Affine Processes are regular

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Part I

Introduction

Affine Processes

We consider a stochastic process X that is

- A time-homogeneous Markov process,
- stochastically continuous,
- takes values in $D = \mathbb{R}_+^m \times \mathbb{R}^n$,
- and has the following property:

Affine Property

There exists functions ϕ and ψ, taking values in $\mathbb{C}$ and $\mathbb{C}^{m+n}$ respectively, such that

$$\mathbb{E}^x \left[\exp \langle X_t, u \rangle \right] = \exp \Big(\underbrace{\phi(t,u) + \langle x, \psi(t,u) \rangle}_{\text{affine in } x} \Big)$$

for all $x \in D$, and for all $(t,u) \in \mathbb{R}_+ \times \mathcal{U}$, where

$$\mathcal{U} = \{ u \in \mathbb{C} : \mathrm{Re}\, \langle x, u \rangle \leq 0 \text{ for all } x \in D \} .$$

A short history of Affine Processes

- Affine Processes on $D = \mathbb{R}_+$ have been obtained as continuous-time limits of branching processes, and studied under the name CBI-process (continously branching with immigration) by [Kawazu and Watanabe, 1971].
- Jump-diffusions with the 'affine property' have been studied by [Duffie, Pan, and Singleton, 2000] with a view towards applications in finance.
- [Duffie, Filipovic, and Schachermayer, 2003] give a full characterization of the class of affine processes on $D = \mathbb{R}_+^m \times \mathbb{R}^n$ under a regularity condition.
- [Cuchiero, Filipovic, Mayerhofer, and Teichmann, 2009] are currently characterizing the class of affine processes taking values in the cone of positive semidefinite matrices.

Examples of Affine Processes

The following processes are affine:

- All Lévy processes;
- The Gaussian Ornstein-Uhlenbeck process, and Levy-driven OU-processes;
- The CIR process (jumps can be added);
- Log-Price & Variance in the Heston model, the Bates model, the Barndorff-Nielsen-Shephard model, and in other time-change models for stochastic volatility;
- On matrix state spaces: The Wishart process, matrix subordinators, matrix OU-processes

The Semi-flow Equations

Define $f_u(x) = \exp(\langle x, u \rangle)$, and $P_t f(x) = \mathbb{E}^x\left[f(X_t)\right]$.
We have that

$$
\begin{aligned}
P_{t+s} f_u(x) &= \exp\left(\phi(t+s,u) + \langle x, \psi(t+s,u)\rangle\right) \\
P_{t+s} f_u(x) &= P_s P_t f_u(x) = e^{\phi(t,u)} \cdot P_s f_{\psi(t,u)}(x) = \\
&= \exp(\phi(t,u) + \phi(s,\psi(t,u)) + \langle x, \psi(s,\psi(t,u))\rangle)\,;
\end{aligned}
$$

Comparison yields:

Semi-flow equations

$$
\begin{aligned}
\psi(t+s,u) &= \psi(s,\psi(t,u)), & \psi(0,u) &= u \\
\phi(t+s,u) &= \phi(t,u) + \phi(s,\psi(t,u)), & \phi(0,u) &= 0\ ,
\end{aligned}
$$

for all $t, s \geq 0$ and $u \in \mathcal{U}$.

An eq of the second type is often called a ‘cocycle’ of the first.

The Regularity Assumption

At this point [Duffie et al., 2003] introduce the following regularity assumption:

Regularity

The process X is called regular, if the derivatives

$$F(u) = \frac{\partial}{\partial t}\phi(t,u)\Big|_{t=0}, \quad R(u) = \frac{\partial}{\partial t}\psi(t,u)\Big|_{t=0}$$

exist, and are continuous at $u = 0$.

Under this condition the semi-flow eqs can be differentiated, to give

The generalized Riccati equations

$$\begin{aligned} \partial_t\phi(t,u) &= F(\psi(t,u)), & \phi(0,u) &= 0, \\ \partial_t\psi(t,u) &= R(\psi(t,u)), & \psi(0,u) &= u\ . \end{aligned}$$

Main result of [Duffie et al., 2003]

[Duffie et al., 2003] then proceed to show their main result:

Theorem (Duffie et al. [2003])

Let X be a regular affine process. Then F, R are of the Levy-Khintchine form

$$F(u) = \left\langle \frac{a}{2} u, u \right\rangle + \langle b, u \rangle - c + \int_D \left(e^{\langle \xi, u \rangle} - 1 - \langle h_F(\xi), u \rangle \right) m(d\xi)$$

$$R_i(u) = \left\langle \frac{\alpha_i}{2} u, u \right\rangle + \langle \beta_i, u \rangle - \gamma_i + \int_D \left(e^{\langle \xi, u \rangle} - 1 - \left\langle h_R^i(\xi), u \right\rangle \right) \mu_i(d\xi)$$

where h_F, h_R are suitable truncation functions, and the parameters $(a, \alpha_i, b, \beta_i, c, \gamma_i, m, \mu_i)_{i=1,\dots,d}$ satisfy additional 'admissibility conditions'.
Moreover $(X_t)_{t\geq 0}$ is a Feller process, and its generator given by... $\hookrightarrow$

Main result of [Duffie et al., 2003] (2)

Theorem (continued)

$$\begin{aligned}\mathcal{A}f(x) = \frac{1}{2}\sum_{k,l=1}^{d}\left(a_{kl}+\sum_{i=1}^{m}\alpha_{kl}^{i}x_i\right)\frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle b + \sum_{i=1}^{d}\beta^i x_i, \nabla f(x)\rangle - \\ -(c+\langle x,\gamma\rangle)+ \\ + \int_{D\setminus\{0\}}\left(f(x+\xi)-f(x)-\langle h_F(\xi),\nabla f(x)\rangle\right)m(d\xi)+ \\ + \sum_{i=1}^{m}x_i\int_{D\setminus\{0\}}\left(f(x+\xi)-f(x)-\left\langle h_R^i(\xi),\nabla f(x)\right\rangle\right)\mu^i(d\xi)\end{aligned}$$

for $f \in C_c^\infty(D)$. Conversely, for each admissible parameter set there exists a regular affine process on D with generator $\mathcal{A}$.

Is the regularity assumption necessary?

- There was no known counterexample of a non-regular affine process[1].
- Suppose $\psi(t, u) = u$ (stationary flow). Then the cocycle equation becomes

  $$\phi(t + s, u) = \phi(t, u) + \phi(s, u)\ .$$

  This is Cauchy's functional equation with the unique continuous solution $\phi(t, u) = tm(u)$.

  The regularity condition is automatically fulfilled! It turns out that this is exactly the case of X being a Levy process (killed at a constant rate.)

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[1] If stochastic continuity is dropped, there are plenty

Is the regularity assumption necessary? (2)

- In the article of [Kawazu and Watanabe, 1971] on CBI-processes the regularity condition is also automatically fulfilled. The proof, however, only works for $D = \mathbb{R}_+$.
- [Dawson and Li, 2006] show that an affine process on $\mathbb{R}_+ \times \mathbb{R}$ is automatically regular under a moment condition.

Conjecture: Every affine process is regular.

Part II

Regularity

The semi-flow equations revisited

We take a closer look at the semi-flow equations for ϕ and ψ:

$$\begin{aligned} \psi(t+s,u) &= \psi(s,\psi(t,u)), & \psi(0,u) &= u \\ \phi(t+s,u) &= \phi(t,u)+\phi(s,\psi(t,u)), & \phi(0,u) &= 0\ . \end{aligned}$$

Insight

We can ignore the co-cycle equation for ϕ and concentrate on the (simpler) equation for ψ.

Extend $\mathcal{U}$ by one dimension to $\widehat{\mathcal{U}} = \mathbb{C}_- \times \mathcal{U}$, and for $\widehat{u} = (u_0, u)$ define the 'big flow'

$$\Upsilon : \mathbb{R}_+ \times \widehat{\mathcal{U}} \to \widehat{\mathcal{U}}, \quad (t, \widehat{u}) \mapsto \begin{pmatrix} \phi(t,u) + u_0 \\ \psi(t,u) \end{pmatrix}.$$

(From now on we omit the $\widehat{\text{hat}}$.)

The semi-flow equations revisited (2)

The big flow Υ satisfies the same equation as ψ:

Semiflow equation

$$\Upsilon(t+s,u) = \Upsilon(t,\Upsilon(s,u)), \quad \Upsilon(0,u) = u$$

- For fixed t, $u \mapsto \Upsilon(t,u)$ is a **continuous transformation** of $\mathcal{U}$ onto itself.
- The family of transformations $(u \mapsto \Upsilon(t,u))_{t\geq 0}$ forms a **semi-group of transformations** of $\mathcal{U}$.

This provides a connection to Hilbert's fifth problem.

Hilbert's 5th Problem

Hilbert (1900): „Lieſ Begriff der continuirlichen Tranſformationſgruppe ohne die Annahme der Differenzirbarkeit der die Gruppe definirenden Functionen.“

Hilbert's fifth problem, modern formulation

Let $(\Upsilon_t)_{t \in G}$ be a topological group of continuous transformations (homeomorphisms) of a Hausdorff space U into itself. Suppose that U is a smooth (C^k, real analytic, ...) manifold, and each Υ_t a smooth mapping. Can we conclude that G is a Lie group (i.e. a group with a smooth parametrization)?

Extremely simplified version

Does the group property of $\Upsilon(t, u)$ transfer smoothness from the u-parameter to the t-parameter?

The answer to these questions is YES!, as shown by [Montgomery and Zippin, 1955].

Hilbert vs. Us

Our setting is not the same, but it is comparable to the setting of Hilbert's 5th problem:

Hilbert's 5th problem	Affine Processes
group	1-parameter semigroup
U: differentiable manifold	$\mathcal{U}$: diff. manifold with boundary
$u \mapsto \Upsilon(t,u)$: homeomorphisms	$u \mapsto \Upsilon(t,u)$: non-invertible
$u \mapsto \Upsilon(t,u)$ smooth	???

If $\Upsilon(t,u)$ – or equivalently $\phi(t,u)$ and $\psi(t,u)$ – are smooth in u (e.g C^1), then the idea of Montgomery & Zippin's proof can be applied in our setting.

Differentiability of $u \mapsto (\phi(t,u), \psi(t,u))$

- $\phi(t,u)$ and $\psi(t,u)$ are differentiable in the interior of $\mathcal{U}$ in the directions corresponding to the positive part $\mathbb{R}_+^m$ of the state space.
- $\phi(t,u)$ and $\psi(t,u)$ are not necessarily differentiable in the directions corresponding to the real-valued part $\mathbb{R}^n$ of the state space.
- If we impose moment conditions on the process X, we can make $\phi(t,u)$ and $\psi(t,u)$ continuously differentiable on all of $\mathcal{U}$ in all directions. This is essentially the idea of [Dawson and Li, 2006]:

How to proceed without moment or other conditions?

Strategy of the proof

The strategy of our proof is the following:

- (A) 'Split the problem': Apply different strategies to the '$\mathbb{R}_+^m$-part' and the '$\mathbb{R}^n$-part';
- (B) Show useful properties of ψ in the 'Key Lemma';
- (C) Use the Key Lemma to reduce the regularity problem to a simpler problem:
  Regularity of a 'partially additive affine process';
- (D) Solve the simpler problem using the ideas of Montgomery & Zippin's solution of Hilbert's fifth problem.

Splitting the state space

First some notation:

$$\begin{tikzcd}
& \mathbb{R}^m_+ \times \mathbb{R}^n \arrow[dl] \arrow[dr] & \\
I = \{1, \dots, m\} & & J = \{m+1, \dots, m+n\}
\end{tikzcd}$$

$$\begin{aligned}
u &= (u_I \ , \ u_J) \\
\mathcal{U} &= \mathcal{U}_I \times \mathcal{U}_J \\
\psi(t,u) &= (\psi_I(t,u) \ , \ \psi_J(t,u))
\end{aligned}$$

The Key Lemma

The Key Lemma

(a) $\psi(t,.)$ maps $\mathcal{U}^\circ$ to $\mathcal{U}^\circ$.

(b) $\psi_J(t,u) = e^{\beta t} u_J$ for all $(t,u) \in \mathcal{U}$, with β a real $n \times n$-matrix.

- Part (a) allows us to 'ignore' the boundary $\partial\mathcal{U}$. (Remember, $\psi(t,u)$ is differentiable in direction u_I *only in the interior of* $\mathcal{U}$.)
- Part (b) shows t-differentiability of ψ_J and allows to split the problem. (Note that $\psi_J(t,u)$ depends only on u_J, not on u_I.)
- The Key Lemma can also be used for a simple proof that X is a Feller process.

The Key Lemma (2)

We give a (hopefully) intuitive explaination for part (b) of the Key Lemma:

Consider the state space $D = \mathbb{R}$, and assume that $\phi(t,u) = 0$. Then both

$$y \mapsto \mathbb{E}^x\left[f_{iy}(X_t)\right] = e^{x\psi(t,iy)} \quad \text{and} \quad y \mapsto \mathbb{E}^{-x}\left[f_{iy}(X_t)\right] = e^{-x\psi(t,iy)}$$

are characteristic functions for any $x \in \mathbb{R}$. By a well-known result, the only characteristic functions, whose reciprocals are also characteristic functions correspond to degenerate distributions.

Here, this implies that $\psi(t,u) = um(t)$, for $m(t)$ a deterministic function. Moreover, by the Markov property $m(t+s) = m(t)m(s)$, which is Cauchy's functional equation with the unique continuous solution $m(t) = e^{\lambda t}m(0)$. Hence, $\psi(t,u) = e^{\lambda t}\,u$ and satisfies (b).

Reduction to a partially additive process

- We call an affine process **partially additive**, if $\psi_J(t,u) = u_J$.
- 'partial' refers to the fact that there are no assumptions on $\psi_I(t,u)$.

Reduction to a partially additive process

Let X be an affine process on D. By the key Lemma $\psi_J(t,u) = e^{\beta t} u_J$. Define the $d \times d$ matrix $K = \begin{pmatrix} \mathrm{id}_m & 0 \\ 0 & \beta \end{pmatrix}$, and the transformed process

$$\widetilde{X}_t = X_t - K^\top \int_0^t X_s \, ds.$$

Then $\widetilde{X}$ is a partially additive affine process on D, with $\widetilde{\phi}(t,u) = \phi(t,u)$ and $\widetilde{\psi_I}(t,u) = \psi_I(t,u)$.

Under the name 'method of the moving frame' this transformation has also proven useful in the context of SPDEs.

Regularity of partially additive processes

- The regularity of a partially additive affine process can now be shown by a combination of Montgomery-Zippin's method and part (a) of the Key Lemma
- The differentiability of $\psi_I(t, u)$ with respect to u_J (which we cannot guarantee) is not needed in the proof, because of the stationarity of $\psi_J(t, u)$.
- The 'moving-frame'-transformation can be inverted, while preserving the regularity of $\widetilde{X}$.

Thus we have shown:

Every affine process is regular.

Summary and Outlook

- The regularity problem for affine processes has interesting connections to more abstract questions on the differentiability of transformation groups (Hilbert's 5th problem).
- Our solution of the regularity problem uses an eclectic mixture of analytic, probabilistic and algebraic techniques.
- The idea of the proof has recently been successfully applied to show that also affine processes taking values in the positive semi-definite matrices are regular [Cuchiero et al., 2009].

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