of an ODE of generalized Riccati type.
- The long-term behavior of affine stochastic volatility models can also be studied through qualitative analysis of the generalized Riccati equations.
- All the above results have applications to the asymptotics of the implied volatility smile, both for long-time and large-strike behavior.

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