Yield Curve Shapes and the Asymptotic Short Rate Distribution in Affine One-Factor Models

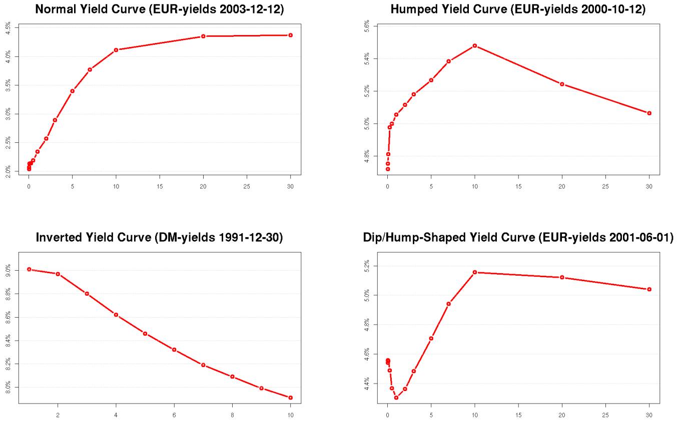
Martin Keller-Ressel
(joint work with Thomas Steiner)

Vienna University of Technology
Research Group for Financial and Actuarial Mathematics (FAM)
http://www.fam.tuwien.ac.at

March, 26th 2007
Frankfurt MathFinance Workshop

1. Motivation
2. Preliminaries
   - Affine Processes
   - Bond Prices and the Yield Curve
3. Theoretical Results
   - The Shape of the Yield Curve
   - The Asymptotic Short Rate Distribution
4. Applications
   - The Vasiček Model
   - The CIR Model
   - The JCIR Model
   - The Gamma Model

Real-world yield curves come in many different shapes:



The Vasiček model, for example, can only reproduce 3 of them: normal, inverse and humped.

The Cox-Ingersoll-Ross (CIR-)model can . . .

The Cox-Ingersoll-Ross (CIR-)model can ...

- ... produce only normal, inverse or humped shapes.

[Cox et al.(1985)]

The Cox-Ingersoll-Ross (CIR-)model can . . .

- . . . produce only normal, inverse or humped shapes.

  [Cox et al.(1985)]

- *'. . . produce [also] yield curves with one hump or one dip'*

  [Carmona and Tehranchi(2006)]

The Cox-Ingersoll-Ross (CIR-)model can ...

- ... produce only normal, inverse or humped shapes.

  [Cox et al.(1985)]

- *'... produce [also] yield curves with one hump or one dip'*

  [Carmona and Tehranchi(2006)]

- ... not reproduce *'some typical shapes, like that of an inverted yield curve.*

  [Brigo and Mercurio(2006)]

The Cox-Ingersoll-Ross (CIR-)model can . . .

- . . . produce only normal, inverse or humped shapes.

  [Cox et al.(1985)]

- *'. . . produce [also] yield curves with one hump or one dip'*

  [Carmona and Tehranchi(2006)]

- . . . not reproduce *'some typical shapes, like that of an inverted yield curve.*

  [Brigo and Mercurio(2006)]

Several sources make contradictory statements on this topic; none of them gives a proof $\Longrightarrow$ This must be an interesting question!

Other questions quickly come to mind:

- What happens if jumps are added to the CIR model, or if the driving Brownian Motion of the Vasiček model is replaced by a Lévy process?
- How is the shape of the yield curve determined by the model parameters?
- How is the shape of the yield curve influenced by the current short rate?

Affine Processes

The class of affine processes (introduced by [Duffie et al.(2003)]) contains all models mentioned so far.

Definition (One-dimensional affine process)

A time-homogenous Markov process $(r_t)_{t\geq 0}$ with state space $D = \mathbb{R}_{\geqslant 0}$ or $\mathbb{R}$ and its semi-group $(P_t)_{t\geq 0}$ are called *affine*, if the characteristic function of its transition kernel $p_t(x,.)$, given by

$$\widehat{p}_t(x,u) = \int_D e^{u\xi}\, p_t(x,d\xi)$$

and defined on some set $\mathcal{U} \subseteq \mathbb{C}$ is *exponentially affine* in x. That is, there exist $\mathbb{C}$-valued functions $\phi(t,u)$ and $\psi(t,u)$, defined on $\mathbb{R}_{\geqslant 0} \times \mathcal{U}$, such that

$$\widehat{p}_t(x,u) = \exp\left(\phi(t,u) + x\psi(t,u)\right) \quad \text{for all} \quad x \in D, u \in \mathcal{U}\ .$$

An affine process is called **regular** if it is stochastically continuous and the one-sided ∂_t-derivatives of $\phi(t,u)$ and $\psi(t,u)$ exist at $t = 0$ for all $u \in \mathcal{U}$, and are continuous at $u = 0$.
It is called **conservative** if $p_t(x, D) = 1$ for all $t > 0, x \in D$.

Closely related to affine processes, is the notion of an Ornstein-Uhlenbeck (OU-)type process:

Definition (OU-type Process)

A conservative, regular affine process is called OU-type process, if there exists a $\beta \in \mathbb{R}$ such that $\psi(t,u) = ue^{\beta t}$.

OU-type process have been studied for longer than affine processes and usually offer good analytic tractability.

Admissibility Conditions

The parameters $(a, \alpha, b, \beta, m, \mu)$ are called *admissible* for an affine process with state space $\mathbb{R}_{\geqslant 0}$ if

$$a = 0; \quad \alpha, b \in \mathbb{R}_{\geqslant 0}; \quad \beta \in \mathbb{R}; \quad \int_{(0,\infty)} (x \wedge 1)\, m(dx) < \infty \,,$$

and admissible for a process with state space $\mathbb{R}$ if

$$a \in \mathbb{R}_{\geqslant 0}; \quad b, \beta \in \mathbb{R}; \quad \alpha = 0, \mu \equiv 0 \,.$$

Moreover define the functions $F(u)$, $R(u)$ for $u \in \mathcal{U}$ as

$$F(u) = au^2 + bu + \int_{D\setminus\{0\}} \left(e^{u\xi} - 1 - uh_F(\xi)\right)\, m(d\xi) \,,$$

$$R(u) = \alpha u^2 + \beta u + \int_{D\setminus\{0\}} \left(e^{u\xi} - 1 - uh_R(\xi)\right)\, \mu(d\xi) \,.$$

Theorem ([Duffie et al.(2003)])

Suppose $(r_t)_{t \geq 0}$ is a one-dimensional, conservative, regular affine process. Then it is a Feller process and there exist some admissible parameters $(a, \alpha, b, \beta, m, \mu)$ such that the generator of $(r_t)_{t \geq 0}$ is given by

$$\mathcal{A}f(x) = (a + \alpha x) f''(x) + (b + \beta x) f'(x) + \\ + \int_{D \setminus \{0\}} \left(f(x+\xi) - f(x) - f'(x) h_F(\xi) \right) \, m(d\xi) + \\ + x \int_{D \setminus \{0\}} \left(f(x+\xi) - f(x) - f'(x) h_R(\xi) \right) \, \mu(d\xi) \,.$$

Moreover, the functions $\phi(t,u)$ and $\psi(t,u)$ are the unique solutions of the **generalized Riccati equations**

$$\partial_t \, \phi(t,u) = F\left(\psi(t,u)\right), \qquad \phi(0,u) = 0 \,,$$
$$\partial_t \, \psi(t,u) = R\left(\psi(t,u)\right), \qquad \psi(0,u) = u \,.$$

Quantities of the type

$$\mathbb{E}\left[\exp\left(-\int_0^t r_s\,ds\right) f(r_t)\middle| r_0 = x\right]$$

can be calculated by applying the Feynman-Kac formula for Feller semigroups.

To guarantee the global existence of bond prices, we will need the following condition:

Condition A

The one-dimensional affine process $(r_t)_{t\geq 0}$ is assumed to be regular and conservative. In addition, if the process has state space $\mathbb{R}$, such that $R(u) = \beta u$, we require that

$$F(u) < \infty \quad \text{for all} \quad u \in \begin{cases} (1/\beta, 0] & \text{if} \quad \beta < 0\,, \\ (-\infty, 0] & \text{else}\,. \end{cases}$$

Bond Prices for Affine Processes

Bond Prices

Let the short rate be given by a one-dimensional affine process $(r_t)_{t\geq 0}$ satisfying Condition A.
Then the bond price $P(t, t+x)$ exists for all $t, x \geq 0$ and is given by

$$P(t, t+x) = \exp\left(A(x) + r_t B(x)\right)$$

where A and B are the unique, globally defined solutions of the generalized Riccati equations

$$\begin{aligned} \partial_x A(x) &= F\left(B(x)\right) & \qquad A(0) &= 0\,, \\ \partial_x B(x) &= R\left(B(x)\right) - 1 & \qquad B(0) &= 0\,. \end{aligned}$$

The Yield Curve

Definition (Yield Curve)

The (zero-coupon) yield $Y(r_t, x)$ is given by $Y(r_t, 0) := r_t$ and

$$Y(r_t, x) := -\frac{\log P(t, t+x)}{x} = -\frac{A(x)}{x} - r_t \frac{B(x)}{x} \quad \text{for all} \quad x > 0 \,.$$

For r_t fixed, we call the function $Y(r_t, .)$ the **yield curve**.

Definition (Forward Rate Curve)

The (instantaneous) forward rate $f(r_t, x)$ is given by $f(r_t, 0) := r_t$ and

$$f(r_t, x) := -\partial_x \log P(t, t+x) = -A'(x) - r_t B'(x) \quad \text{for all} \quad x > 0 \,.$$

For r_t fixed, we call the function $f(r_t, .)$ the **forward rate curve**.

Quasi-Mean-Reversion

We define the following quantity, that generalizes the rate of mean-reversion of a OU-type process:

Definition (Quasi-mean-reversion)

Given a one-dimensional conservative affine process $(r_t)_{t \geq 0}$, define the *quasi-mean-reversion* λ as the positive solution of

$$R(-1/\lambda) = 1 \; .$$

If there is no positive solution we set $\lambda := 0$.

The long end of the yield curve

The first quantity associated to the yield curve that we consider, is the asymptotic level b_{asymp} of the yield curve as $x \to \infty$, also known as long-term yield, consol yield or simply 'long end'.

Theorem 1 – [K. & S.(2007)]

Let the short rate process be given by a one-dimensional affine process $(r_t)_{t \geq 0}$ satisfying Condition A with quasi-mean-reversion λ. If $\lambda > 0$ then

$$b_{\text{asymp}} = \lim_{x \to \infty} Y(r_t, x) = \lim_{x \to \infty} f(r_t, x) = -F(-1/\lambda) \ .$$

If $\lambda = 0$ then

$$b_{\text{asymp}}(r_t) = -F(-\infty) + r_t \left(1 - R(-\infty)\right)$$

where $F(-\infty)$ and $R(-\infty)$ are understood as limits.

The Dybvig-Ingersoll-Ross-Theorem

As an immediately Corollary we get the Dybvig-Ingersoll-Ross theorem for affine processes:

Theorem – 'Long forward rates never fall'

Under the conditions of Theorem 1, the long forward rate $b_{\mathrm{asymp}}(r_t)$ is non-decreasing in t almost surely.

The above theorem has been stated originally by [Dybvig et al.(1996)], with a general proof given by [Hubalek et al.(2002)].

Yield Curve Shapes

We distinguish the following yield curve shapes

Definition

The yield curve $Y(r_t, x)$ is called

- **normal** if it is a strictly increasing function of x,
- **inverse** if it is a strictly decreasing function of x,
- **humped** if it has exactly one local maximum and no minimum on $(0, \infty)$.

In addition we call the yield curve **flat** if it is constant over all $x \in \mathbb{R}_{\geqslant 0}$.

This is our main result on yield curve shapes:

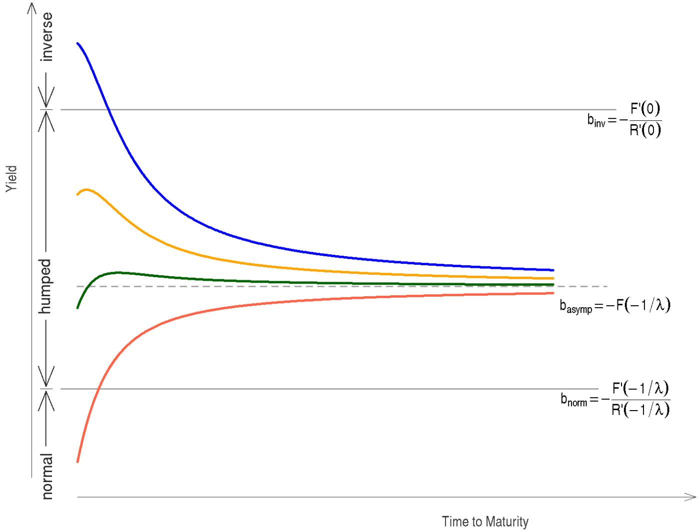
Theorem 2 – [K. & S.(2007)]

Let the short rate process be given by a one-dimensional affine process $(r_t)_{t\geq 0}$ satisfying Condition A. In addition suppose that $\lambda > 0$, $F \neq 0$ and that either F or R is non-linear. Then the following holds:

- The yield curve $Y(r_t, .)$ can only be normal, inverse or humped.
- Define

$$b_{\mathrm{norm}} := -\frac{F'(-1/\lambda)}{R'(-1/\lambda)}, \qquad b_{\mathrm{inv}} := \begin{cases} -\dfrac{F'(0)}{R'(0)} & \text{if } R'(0) < 0 \\ +\infty & \text{if } R'(0) \geq 0 \,. \end{cases}$$

The yield curve is normal if $r_t \leq b_{\mathrm{norm}}$, humped if $b_{\mathrm{norm}} < r_t < b_{\mathrm{inv}}$ and inverse if $r_t \geq b_{\mathrm{inv}}$.



An illustration of Theorem 2

Theorem 2 leads to several interesting Corollaries:

Corollary 1

Under the conditions of Theorem 2 the instantaneous forward rate curve has the same global behavior as the yield curve, i.e.

$Y(r_t, .)$ is inverse $\iff$ $f(r_t, .)$ is strictly decreasing

$Y(r_t, .)$ is humped $\iff$ $f(r_t, .)$ has exactly one local maximum and no local minimum

$Y(r_t, .)$ is normal $\iff$ $f(r_t, .)$ is strictly increasing .

In the second case the maximum of the forward rate curve is $f(r_t, x_*)$, where x_* solves

$$r_t = -\frac{F'(B(x))}{R'(B(x))}, \quad x \in (0, \infty) .$$

Corollary 2

Under the conditions of Theorem 2 it holds that

$$b_{\mathrm{norm}} < b_{\mathrm{asymp}} < b_{\mathrm{inv}} \quad \text{and that} \quad D \cap (b_{\mathrm{norm}}, b_{\mathrm{inv}}) \neq \emptyset \,.$$

The occurrence of a humped yield curve is a necessary and sufficient sign of randomness in the short rate model:

Corollary 3

Let $(r_t)_{t \geq 0}$ satisfy Condition A with $F \neq 0$ and $\lambda > 0$. Then the following statements are equivalent:

(i) There exists a $r_t \in D$ such that $Y(r_t, .)$ is flat.

(ii) There exists no $r_t \in D$ such that $Y(r_t, .)$ is humped.

(iii) The short rate process $(r_t)_{t \geq 0}$ is deterministic.

(iv) $F(u) = bu$ and $R(u) = \beta u$.

The Asymptotic Short Rate Distribution

We recall two properties of $\mathbb{R}$-valued random variables:

Infinitely Divisible

A $\mathbb{R}$-valued random variable X is called infinitely divisible, if for every $n \in \mathbb{N}$, there exist i.i.d random variables $(X_1^n, \dots, X_n^n)$ such that

$$X \overset{d}{=} X_1^n + \dots + X_n^n .$$

Self-Decomposable

A $\mathbb{R}$-valued random variable X is called self-decomposable, if for every $c \in (0,1)$, there exists a random variable X_c, independent of X, such that

$$X \overset{d}{=} cX + X_c .$$

Every self-decomposable random variable is also infinitely divisible, but the reverse is not true.

For OU-type processes, limit distributions have been studied for some time; the following result goes back to [Jurek and Vervaat(1983)] and [Sato and Yamazato(1984)].

Theorem ([Jurek and Vervaat(1983)] and others)

Let $(r_t)_{t\geq 0}$ a OU-type process on $\mathbb{R}$ (i.e. a conservative, regular affine process with $R(u) = \beta u$). If $\beta < 0$ and

$$\exists\, \epsilon > 0 \text{ such that } \int_0^\epsilon \frac{|F(s)|}{s}\, ds < \infty$$

then $(r_t)_{t\geq 0}$ converges to a limit distribution L, such that:

(i) *L is self-decomposable.*

(ii) *The cumulant generating function $\kappa(u) = \log \mathbb{E}\left[e^{uL}\right]$ exists for all $u \in \mathcal{U}$ and is given by*

$$\kappa(u) = \frac{1}{\beta} \int_u^0 \frac{F(s)}{s} ds \qquad u \in \mathcal{U}\ .$$

The result can be extended to affine processes:

Theorem 3 – [K. & S.(2007)]

Let $(r_t)_{t\geq 0}$ be a one-dimensional, regular, conservative affine process with state space $\mathbb{R}_+$. If $R'(0) < 0$ and

$$\exists\ \epsilon > 0 \text{ such that } \int_{-\epsilon}^{0} \frac{F(s)}{R(s)}\, ds < \infty$$

then $(r_t)_{t\geq 0}$ converges in law to a limit distribution L, such that:

(i) L is infinitely divisible.

(ii) The cumulant generating function $\kappa(u) = \log \mathbb{E}\left[e^{uL}\right]$ of L exists for all $u \in \mathcal{U}$ and is given by

$$\kappa(u) = \int_{u}^{0} \frac{F(s)}{R(s)} ds \qquad u \in \mathcal{U}\ .$$

Some remarks:

- In the OU-type case, the limit distribution is self-decomposable, in the affine case only infinitely divisible.
- In the OU-type case, also a converse result exists: If L is a self-decomposable distribution on $\mathbb{R}$, there exists a OU-type process, converging to L.
- In the affine case no such result is known yet.

Applications – Overview

- The Vasiček Model
- The Cox-Ingersoll-Ross Model
- The JCIR Model
- The Gamma Model

Warm up - The Vasiček model

The short rate is given by a Gaussian OU process:

The Vasiček Model

$$dr_t = -\lambda(r_t - \theta)\,dt + \sigma\,dW_t, \quad r_0 \in \mathbb{R}$$

From the form of the generator of $(r_t)_{t\geq 0}$ we get

$$F(u) = \lambda\theta u + \frac{\sigma^2}{2}u^2 \quad \text{and} \quad R(u) = -\lambda u$$

It is calculated immediately that

$$b_{\mathrm{inv}} = -\frac{F'(0)}{R'(0)} = \theta$$

$$b_{\mathrm{asymp}} = -F(-1/\lambda) = \theta - \frac{\sigma^2}{2\lambda^2}$$

$$b_{\mathrm{norm}} = -\frac{F'(-1/\lambda)}{R'(-1/\lambda)} = \theta - \frac{\sigma^2}{\lambda^2}$$

Note that for the Vasiček model b_{asymp} is the *arithmetic* average of b_{norm} and b_{inv}.

From Theorem 3 we find that the cumulant generating function of the limit distribution is

$$\kappa(u) = \int_u^0 \frac{F(s)}{R(s)}\, ds = \int_0^u \theta + \frac{\sigma^2}{2\lambda} s\, ds = u\theta + \frac{u^2}{2}\frac{\sigma^2}{2\lambda}$$

which corresponds to a Gaussian distribution with mean θ and variance $\frac{\sigma^2}{2\lambda}$.

Note that for the Vasiček model b_{asymp} is the *arithmetic* average of b_{norm} and b_{inv}.

From Theorem 3 we find that the cumulant generating function of the limit distribution is

$$\kappa(u) = \int_u^0 \frac{F(s)}{R(s)}\, ds = \int_0^u \theta + \frac{\sigma^2}{2\lambda} s\, ds = u\theta + \frac{u^2}{2}\frac{\sigma^2}{2\lambda}$$

which corresponds to a Gaussian distribution with mean θ and variance $\frac{\sigma^2}{2\lambda}$.

All this has been well-known in case of the Vasiček model.

The CIR model

The short rate is given by square root diffusion process. It has state space $\mathbb{R}_+$, i.e. the short rate can never become negative.

The Cox-Ingersoll-Ross Model

$$dr_t = -a(r_t - \theta)dt + \sigma\sqrt{r_t}\,dW_t, \qquad r_0 \geq 0$$

We find

$$F(u) = a\theta u, \qquad R(u) = \frac{\sigma^2}{2}u^2 - au\ .$$

The quasi-mean-reversion is given by

$$\lambda = \frac{1}{2}\left(\sqrt{a^2 + 2\sigma^2} + a\right)\ .$$

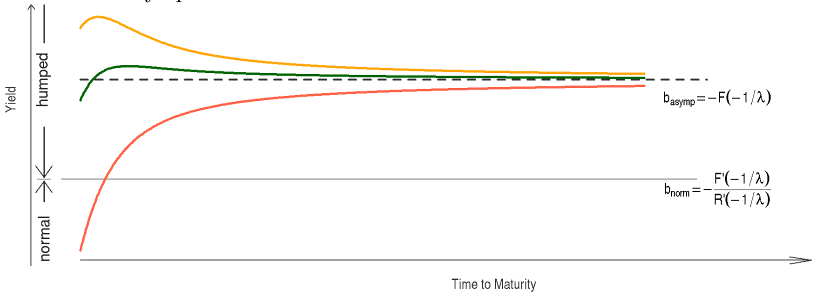
It is easily calculated that

$$b_{\mathrm{inv}} = \theta,$$
$$b_{\mathrm{asymp}} = \frac{2a\theta}{\sqrt{a^2 + 2\sigma^2} + a},$$
$$b_{\mathrm{norm}} = \frac{a\theta}{\sqrt{a^2 + 2\sigma^2}}.$$

Note that b_{asymp} is the *harmonic* mean of b_{inv} and b_{norm}. We quote now from page 394 of [Cox et al.(1985)]:

> *'When the spot rate is below the long-term yield* $[= b_{\mathrm{asymp}}]$*, the term structure is uniformly rising. With an interest rate in excess of* θ $[= b_{\mathrm{inv}}]$*, the term structure is falling. For intermediate values of the interest rate, the yield curve is humped.'*

In other words, [Cox et al.(1985)] claim that any yield curve starting below b_{asymp} is normal. This claim is repeated e.g. in [Rebonato(1998)]. It stands, however, in contradiction to Theorem 2 and its Corollaries, which state that all yield curves starting in $(b_{\mathrm{norm}}, b_{\mathrm{inv}})$ are humped, and that $b_{\mathrm{norm}} < b_{\mathrm{asymp}} < b_{\mathrm{inv}}$.



The middle yield curve provides a counter-example to the claims of [Cox et al.(1985)].

The asymptotic distribution of the short rate can be calculated by Theorem 3. It yields

$$\kappa(u) = \int_u^0 \frac{F(s)}{R(s)}\,ds = \int_0^u \frac{\theta}{1 - \frac{\sigma^2}{2a}s}\,ds = -\frac{2a\theta}{\sigma^2}\log\left(1 - \frac{\sigma^2}{2a}u\right)$$

which is the cumulant generating function of a gamma distribution with shape parameter $2a\theta/\sigma^2$ and scale parameter $\sigma^2/2a$.

JCIR – The CIR model with added jumps.

The JCIR model is used for interest rate modelling in [Brigo and Mercurio(2006)]. The short rate is given by

The JCIR Model

$$dr_t = -a(r_t - \theta)dt + \sigma\sqrt{r_t}\,dW_t + dJ_t, \qquad r_0 \geq 0$$

where $(J_t)_{t\geq 0}$ is a compound Poisson process with exponentially distributed jumps with mean ν and intensity c. We find

$$F(u) = a\theta u + \frac{cu}{\nu - u}, \qquad R(u) = \frac{\sigma^2}{2}u^2 - au\ .$$

By Theorem 1 and 2 we obtain

$$\begin{aligned} b_{\mathrm{inv}} &= \theta + \frac{c}{a\nu} \\ b_{\mathrm{asymp}} &= \frac{2a\theta}{a+\gamma} + \frac{2c}{\nu(a+\nu)+2}; \qquad (\gamma = \sqrt{a^2+2\sigma^2}) \\ b_{\mathrm{norm}} &= \frac{a\theta}{\gamma} + \frac{c\nu\sigma^4}{\gamma(\sigma^2\nu+\gamma-a)^2}\,. \end{aligned}$$

Setting $c = 0$ the values of the CIR model are recovered.

For the cumulant generating function of the asymptotic distribution we get (with $\rho := \sigma^2/2$ and $\Delta := a - \nu\rho$)

$$\kappa(u) = \left(\frac{c}{\Delta} - \frac{a\theta}{\rho}\right)\log\left(1 - \frac{\rho}{a}u\right) - \frac{c}{\Delta}\log\left(1 - \frac{u}{\nu}\right)\ .$$

Under the conditions $\Delta > 0$ and $a\theta/\rho > c/\Delta$ this limit distribution can we written as the convolution of a gamma distribution with parameters $(a\theta/\rho - c/\Delta, \rho/a)$ and a gamma distribution with parameters $(c/\Delta, \nu^{-1})$.

If the above conditions are not fulfilled, it is possible to show that the resulting limit distribution is *infinitely divisible*, but not *selfdecomposable*.

Building new models – The Gamma Model

We want to build a model of OU-type, with the same asymptotic distribution as the CIR model (a gamma distribution):

OU-type implies $R(u) = -\lambda u$; from Theorem 3 we get

$$-k \log(1 - \theta u) = \kappa(u) = \frac{1}{\lambda} \int_0^u \frac{F(s)}{s} ds \, .$$

Solving for F we find

$$F(u) = \frac{\lambda \theta k u}{1 - \theta u}$$

and calculate. . .

$$b_{\mathrm{inv}} = k\theta,$$
$$b_{\mathrm{asymp}} = \frac{k}{1/\theta + 1/\lambda},$$
$$b_{\mathrm{norm}} = \frac{k/\theta}{(1/\theta + 1/\lambda)^2}\,.$$

Here b_{asymp} is just the *geometric* mean of b_{norm} and b_{inv}. Bond prices can be calculated explicitly from the generalized Ricatti equations:

$$P(t, t+x) = \exp\left\{-\lambda x \frac{\theta k}{\theta + \lambda} + r_t B(x)\right\} (\theta B(x) - 1)^{\frac{\lambda k}{\theta + \lambda}}\,.$$

Summary

- Term structure models where the short rate is given by a one-dimensional, time-homogenous affine process produce only inverse, normal or humped yield curves.
- This class of models includes the 'classical' Vasiček and CIR model, but also extensions of these models that are obtained by adding jumps.
- The boundaries between inverse, normal and humped shapes, and the long-term yield, can be calculated directly from the model specification.
- Under some conditions, the short rate process converges to a limit distribution. The cumulant generating function of this distribution can also be calculated directly from the model specification.

Damiano Brigo and Fabio Mercurio.
Interest Rate Models - Theory and Practice.
Springer Finance. Springer, 2nd edition, 2006.

René Carmona and Michael Tehranchi.
Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective.
Springer Finance. Springer, 2006.

John Cox, Jonathan Ingersoll, and Stephen Ross.
A theory on the term structure of interest rates.
Econometrica, 53(2):385–407, 1985.

Darrell Duffie, Damir Filipović, and Walter Schachermayer.
Affine processes and applications in finance.
The Annals of Applied Probability, 13(3):984–1053, 2003.

Philip Dybvig, Jonathan Ingersoll, and Stephen Ross.
Long forward and zero-coupon rates can never fall.

The Journal of Business, 69(1):1–25, 1996.

- Friedrich Hubalek, Irene Klein, and Josef Teichmann.
  A general proof of the Dybvig-Ingersoll-Ross-Theorem: Long forward rates can never fall.
  Mathematical Finance, 12(4):447–451, 2002.

- Zbigniew J. Jurek and Wim Vervaat.
  An integral representation for self-decomposable Banach space valued random variables.
  Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 62:247–262, 1983.

- Martin Keller-Ressel and Thomas Steiner.
  Yield curve shapes and the asymptotic short rate distribution.
  Submitted 2007.

- Riccardo Rebonato.
  Interest-Rate Option Models.
  Wiley, 2nd edition, 1998.

Ken-iti Sato and M. Yamazato.
Operator-selfdecomposable distributions as limit distributions of processes of Ornstein-Uhlenbeck type.
Stochastic Processes and Applications, 17:73–100, 1984.