

Smile Asymptotics for Affine Stochastic Volatility Models

Martin Keller-Ressel

16. November, 2007
TU Wien, Vienna

- 1 Introduction
- 2 Long-time Asymptotics
- 3 Short-time Asymptotics
- 4 Extreme-Strike Asymptotics
- 5 The Forward Smile

affine stochastic volatility models (ASVMs)

X_t ... log-price-process

V_t ... stochastic variance process

- (X_t, V_t) ist a Markov process
- The joint cumulant generating function is affine:

$$\begin{aligned}\Phi(t, u, w) &:= \log \mathbb{E}[\exp(uX_t + wV_t)] = \\ &= \phi(t, u, w) + V_0\psi(t, u, w) + X_0u\end{aligned}$$

- There exists a set $\mathcal{U} \subset \mathbb{R}^2$, such that $\Phi(t, u, w) < \infty$ for all $(t, u, w) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$ and $\mathbf{0} \in \mathcal{U}^\circ$.

This result is crucial:

Generalized Riccati Equations

There exist functions $F, R : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{\infty\}$, such that:

$$\partial_t \phi(t, u, w) = F(u, \psi(t, u, w)), \quad \phi(0, u, w) = w$$

$$\partial_t \psi(t, u, w) = R(u, \psi(t, u, w)), \quad \psi(0, u, w) = w$$

$F(u, w)$... state-independent dynamics

$R(u, w)$... state-dependent dynamics

F, R are of Levy-Khintchine form

Mean-reversion function

The mean reversion function $\lambda(u)$ of V_t is given by

$$\lambda(u) := \frac{\partial R}{\partial w}(u, 0).$$

Martingale Condition

$S_t = \exp(X_t)$ is a martingale if and only if $F(1, 0) = R(1, 0) = 0$ and there exists $\epsilon > 0$ such that

$$\int_0^\epsilon \frac{d\eta}{R(1, \eta)} = \infty.$$

A simple sufficient condition for $(S_t)_{t \geq 0}$ to be a martingale is $F(1, 0) = R(1, 0) = 0$ and $\lambda(1) < \infty$.

Heston model

Heston in SDE form

$$dX_t = -\frac{1}{2}V_t dt + \sqrt{V_t} dW_t^1$$

$$dV_t = -\lambda(V_t - \theta) dt + \gamma\sqrt{V_t} dW_t^2$$

$$\langle dW_t^1, dW_t^2 \rangle = \rho dt$$

Heston in dual form

$$F(u, w) = \lambda\theta w$$

$$R(u, w) = -\frac{1}{2}u - \lambda w + \frac{u^2}{2} + \frac{\gamma^2 w^2}{2} + \rho\gamma w u$$

Heston with added jumps ('Bates model')

Bates in SDE form

$$dX_t = \left(\mu - \frac{1}{2} V_t \right) dt + \sqrt{V_t} dW_t^1 + dJ_t$$
$$dV_t = -\lambda(V_t - \theta) dt + \gamma \sqrt{V_t} dW_t^2$$

Bates in dual form

$$F(u, w) = \lambda \theta w + \kappa(u) - u \kappa(1)$$
$$R(u, w) = -\frac{1}{2} u - \lambda w + \frac{u^2}{2} + \frac{\gamma^2 w^2}{2} + \rho \gamma w u$$

$\kappa(u) = \log \mathbb{E}[e^{J_1}]$... char. exponent von J_t .

Bates 2000

Bates 2000 in SDE form

$$dX_t = \left(\mu - \frac{1}{2} \right) V_t dt + \sqrt{V_t} dW_t^1 + dJ_t(V_t)$$
$$dV_t = -\lambda(V_t - \theta) dt + \gamma \sqrt{V_t} dW_t^2$$

The intensity of $J_t(V_t)$ is *proportional* to V_t .

Bates 2000 in dual form

$$F(u, w) = \lambda \theta w$$

$$R(u, w) = -\frac{1}{2} u - \lambda w + \frac{u^2}{2} + \frac{\gamma^2 w^2}{2} + \rho \gamma w u + \kappa(u) - u \kappa(1)$$

Barndorff-Nielsen-Shephard (BNS)

BNS in SDE form

$$\begin{aligned}dX_t &= \left(\mu - \frac{1}{2} V_t^2\right) dt + \sqrt{V_t} dW_t + \rho dJ_{\lambda t} \\dV_t &= -\lambda V_t dt + dJ_{\lambda t}\end{aligned}$$

BNS in dual form

$$\begin{aligned}F(u, w) &= \lambda \kappa(w + \rho u) - u \lambda \kappa(\rho) \\R(u, w) &= \frac{1}{2}(u^2 - u) - \lambda w\end{aligned}$$

- 1 Introduction
- 2 Long-time Asymptotics**
- 3 Short-time Asymptotics
- 4 Extreme-Strike Asymptotics
- 5 The Forward Smile

Lemma

Suppose $(S_t)_{t \geq 0}$ is a martingale and the mean-reversion function $\lambda(u)$ satisfies

$$\lambda(0) < 0 \quad \text{and} \quad \lambda(1) < 0 .$$

Then there exists a unique continuously differentiable function $w(u) : [0, 1] \rightarrow \mathbb{R}_{\leq 0}$ such that

$$R(u, w(u)) = 0 \quad \text{for all } u \in [0, 1]$$

and $w(0) = w(1) = 0$.

Long-Term Asymptotics of ϕ, ψ

$w(u)$ and $m(u) := F(u, w(u))$ are cumulant generating functions of infinitely divisible random variables and

$$\lim_{t \rightarrow \infty} \psi(t, u) = w(u),$$
$$\lim_{t \rightarrow \infty} \frac{1}{t} \phi(t, u) = m(u)$$

for all $u \in [0, 1]$.

Write the price of a European call as Fourier Integral, and use a saddlepoint approximation:

$$\begin{aligned} \frac{1}{S_0} C(T, \xi) &= 1 - \frac{e^{(1-u_*)\xi}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iz\xi} \exp(\Phi_T(u_* + iz))}{(z + i(1-u_*))(z - iu_*)} dz = \\ &= 1 - \frac{\exp((1-u_*)\xi + Tm(u_*) + C)}{2u_*(1-u_*)\pi\sqrt{T}} \sqrt{\frac{1}{2\pi m''(u_*)}} + \\ &\quad + \mathcal{O}\left(\frac{1}{T}\right) \end{aligned}$$

under the condition $m'(u_*) = 0$.

Comparing with a Black-Scholes price yields the following result:

Long-term Asymptotics for the volatility smile

Let u_* be the solution of

$$m'(u_*) = 0 .$$

then

$$\begin{aligned}\sigma_{\text{imp}}^2(T, \xi) \Big|_{\xi=0} &= -8m(u_*) + \mathcal{O}(T^{-1}) \\ \frac{\partial}{\partial \xi} \sigma_{\text{imp}}^2(T, \xi) \Big|_{\xi=0} &= \frac{1}{T} (8u_* - 4) + \mathcal{O}(T^{-2})\end{aligned}$$

for $T \rightarrow \infty$.

- Approach has been used by Lewis for diffusion models.
- In the complex plane u_* is a saddlepoint of m .
- Problem: $m'(u_*) = 0$ can not be solved explicitly in most cases.
- Even if it can (e.g. Heston model), the resulting expression is complicated.
- Idea: Expand in some model parameter

In the Heston model, expand in γ : (vol-of-vol parameter)

$$\begin{aligned} \text{ATMVariance} &\sim \theta + \frac{\theta\rho}{2\lambda}\gamma + \frac{\theta(5\rho^2 - 1)}{16\lambda^2}\gamma^2 + \mathcal{O}(\gamma^3) \\ T \cdot \text{ATMVarSkew} &\sim \frac{\rho}{\lambda}\gamma + \frac{\rho^2}{2\lambda^2}\gamma^2 + \mathcal{O}(\gamma^3) \end{aligned}$$

In the BNS model, expand in a scale parameter s of the jump distribution ($\kappa(u) = \kappa_0(su)$):

$$\begin{aligned} \text{ATMVariance} &\sim \kappa'_0(0)s + \frac{16\lambda^2\rho^2 + 8\lambda\rho - 1}{16\lambda}\kappa''_0(0)s^2 + \mathcal{O}(s^3) \\ T \cdot \text{ATMVarSkew} &\sim \rho \frac{\kappa''_0(0)}{\kappa'_0(0)}s + \mathcal{O}(s^2) \end{aligned}$$

In the Bates 2000 model expand in γ **and** in a scale parameter s :

$$\begin{aligned} \text{ATMVariance} &\sim \theta + \frac{\theta\rho}{2\lambda}\gamma + \theta\kappa''_0(0)s^2 + \mathcal{O}(\gamma^2 + s^2\gamma) \\ T \cdot \text{ATMVarSkew} &\sim \frac{\rho}{\lambda}\gamma + \frac{\rho^2}{2\lambda^2}\gamma^2 + \mathcal{O}(\gamma^3 + \gamma^2s^2) \end{aligned}$$

- 1 Introduction
- 2 Long-time Asymptotics
- 3 Short-time Asymptotics**
- 4 Extreme-Strike Asymptotics
- 5 The Forward Smile

Smile Asymptotics: $T \rightarrow 0$

Medvedev & Scaillet consider a jump-diffusion Modell:

$$dX_t = \left(-\frac{\sigma_t^2}{2} - \mu_t(\sigma_t) \right) dt + \sigma_t dW_t^1 + dJ_t(\sigma_t)$$
$$d\sigma_t = a(\sigma_t)dt + b(\sigma_t)dW_t^2$$

ρ ... Correlation of W_t^1, W_t^2 .

$\lambda(\sigma_t)$... jump intensity of $J_t(\sigma_t)$

- Jumps can be added to the volatility process; do not influence short-term asymptotics.

For $T \rightarrow 0$ the following holds:

At-the-Money Vol

$$\sigma_{\text{imp}}|_{\xi=0} = \sigma_0 + \mathcal{O}(\sqrt{T})$$

At-the-Money Skew

$$\frac{\partial \sigma_{\text{imp}}}{\partial \xi} \Big|_{\xi=0} = \left[\frac{b\rho}{2\sigma_0} + \frac{\mu}{\sigma_0} \right] + \mathcal{O}(\sqrt{T})$$

μ ... jump compensator of J_t :

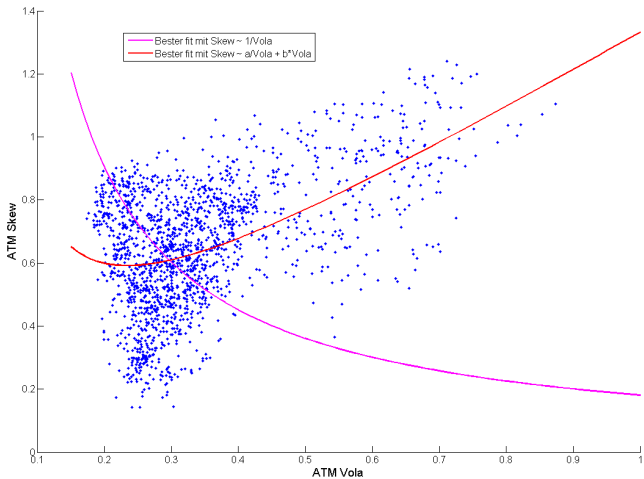
$$\mu = \lambda \int_{-\infty}^{\infty} (e^x - 1) \nu(dx)$$

Implication for the relationship of Skew vs. Volatility:

$$\text{Skew} \propto \frac{a}{\text{Vola}} \qquad \text{In Heston, BNS, Bates}$$

$$\text{Skew} \propto \frac{a}{\text{Vola}} + b \cdot \text{Vola} \qquad \text{In Bates2000}$$

- Is the b -term needed?
- Crucial for the valuation of exotic options depending on forward skew, such as Cliquets with local caps and floors.



Asymptotik für das Bates2000-Modell



- 1 Introduction
- 2 Long-time Asymptotics
- 3 Short-time Asymptotics
- 4 Extreme-Strike Asymptotics**
- 5 The Forward Smile

Lee's Moment Formula

Define

$$\omega_+(T) := \sup \{ \omega : \mathbb{E} S_T^\omega < \infty \}$$

$$\omega_-(T) := \inf \{ \omega : \mathbb{E} S_T^\omega < \infty \} .$$

Then

$$\limsup_{\xi \rightarrow \infty} \frac{\sigma_{\text{imp}}^2(T, \xi)}{|\xi|} = \frac{\varsigma(\omega_+(T) - 1)}{T}$$

$$\limsup_{\xi \rightarrow -\infty} \frac{\sigma_{\text{imp}}^2(T, \xi)}{|\xi|} = \frac{\varsigma(\omega_-(T))}{T}$$

where $\varsigma(x) = 2 - \left(4\sqrt{x^2 + x} - x \right)$.

- We call ω_+ upper critical moment and ω_- lower critical moment.
- From conservativeness and the martingale property it follows that $\omega_- \leq 0$ and $\omega_+ \geq 1$.
- The inverse function of $\omega_{\pm}(T)$ is $T_*(\omega)$, the time of moment explosion:

$$T_*(\omega) := \sup \{ T \geq 0 : \mathbb{E}[S_T^\omega] < \infty \} \quad (\omega \in \mathbb{R} \setminus [0, 1]) .$$

- For CEV-type stochastic volatility models, moment explosions have been studied in a recent paper of Andersen & Piterberg.

Moment Explosions in ASVMs

Let $\omega \in \mathbb{R} \setminus [0, 1]$ and suppose that $(S_t)_{t \geq 0}$ is a martingale and that $\lambda(\omega) \neq \pm\infty$. Define

$$f_+ := \sup \{ \eta \geq 0 : F(\omega, \eta) < \infty \}$$

$$r_+ := \sup \{ \eta \geq 0 : R(\omega, \eta) < \infty \}$$

$$r_0 := \inf \{ \eta \geq 0 : R(\omega, \eta) = 0 \} .$$

- If $F(\omega, 0) < \infty$, $R(\omega, 0) < \infty$ and $r_0 < \min(f_+, r_+)$, then

$$T_*(\omega) = +\infty .$$



Moment Explosions (ctd.)

- If $F(\omega, 0) < \infty$, $R(\omega, 0) < \infty$ and $r_0 \geq \min(f_+, r_+)$, then

$$T_*(\omega) = \int_0^{\min(f_+, r_+)} \frac{1}{R(\omega, \eta)} d\eta .$$

- If $F(\omega, 0) = \infty$ or $R(\omega, 0) = \infty$, then

$$T_*(\omega) = 0 .$$

$T_*(\omega)$ can be inverted to give the upper critical moment $\omega_+(T)$ and the lower critical moment $\omega_-(T)$.

Moment Explosions in the Heston model

In the Heston model, the explosion time is given by

$$T_*(\omega) = \begin{cases} +\infty & b < 0, \Delta > 0 \\ \frac{1}{\sqrt{\Delta}} \log \left(\frac{b + \sqrt{\Delta}}{b - \sqrt{\Delta}} \right) & b > 0, \Delta > 0 \\ \frac{2}{\sqrt{-\Delta}} \left(\arctan \frac{\sqrt{-\Delta}}{b} + \pi \mathbf{1}_{\{b < 0\}} \right) & \Delta < 0. \end{cases}$$

where $b = \rho\omega\gamma - \lambda$ and $\Delta = b^2 - \gamma^2\omega(\omega - 1)$.

Moment Explosions in the BNS model

Let $\kappa_+ := \sup \{u > 0 : \kappa(u) < \infty\}$. Then

$$T_* = \int_0^{f_+ \wedge r_+} \frac{d\eta}{R(\omega, \eta)} = -\frac{1}{\lambda} \log \left(1 - \frac{2\lambda(\kappa_+ - \rho\omega)}{\omega(\omega - 1)} \right).$$

The critical moment functions $\omega_{\pm}(T)$ are given by

$$\begin{aligned} \omega_{\pm}(T) &= \frac{1}{2} - \rho \frac{\lambda}{1 - e^{-\lambda T}} \pm \\ &\pm \sqrt{\frac{1}{4} + (2\kappa_+ - \rho) \frac{\lambda}{1 - e^{-\lambda T}} + \rho^2 \frac{\lambda^2}{(1 - e^{-\lambda T})^2}}. \end{aligned}$$

Forward Start Options

Today ($t = 0$) the seller and buyer agree on

- A start date τ .
- A time-to-maturity T .
- A strike ratio M .

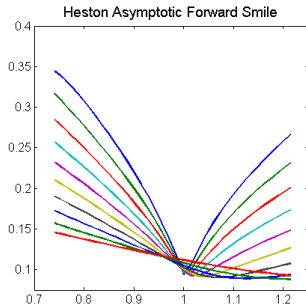
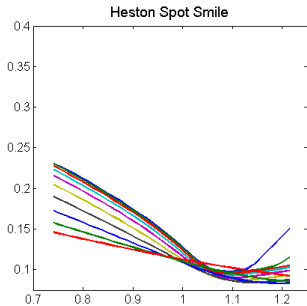
The payoff at time $\tau + T$ (for a call-type contract) is

$$\left(\frac{S_{\tau+T}}{S_{\tau}} - M \right)_+$$

For forward-start options we define moneyness ξ as
 $\xi = \log M + rT$.

- Several Exotic options have forward-start features built in.
- By comparing with BS-prices, we can for each τ define a 'forward smile' parameterized by (ξ, T) .
- In an exp-Levy model: forward smile = spot smile
- In a local vol model the forward smile is flatter than the spot smile
- In an ASVM the forward smile is steeper than the spot smile and converges¹ to an 'asymptotic smile' as $\tau \rightarrow \infty$.

¹under mild conditions



How to calculate the asymptotic forward smile?

Lemma

Suppose that $\lambda(0) < 0$ and that $F(u, w)$ satisfies a logarithmic moment condition. Then $(V_t)_{t \geq 0}$ converges in law to its stationary distribution with cumulant generating function

$$l(w) = - \int_0^w \frac{F(0, \eta)}{R(0, \eta)} d\eta .$$

The asymptotic forward smile

Let $C(\tau, T, \xi)$ be the price of a forward call.

Then

$$\lim_{\tau \rightarrow \infty} e^{r\tau} C(\tau, T, \xi) = \mathbb{E} \left[(e^{\tilde{X}_T} - e^\xi)_+ \right],$$

where \tilde{X}_T is a random variable with cumulant generating function

$$\log \mathbb{E}[e^{\tilde{X}_T u}] = \phi(T, u) + l(\psi(T, u)).$$

Consider 'forward moments' $\mathbb{E}[e^{\tilde{X}_T \omega}]$, and analyze the extreme-strike behaviour of the asymptotic forward smile as we did for the spot smile.

Forward Moment Explosions in ASVMs

Let $\omega \in \mathbb{R} \setminus [0, 1]$ and suppose that $(S_t)_{t \geq 0}$ is a martingale and that $\lambda(\omega) \neq \pm\infty$. Define

$$f_+ := \sup \{ \eta \geq 0 : F(\omega, \eta) < \infty \}$$

$$r_+ := \sup \{ \eta \geq 0 : R(\omega, \eta) < \infty \}$$

$$l_+ := \sup \{ \eta \geq 0 : l(\eta) < \infty \}$$

$$r_0 := \inf \{ \eta \geq 0 : R(\omega, \eta) = 0 \} .$$

- If $F(\omega, 0) < \infty$, $R(\omega, 0) < \infty$ and $r_0 < \min(f_+, r_+, l_+)$, then

$$T_*^F(\omega) = +\infty .$$



Forward Moment Explosions (ctd.)

- If $F(\omega, 0) < \infty$, $R(\omega, 0) < \infty$ and $r_0 < \min(f_+, r_+, l_+)$, then

$$T_*^F(\omega) = \int_0^{\min(f_+, r_+, l_+)} \frac{1}{R(\omega, \eta)} d\eta .$$

- If $F(\omega, 0) = \infty$ or $R(\omega, 0) = \infty$, then

$$T_*^F(\omega) = 0 .$$

- In the Heston model $f_+ = r_+ = \infty$, but $l_+ = \frac{2\lambda}{\gamma}$.
- Instead of integrating from 0 to ∞ , we now integrate only from 0 to $\frac{2\lambda}{\gamma}$.
- Idea: Explosion time getting smaller $\Leftrightarrow |\omega_{\pm}^F|$ getting smaller \Leftrightarrow Smile getting steeper.
- These statements can be quantified for the Heston and other ASVMs.