A remark on Gatheral's 'most-likely path approximation' of implied volatility

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We give a rigorous proof of the representation of implied volatility as a time-average of weighted expectations of local or stochastic volatility. With this proof we fix the problem of a circular definition in the original derivation of Gatheral, who introduced this implied volatility representation in his book 'The Volatility Surface'.

1 Gatheral's most-likely path approximation

In his book 'The Volatility Surface – A Practitioners Guide', Jim Gatheral presents an approximation formula for the implied volatility of a European call, when the underlying stock follows a general diffusion process

$$\frac{dS_t}{S_t} = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dW_t \tag{1}$$

under the pricing measure $\mathbb{P}$. Here, the volatility term of the diffusion can be time- and state-dependent as in a local volatility model, but also random as in a stochastic volatility model. The 'most-likely path approximation' to implied Black-Scholes volatility in this model consists of two parts: The first part is the assertion that implied variance – the square of implied volatility – can be written as a time-average of weighted expectations of $\sigma^2(t, S_t)$:

$$\sigma^2_{\mathrm{imp}}(K, T) = \frac{1}{T} \int_0^T \mathbb{E}^{\mathbb{G}_t}\left[\sigma^2(t, S_t)\right] dt\ . \tag{2}$$

Here, the measures $\mathbb{G}_t$ are given by their Radon-Nikodym derivatives with respect to the pricing measure,

$$\frac{d\mathbb{G}_t}{d\mathbb{P}} = \frac{S_t^2 \Gamma_{\mathrm{BS}}(S_t, \overline{\sigma}_{K,T}(t))}{\mathbb{E}\left[S_t^2 \Gamma_{\mathrm{BS}}(S_t, \overline{\sigma}_{K,T}(t))\right]}, \tag{3}$$

where $\overline{\sigma}_{K,T}(t)$ is a function that is yet to be specified and Γ_{BS} denotes the Black-Scholes Gamma. Let us emphasize that (2) is an exact formula, and that it is the second part of the method where the approximation happens: Gatheral argues that the density (3) is concentrated (as a function of (t, S)) close to a narrow ridge connecting today's stock price S_0 to the strike price K at time T, and claims that a good approximation to (2) is to evaluate it as if the density was *entirely concentrated* on this ridge[1]. Adopting the terminology of Gatheral (2009) we call this ridge the most-likely path and the described approximation method the most-likely path approximation.

In this note we will only be concerned with the first part of Gatheral's method, i.e. the derivation of the exact equation (2), and in particular the definition of the yet unknown function $\overline{\sigma}_{K,T}(t)$. Gatheral (2006) defines on page 27 first the 'Black-Scholes forward implied variance' $v_{K,T}(t)$ by

$$v_{K,T}(t) = \frac{\mathbb{E}\left[\sigma^2(t, S_t) S_t^2 \Gamma_{\mathrm{BS}}(S_t, \overline{\sigma}_{K,T}(t))\right]}{\mathbb{E}\left[S_t^2 \Gamma_{\mathrm{BS}}(S_t, \overline{\sigma}_{K,T}(t))\right]}, \tag{4}$$

and then, in the equation below, the quantity $\overline{\sigma}_{K,T}(t)$ by

$$\overline{\sigma}^2_{K,T}(t) = \frac{1}{T-t} \int_t^T v_{K,T}(u) du \; . \tag{5}$$

There is, however, a problem with this definition – it is circular: $v_{K,T}(t)$ is defined as a function of $\overline{\sigma}_{K,T}(t)$, but then $\overline{\sigma}_{K,T}(t)$ is defined as a function of $v_{K,T}(t)$. At best, it can be seen as an implicit definition, which leaves the question open whether (and under which conditions) the quantities $v_{K,T}(t)$ and $\overline{\sigma}_{K,T}(t)$ actually exist[2]. We will show that a simpler definition of $\overline{\sigma}_{K,T}(t)$ can be given, which clarifies the problem of existence, implies both equations (4) and (5) and finally leads to a proof of the implied volatility representation (2).

2 A rigorous proof of the implied volatility representation

2.1 Defining $\overline{\sigma}_{K,T}(t)$

Consider the price of a call maturing at time T with strike K, written on an underlying stock S_t that follows (1). We denote this price by $C(K, T)$. Moreover we denote by $C_{\mathrm{BS}}(t, S_t, K, T; \sigma)$ the Black-Scholes price at time t with a current spot price of S_t for a call option with strike K, maturity T, and with volatility parameter σ. We claim the following:

Proposition 2.1. *There exists a unique positive deterministic function $\overline{\sigma}_{K,T}(t)$, such that the equality*

$$\mathbb{E}\left[C_{BS}(t, S_t, K, T; \overline{\sigma}_{K,T}(t))\right] = C(K, T) \tag{6}$$

is satisfied for all $t \in [0, T)$.

[1] See Gatheral (2006, Page 29ff) for details.

[2] See also Lee (2004, Sec. 2.3), who remarks that the proof in Gatheral (2006) hinges upon the assumption of the existence of $v_{K,T}(t)$.

Proof. For $\sigma = 0$, the Black-Scholes price $C_{\text{BS}}(t, S_t, K, T; \sigma)$ equals $(S_t - K)_+$. Since S_t is a martingale, we have by Jensen's inequality – or by positivity of calendar spreads – that

$$\mathbb{E}\left[C_{\text{BS}}(t, S_t, K, T; 0)\right] = \mathbb{E}\left[(S_t - K)_+\right] \le \mathbb{E}\left[(S_T - K)_+\right] = C(K,T) \ .$$

For $\sigma \to \infty$ the Black-Scholes price $C_{\text{BS}}(t, S_t, K, T; \sigma)$ approaches S_t. In this case we get

$$\mathbb{E}\left[C_{\text{BS}}(t, S_t, K, T; \infty)\right] = \mathbb{E}\left[S_t\right] = \mathbb{E}\left[S_T\right] \ge C(K,T) \ .$$

In addition $\sigma \mapsto C_{\text{BS}}(t, S_t, K, T; \sigma)$ is for any given S_t a continuous and strictly monotone increasing function, such that also $\sigma \mapsto \mathbb{E}\left[C_{\text{BS}}(t, S_t, K, T; \sigma)\right]$ is. We conclude that (6) has a unique solution $\overline{\sigma}_{K,T}$ for each $t \in [0, T)$. □

Note that for $t = 0$ equation (6) becomes

$$C_{\text{BS}}(0, S_0, K, T; \overline{\sigma}_{K,T}(0)) = C(K,T) \ ,$$

such that $\overline{\sigma}_{K,T}(0)$ is simply the Black-Scholes implied volatility of the call $C(K,T)$. Thus, if we define

$$v_{K,T}(t) = -\frac{\partial}{\partial t}\Big(\overline{\sigma}^2_{K,T}(t) \cdot (T - t)\Big) \ ,$$

$\overline{\sigma}^2_{K,T}(t)$ satisfies (5), and we can recover implied variance from $v_{K,T}(t)$ by the integration

$$\sigma^2_{\text{imp}}(K,T) = \overline{\sigma}^2_{K,T}(0) = \frac{1}{T}\int_0^T v_{K,T}(t)\, dt \ . \tag{7}$$

2.2 An Interpretation of $\overline{\sigma}_{K,T}(t)$

There is a nice interpretation to the definition of $\overline{\sigma}_{K,T}(t)$ through equation (6), in terms of a state-switching pricing model. Consider, for τ between 0 and T, the price process $\widetilde{S}^\tau_t$ given by

$$\begin{aligned} \widetilde{S}^\tau_t &= S_t \qquad t \le \tau \\ d\widetilde{S}^\tau_t / \widetilde{S}^\tau_t &= \sigma_\tau dW_t, \qquad t \ge \tau \ . \end{aligned} \tag{8}$$

The process $\widetilde{S}^\tau_t$ switches from dynamics of the type (1) before time τ to Black-Scholes dynamics with fixed (constant and deterministic) volatility σ_τ after time τ. Now we can ask ourselves: Which volatility σ_τ do we have to choose, such that $\widetilde{S}^\tau_t$ yields the same call option price $C(K,T)$ as the original model (1)? For $\tau = 0$, the answer is the Black-Scholes implied volatility $\sigma_{\text{imp}}(K,T)$. For $\tau \in (0,T)$ a simple conditioning argument gives that

$$\mathbb{E}\left[(\widetilde{S}^\tau_T - K)_+\right] = \mathbb{E}\left[\mathbb{E}\left[(\widetilde{S}^\tau_T - K)_+ \,\Big|\, \mathcal{F}_\tau\right]\right] = \mathbb{E}\left[C_{\text{BS}}(\tau, S_\tau, K, T; \sigma_\tau)\right] \ .$$

If we set this equal to $C(K,T)$ we arrive precisely at equation (6) and see that we must choose $\sigma_\tau = \overline{\sigma}_{K,T}(\tau)$. In the sense of the state-switching model (8), we can interpret $\overline{\sigma}_{K,T}(t)$ as a 'forward-starting' implied volatility of the call $C(K,T)$.

2.3 Proving the implied volatility representation

Finally, let us derive equation (2) from the definition of $\overline{\sigma}_{K,T}(t)$. The Black-Scholes price $C_{\mathrm{BS}}(t, S_t, K, T; \sigma)$ satisfies, of course, the valuation equation

$$\frac{\partial C_{\mathrm{BS}}}{\partial t} = -\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_{\mathrm{BS}}}{\partial S^2}. \tag{9}$$

Moreover, we have the following relationship between Black-Scholes Gamma and Vega:

$$\frac{\partial C_{\mathrm{BS}}}{\partial \sigma} = \sigma S_t^2 (T-t) \frac{\partial^2 C_{\mathrm{BS}}}{\partial S^2}. \tag{10}$$

Define now

$$f(t, S_t) = C_{\mathrm{BS}}(t, S_t, K, T, \overline{\sigma}_{K,T}(t)).$$

By (9), (10) and the definition of $v_{K,T}(t)$ we obtain that $f(t, S_t)$ satisfies

$$\frac{\partial f}{\partial t} = -\frac{1}{2} v_{T,K}(t) S_t^2 \frac{\partial^2 f}{\partial S^2} \,. \tag{11}$$

On the other hand, applying Ito's formula to $f(t, S_t)$, it holds for any $\tau \in [0, T)$, that

$$f(T, S_T) - f(\tau, S_\tau) = \int_\tau^T \left\{ \frac{\partial f}{\partial S}\, dS_t + \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 f}{\partial S^2}\, dt \right\} \,. \tag{12}$$

Taking expectations, the left hand side equals $C(K,T) - \mathbb{E}\left[f(t, S_t)\right]$, which is 0 by (6). On the right hand side we interchange integral and expectation; the dS_t-term contributes 0, and we insert (11) to obtain

$$0 = \frac{1}{2} \int_\tau^T \mathbb{E}\left[\left(\sigma^2(t, S_t) - v_{K,T}(t)\right) S_t^2 \frac{\partial^2 f}{\partial S^2} \right] dt \,.$$

Since τ was arbitrary in $[0, T)$, we conclude that also the integrand must equal 0 for all $t \in [0, t)$. Rearranging terms and rewriting $\frac{\partial^2 f}{\partial S^2}$ as Black-Scholes Gamma evaluated at a volatility of $\overline{\sigma}_{K,T}(t)$, we get the equality

$$v_{K,T}(t) = \frac{\mathbb{E}\left[\sigma^2(t, S_t) S_t^2 \Gamma_{\mathrm{BS}}(S_t, \overline{\sigma}_{K,T}(t))\right]}{\mathbb{E}\left[S_t^2 \Gamma_{\mathrm{BS}}(S_t, \overline{\sigma}_{K,T}(t))\right]}. \tag{13}$$

Defining the measures $\mathbb{G}_t$ by the Radon-Nikodym derivatives (3), we can write this equation as $v_{K,T}(t) = \mathbb{E}^{\mathbb{G}_t}\left[\sigma^2(t, S_t)\right]$. Inserting into (7) then yields the desired implied volatility representation (2).

References

Jim Gatheral. *The Volatilty Surface.* Wiley Finance, 2006.

Jim Gatheral. Presentation at the Copenhagen Volatility Surface Seminar, 2009.

Roger Lee. Implied volatility: Statics, dynamics, and probabilistic interpretation. In *Recent Advances in Applied Probability.* Springer, 2004.