MOMENT EXPLOSIONS IN STOCHASTIC VOLATILITY MODELS

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ABSTRACT. Let $(S_t)_{t\geq 0}$ be the discounted price process in a stochastic volatility model. A moment explosion takes place, if the moment $\mathbb{E}[S_t^u]$ of some given order $u \in \mathbb{R}$ becomes infinite ('explodes') after some finite time $T_*(u)$. Moment explosions are closely related to the shape of the implied volatility surface, where they can be used to obtain approximations for deep in-the-money and out-of-the-money strikes. Furthermore moment explosions may lead to infinite prices of derivatives with super-linear payoff, and to the breakdown of error estimates for numerical approximation schemes. Comparison results for parabolic PDEs combined with an exponentially-affine ansatz for the solution, allow to link explosion times to the blow-up time of ordinary differential equations of the (generalized) Riccati type.

Let $(S_t, V_t)_{t\geq 0}$ be a Markov process, representing a (not necessarily purely continuous) stochastic volatility model. $(S_t)_{t\geq 0}$ is the (discounted) price of a traded asset, such as a stock, and $(V_t)_{t\geq 0}$ represents a latent factor, such as stochastic volatility, stochastic variance, or the stochastic arrival rate of jumps. A **moment explosion** takes place, if the moment $\mathbb{E}[S_t^u]$ of some given order $u \in \mathbb{R}$ becomes infinite ('explodes') after some finite time $T_*(u)$. This time is called time of moment explosion, and formally defined by

$$T_*(u) = \sup\{t \geq 0 : \mathbb{E}[S_t^u] < \infty\} \ .$$

We say that no moment explosion takes place for some given order u, if $T_*(u) = \infty$.

Moment explosions can be considered both under the physical and the pricing measure, with most applications belonging to the latter. If $(S_t)_{t\geq 0}$ is a martingale, then Jensen's inequality implies that moment explosions can only occur for moments of order $u \in \mathbb{R} \setminus [0, 1]$.

Key words and phrases. moment explosion, stochastic volatility, Heston model, SABR model, affine stochastic volatility model, implied volatility surface, smile asymptotics, (generalized) Riccati equation.

Conceptually, the notion of a moment explosion has to be distinguished from an explosion of the process itself, which refers to the situation that the process $(S_t)_{t\geq 0}$, not one of its moments, becomes infinite with some positive probability.

Applications. In *equity* and *foreign exchange* models, where $(S_t)_{t\geq 0}$ represents a stock price or an exchange rate, moment explosions are closely related to the shape of the implied volatility surface, and can be used to obtain approximations for the implied volatility of deep in-the-money and out-of-the-money options (cf. [eqf08-013, large strike asymptotics] and the references therein). According to Lee [2004], Benaim and Friz [2008], the asymptotic shape of the implied volatility surface for some fixed maturity T is determined by the smallest and largest moment of S_T that is still finite. These critical moments $u_-(T)$ and $u_+(T)$ are the piecewise inverse functions[1] of the moment explosion time. Often the explosion time is easier to calculate, such that a feasible approach is to first calculate explosion times, and then to invert to obtain the critical moments. Let us note that finite critical moments of the underlying S_T correspond, in essence, to exponential tails of $\log(S_T)$. There is evidence that refined knowledge of how moment explosion occurs (or the asymptotic behaviour of $u \mapsto \mathbb{E}[S_T^u]$ in the case of non-explosion) can lead to refined results about implied volatility, see Benaim, Friz, and Lee [2008], Gulisashvili and Stein [2009] for some examples of SABR type.

In *fixed income* markets $(S_t)_{t\geq 0}$ might represent a forward LIBOR rate or swap rate. Andersen and Piterbarg [2007] give examples of derivatives with super-linear payoff, whose pricing involves calculation of the second moment of S_T. It is clear that an explosion of the second moment will lead to infinite prices of such derivatives.

For *numerical procedures*, such as discretization schemes for SDEs, error estimates that depend on higher-order moments of the approximated process, may break down if moment explosions occur (see e.g. Alfonsi [2008]). Moment explosions may also lead to *infinite expected utility* in utility maximization problems,

[1] on the intervals $(-\infty, 0)$ and $(1, \infty)$ respectively

for a discussion in the context of affine stochastic volatility models see Kallsen and Muhle-Karbe [2008]).

Moment Explosions in the Black-Scholes and exponential Lévy model. In the Black-Scholes model, moment explosions never occur, since moments of all orders exist for all times. In an exponential-Lévy model [eqf08-031, exponential Levy model], S_t is given by $S_t = S_0 \exp(X_t)$, where X_t is a Lévy process. It holds that $\mathbb{E}[S_t^u] = e^{t\kappa(u)}$, where $\kappa(u)$ is the cumulant generating function of X_1. Thus in an exponential-Lévy model the time of moment explosion is given by

$$T_*(u) = \begin{cases} +\infty & \kappa(u) < \infty \, , \\ 0 & \kappa(u) = \infty \, . \end{cases}$$

Let us remark that, from Theorem 25.3 in Sato [1999], $\kappa(u) < \infty$ iff $\int e^{ux} 1_{|x|>1} \nu(dx) < \infty$ where $\nu(dx)$ denotes the Lévy measure of X.

Moment Explosions in the Heston model. The situation becomes more interesting in a true stochastic volatility model, like the Heston model cf. [eqf08-018, Heston-model],

$$\begin{aligned} dS_t &= S_t \sqrt{V_t}\, dW_t^1, \quad S_0 = s \, , \\ dV_t &= -\lambda(V_t - \theta)\, dt + \eta \sqrt{V_t}\, dW_t^2 \, , \quad V_0 = v, \quad \langle dW_t^1, dW_t^2 \rangle = \rho\, dt. \end{aligned}$$

We now discuss how to compute the moments of S_t (equivalently: the moment generating function of $X_t = \log S_t/S_0$). The joint process $(X_t, V_t)_{t \geq 0}$ is a (time-homogenous) diffusion, started at $(0, v)$, with generator

$$\mathcal{L} = \frac{v}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) + \frac{v}{2}\eta^2 \frac{\partial^2}{\partial v^2} + \lambda(\theta - v)\frac{\partial}{\partial v} + \rho\eta v \frac{\partial^2}{\partial x \partial v} \, .$$

Note that $(X_t, V_t)_{t\geq 0}$ has affine structure in the sense that the coefficients of $\mathcal{L}$ are affine-linear in the state-variables[2]. Now

$$\mathbb{E}\left[e^{uX_T}|X_t = x, V_t = v\right] = e^{ux}\mathbb{E}\left[e^{uX_T}|X_t = 0, V_t = v\right] \tag{1}$$

satisfies, as function of (t, x, v) the backward equation $\partial_t + \mathcal{L} = 0$ with terminal data e^{ux} and after replacing $T-t$ with t we can rewrite this as initial value problem. Indeed, setting $f = f(t, v; u) := \mathbb{E}\left[e^{uX_t}|X_0 = 0, V_0 = v\right]$, and noting that

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)(e^{ux} f) = e^{ux}\left(u^2 - u\right) f \quad \text{and} \quad \frac{\partial^2}{\partial x \partial v}(e^{ux} f) = e^{ux} u \frac{\partial}{\partial v} f,$$

we see that f satisfies a parabolic partial differential equation

$$\partial_t f = \mathcal{A} f := \left(\frac{v}{2}\eta^2 \frac{\partial^2}{\partial v^2} + [\lambda(\theta - v) + \rho\eta u v]\frac{\partial}{\partial v} + \frac{v}{2}\left(u^2 - u\right)\right) f$$

with initial condition $f(0, \cdot; u) = 1$, in which (again) all coefficients depend in an affine-linear way on v. The exponentially-affine ansatz $f(t, v; u) = \exp\left(\phi(t, u) + v\psi(t, u)\right)$ then immediately reduces this PDE to a system of ODEs for $\phi(t, u)$ and $\psi(t, u)$,

$$\frac{\partial}{\partial t}\phi(t, u) = F(u, \psi(t, u)), \qquad \phi(0, u) = 0, \tag{2a}$$

$$\frac{\partial}{\partial t}\psi(t, u) = R(u, \psi(t, u)), \qquad \psi(0, u) = 0, \tag{2b}$$

where $F(u, w) = \lambda\theta w$ and $R(u, w) = \frac{w^2}{2}\eta^2 + (\rho\eta u - \lambda)w + \frac{1}{2}(u^2 - u)$. Equation (2b) is a Riccati differential equation, whose solution blows up at finite time, corresponding to the moment explosion of S_t. Explicit calculations (see Andersen and Piterbarg [2007] for instance) yield[3]

(3)

$$T_*^{\text{Heston}}(u) = \begin{cases} +\infty & \Delta(u) \geq 0, \chi(u) < 0 \\ \frac{1}{\sqrt{\Delta}}\log\left(\frac{\chi(u)+\sqrt{\Delta}}{\chi(u)-\sqrt{\Delta}}\right) & \Delta(u) \geq 0, \chi(u) > 0 \\ \frac{2}{\sqrt{-\Delta(u)}}\left(\arctan\frac{\sqrt{-\Delta(u)}}{\chi(u)} + \pi\mathbf{1}_{\{\chi(u)<0\}}\right) & \Delta(u) < 0, \end{cases}$$

[2]In fact, it does not even depend on x which implies the homogeneity properties (1).
[3]Only $u \notin [0, 1]$ needs to be discussed; in this case $\chi(u) = 0 \implies \Delta(u) < 0$.

where $\chi(u) = \rho\eta u - \lambda$ and $\Delta(u) = \chi(u)^2 - \eta^2(u^2 - u)$. A simple analysis of this condition (cf. Andersen and Piterbarg [2007]) then allows the express the no explosion condition in terms of the correlation parameter ρ. With focus on positive moments of the underlying, $u \geq 1$, we have

$$T_*^{\text{Heston}}(u) = +\infty \iff \rho \leq -\sqrt{\frac{u-1}{u}} + \frac{\lambda}{\eta u}. \tag{4}$$

Similar results for a class of non-affine stochastic volatility models will be discussed below.

Moment Explosions in time-changed exponential Lévy models. Stochastic volatility can also be introduced in the sense of running time at a stochastic "business" clock. For instance, when $\rho = 0$ the (log-price) in Heston model is a Brownian motion with drift, $W_t - t/2$, run at an Cox-Ingersoll-Ross[4] (CIR) clock $\tau(t,\omega) = \tau_t$ where

$$\begin{aligned} dV_t &= -\lambda(V_t - \theta)\, dt + \eta\sqrt{V_t}\, dW_t,\ \ V_0 = v, \\ d\tau_t &= V dt,\ \ \tau_0 = 0. \end{aligned}$$

Since (V,τ) has affine structure, there is a tractable moment generating / characteristic function in the form

$$\mathbb{E}\left(\exp\left(u\tau_T\right)\right) = \mathbb{E}\left(\exp\left(u\int_0^T V(t,\omega)\, dt\right)\right) = \exp\left(A(u,T) + vB(u,T)\right). \tag{5}$$

where[5]

$$\begin{aligned} A(u,t) &= \lambda^2\theta t/\eta^2 - \frac{2\lambda\theta}{\eta^2}\log\left[\sinh(\gamma t/2)\cdot\left(\coth(\gamma t/2) + \frac{\lambda}{\gamma}\right)\right], \\ B(u,t) &= 2u/(\lambda + \gamma\coth(\gamma t/2)),\ \ \gamma = \sqrt{\lambda^2 - 2\eta^2 u}. \end{aligned}$$

We can replace $W_t - t/2$ above by a general Lévy-process $L = L_t$ and run it again at some independent clock $\tau = \tau(t,\omega)$, assuming only knowledge of the

[4]When $u = 1$, (5) is precisely the Cox-Ingersoll-Ross bond pricing formula.
[5]For $u < u^*$ since (5) explodes as $u \uparrow u^*$, where $u^* > 0$ is determined by $I(u^*) \equiv \lambda + \gamma(u)\coth(\gamma(u)t/2) = 0$.

cumulant generating function (cgf) $\kappa_T(u) = \log \mathbb{E}(\exp(u\tau_T))$. If we also set $\kappa_L(u) = \log \mathbb{E}[\exp(uL_1)]$, a simple conditioning argument shows that the moment generating function of $L \circ \tau$ is given by

$$M(u) = \mathbb{E}\left[\mathbb{E}\left(e^{uL_\tau}|\tau\right)\right] = \mathbb{E}\left[e^{\kappa_L(u)\tau}\right] = \exp\left[\kappa_\tau(\kappa_L(u))\right].$$

From here on, moment explosions of $L \circ \tau$ can be investigated analytically, provided κ_τ, κ_L are known in sufficiently explicit form. For some computations in this context, also with regard to the asymptotic behaviour of the implied volatility smile, see Benaim and Friz [2008].

Moment Explosions in non-affine diffusion models. Both Andersen and Piterbarg [2007] and Lions and Musiela [2007] study existence of u^{th} moments, $u \geq 1$, for (not necessarily affine) diffusion models of the type

$$dS_t = V_t^\delta S_t^\beta \, dW_t^1, \qquad S_0 = s$$

$$dV_t = \eta V_t^\gamma \, dW_t^2 + b(V_t)\, dt, \qquad V_0 = v, \quad \langle dW_t^1, dW_t^2 \rangle = \rho \, dt.$$

where $\delta, \gamma > 0$, $\beta \in [0,1]$ and the function $b(v)$ are subject to suitable conditions that ensure a unique solution. For instance, the SABR-model [eqf08-025, SABR model] falls into this class. Lions and Musiela first show that if $\beta < 1$, no moment explosions occur. For $\beta = 1$ the same reasoning as in the Heston model shows that $f(t,v;u) = \mathbb{E}[(S_t/s)^u]$ satisfies the PDE[6]

$$\frac{\partial}{\partial t} f = \mathcal{A} f := \frac{v^{2\gamma}}{2}\eta^2 \frac{\partial^2 f}{\partial v^2} + \left[b(v) + \eta\rho u v^{\delta+\gamma}\right] \frac{\partial f}{\partial v} + \frac{v^{2\delta}}{2}\left(u^2 - u\right) f \tag{6}$$

with initial condition $f(0,\cdot;u) \equiv 1$. Note that the Heston model is recovered as the special case $\beta = 1$, $\delta = \gamma = 1/2$, $b(v) = -\lambda(v - \theta)$. Using the (exponentially-affine in v^q) ansatz $f(t,v;u) = \exp(\phi(t,u) + v^q \psi(t,u))$, with suitably chosen q, ϕ and ψ, Lions and Musiela construct supersolutions of (6), leading to lower bounds for

[6]Care is necessary since f can be $+\infty$; cf. Lions and Musiela [2007] for a proper discussion via localization.

$T_*(u)$, and then subsolutions, leading to matching upper bounds[7]. We report the following results from Lions and Musiela [2007]:

(i) $\beta < 1$: no moment explosion occurs, i.e. $\mathbb{E}\left(S_t^u\right) < \infty$ for all $u \geq 1, t \geq 0$;

(ii) $\beta = 1, \gamma + \delta < 1$: as in (i), no moment explosion occurs;

(iii) $\beta = 1, \gamma + \delta = 1$: If $\gamma = \delta = \frac{1}{2}$, then this choice of parameters yields a Heston-type model, where the mean-reversion term $-\lambda(V_t - \theta)\,dt$ has been replaced by the more general $b(V_t)\,dt$. With λ replaced by $\lim_{v\to\infty} -b(v)/v$ the formula (3) remains valid. If $\gamma \neq \delta$, then the model can be transformed into a Heston-like model by the change of variables $\widetilde{V}_t := V_t^{2\delta}$. The time of moment explosion $T_*(u)$ can be related to the expression in (3), by

$$T_*(u) = \frac{1}{2\delta} T_*^{\text{Heston}}(u).$$

(iv) $\beta = 1, \gamma+\delta > 1$: Let $b_\infty = \lim_{v\to\infty} b(v)/v^{\delta+\gamma}$, and $\rho(u)^* = -\sqrt{(u-1)/u} - b_\infty / (\eta u),$. Then

$$T_*(u) = \begin{cases} +\infty & \rho < \rho^*(u) \\ 0 & \rho > \rho^*(u)\ . \end{cases}$$

The borderline case $\rho = \rho^*(u)$ is delicate and we refer to [Lions and Musiela, 2007, page 13]. Observe that, the condition on $\rho < \rho^*(u)$ is consistent with the Heston model (4), upon setting $\gamma = \delta = 1/2, b_\infty = -\lambda$, while the behaviour of $\rho > \rho^*(u)$ is different in the sense that there is no immediate moment explosion in the Heston model.

Moment Explosions in affine models with jumps. Recall that in the Heston model

$$\mathbb{E}\left[e^{uX_t} | X_0 = x, V_0 = v\right] = e^{ux} \exp\left(\phi(t,u) + v\psi(t,u)\right) \tag{7}$$

[7] A supersolution $\overline{f}$ of (6) satisfies $\mathcal{A}\overline{f} - \frac{\partial \overline{f}}{\partial t} \leq 0$, a subsolution $\underline{f}$ satisfies $\mathcal{A}\underline{f} - \frac{\partial \underline{f}}{\partial t} \geq 0$.

and it was this form of exponentially-affine dependence on x, v that allowed an analytical treatment via Riccati equations. Assuming only validity of (7), for all $u \in \mathbb{C}$ for which the expectation exists, and that $(X_t, V_t)_{t\geq 0}$ is a (stochastically continuous, time-homogenous) Markov process on $\mathbb{R}\times\mathbb{R}_{\geq 0}$ puts us in the framework of affine processes (cf. Duffie, Filipovic, and Schachermayer [2003]), which in fact includes the bulk of analytically tractable stochastic volatility models with and without jumps.

The infinitesimal generator $\mathcal{L}$ of the process $(X_t, V_t)_{t\geq 0}$ now includes integral terms corresponding to the jump effects and thus is a partial integro-differential operator. Nevertheless, the exponentially-affine ansatz $f(t, v; u) = \exp(\phi(t, u) + v\psi(t, u))$ still reduces the Kolmogorov equation to ordinary differential equations of the type (2). The functions $F(u, w)$ and $R(u, w)$ are no longer quadratic polynomials, but of Lévy-Khintchine form (cf. [eqf02-001, infinite divisibility]). The time of moment explosion can be determined by calculating the blow-up time for the solutions of these *generalized* Riccati equations. This approach can be applied to a Heston model with an additional jump term:

$$dX_t = \left(c\,(V_t) - \frac{V_t}{2}\right)dt + \sqrt{V_t}\,dW_t^1 + dJ_t(V_t), \quad X_0 = 0$$
$$dV_t = -\lambda(V_t - \theta)\,dt + \eta\sqrt{V_t}\,dW_t^2, \quad V_0 = v, \quad \langle dW_t^1, dW_t^2\rangle = \rho dt\ .$$

The process $J_t(V_t)$ is a pure-jump process based on a fixed Lévy measure $\nu(dx)$. More precisely, writing $\hat{\mu}$ for the compensated Poisson random measure, independent of (W_t^1, W_t^2), with intensity $\nu\,(dx) \otimes dt$, we assume that

$$dJ_t\,(V_t) = \begin{cases} \int_{|x|<1} x\,\tilde{\mu}\,(dx, dt) + \int_{|x|\geq 1} x\mu\,(dx, dt) & \text{... case (a)} \\ \int_{|x|<1} V_t x\,\tilde{\mu}\,(dx, dt) + \int_{|x|\geq 1} V_t x\mu\,(dx, dt) & \text{... case (b).} \end{cases}$$

In case (a), the process J_t is a genuine (pure-jump) Lévy process; in case (b) jumps are amplified linearly with the variance level, as proposed by Bates [2000]. We focus first on case (a). Assuming $\mathbb{E}\left[e^{J_t}\right] < \infty$, or equivalently $\int e^x 1_{|x|\geq 1}\nu\,(dx) < \infty$, so that $e^{\tilde{J}_t} := e^{J_t + ct}$ is a martingale for suitable drift, $c = -\log \mathbb{E}\left[e^{J_1}\right]$, we have[8]

$$\mathbb{E}\left[e^{uX_t}\right] = \mathbb{E}\left[e^{uX_t^{\text{Heston}}}\right]\mathbb{E}\left[e^{u\tilde{J}_t}\right] = \mathbb{E}\left[e^{uX_t^{\text{Heston}}}\right]e^{t\tilde{\kappa}(u)}.$$

Here, $\tilde{\kappa}(u) = \int (e^{ux} - 1 - u(e^x - 1))\, \nu(dx)$ is well-defined with values in $(-\infty, \infty]$ and finiteness of $\tilde{\kappa}(u) < \infty$ is tantamount to $\int e^{ux} 1_{|x| \geq 1} \nu\,(dx) < \infty$. Hence in case (a) we can link the time of moment explosion $T_*(u)$ to $T_*^{\text{Heston}}(u)$, given by (3), and have

$$T_*(u) = \begin{cases} T_*^{\text{Heston}}(u) & \tilde{\kappa}(u) < \infty\ , \\ 0 & \tilde{\kappa}(u) = \infty\ . \end{cases}$$

In the case (b) the jump process $J_t(V_t)$ depends on V_t and the above argument cannot be used. A direct analysis of the (generalized) Riccati equations (cf. Keller-Ressel [2009]) shows that in the case $\tilde{\kappa}(u) < \infty$ the time of moment explosion is given by formula (3), only now $\Delta(u) = \chi(u)^2 - \eta^2\left(2\tilde{\kappa}(u) + u^2 - u\right)$, and immediate moment explosion happens in the case $\tilde{\kappa}(u) = \infty$.

Also the model introduced by Barndorff-Nielsen and Shephard [2001] (cf. [eqf08-017, BNS model]), which features simultaneous jumps in price and variance, falls into the class of affine models. It is given by

$$\begin{aligned} dX_t &= \left(c - \frac{V_t}{2}\right) dt + \sqrt{V_t}\, dW_t + \rho\, dJ_{\lambda t}, \quad X_0 = 0 \\ dV_t &= -\lambda V_t\, dt + dJ_{\lambda t}, \quad V_0 = v, \end{aligned}$$

where $\lambda > 0$, $\rho < 0$ and $(J_t)_{t \geq 0}$ is a pure jump Lévy process with positive jumps only, and with Lévy measure $\nu(dx)$. The drift parameter c is determined by the martingale condition for $(S_t)_{t \geq 0}$. The time of moment explosion can be calculated (cf. Keller-Ressel [2009]) and is given by

$$T_*(u) = -\frac{1}{\lambda} \log \max\left(0, \left(1 - \frac{2\lambda\, \max(0, \kappa_+ - \rho u)}{u(u-1)}\right)\right),$$

[8] X_t^{Heston} denotes the usual log-price process in the classical Heston model, i.e. with $J \equiv 0$.

where $\kappa_+ := \sup\{u > 0 : \kappa(u) < \infty\}$ and $\kappa(u) = \int_0^\infty (e^{ux} - 1)\, \nu(dx) \in [0, \infty]$.

Moment Explosions in affine diffusion models of Dai-Singleton type. For affine diffusion models with an arbitrary number of stochastic factors, the analysis of moment explosions through the Riccati equations has been studied by Glasserman and Kim [2009]. Without structural restrictions this approach will lead to multiple coupled Riccati differential equations, whose blow-up behavior is tedious to analyze in full generality. However, for concrete specifications this approach can still lead to explicit results. Glasserman and Kim [2009] consider affine models [eqf11-029, affine models] of Dai-Singleton type (cf. Dai and Singleton [2000]), which are given by a diffusion process

$$dY_t = -A^\top(\Theta - Y_t)dt + \sqrt{\mathrm{diag}(b + B^\top Y_t)}\, dW_t, \tag{8}$$

evolving on the state space $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^{n-m}$. The state vector Y is partitioned correspondingly, into components (Y^v, Y^d), called *volatility factors* and *dependent factors*. The vector $b \in \mathbb{R}^n$, and matrices $A, B \in \mathbb{R}^{n\times n}$ are subject to the following structural constraints:

(C1) $A = \begin{pmatrix} A^v & A^c \\ 0 & A^d \end{pmatrix}$, with real and strictly negative eigenvalues.

(C2) The off-diagonal entries of A^v are nonnegative.

(C3) The vector $\Theta = (\Theta^v, \Theta^d)$ has $\Theta^d = 0$, $\Theta^v \geq 0$, and $(-A^\top\Theta)^v \gg 0$.[9]

(C4) $B = \begin{pmatrix} I & B^c \\ 0 & 0 \end{pmatrix}$, and $b = (b^v, b^d)$ with $b^v = 0$ and $b^d = (1, \ldots, 1)$.

Note that condition (C1) assumes strict mean reversion in all components, which is a typical assumption for interest rate models. Most equity pricing models, however, will not satisfy this condition in the strict sense: The Heston model, for example, is of the form (8), but has an eigenvalue of 0 in the matrix A, and thus does not satisfy (C1). Nevertheless, relaxing this condition is in general not a problem, see

[9] following the notation of Dai and Singleton [2000], '$\gg$' denotes strict inequality, simultaneously in all components of the vectors.

e.g. Filipović and Mayerhofer [2009]

Glasserman and Kim [2009] show that the moments of Y_t are represented by the *transform formula*

$$\mathbb{E}[\exp(2u \cdot Y_t)] = \exp\left(-2\int_0^t \Theta^\top A x(s)\,ds + 2\int_0^t |x^d(s)|^2\,ds + 2x(t)\cdot Y_0\right) , \tag{9}$$

where $x(t)$ is a solution to the coupled system of Riccati equations, given by

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} A^v & A^c \\ 0 & A^d \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} I & B^c \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1^2(t) \\ \vdots \\ x_n^2(t) \end{pmatrix} , \tag{10}$$

with initial condition $x(0) = u$. Equation (9) holds in the sense that if either side is well-defined and finite, also the other one is finite, and equality holds. Thus, moment explosions can again be linked to the blow-up time of the ODE (10).

Glasserman and Kim [2009] consider two concrete specifications of the above model, with one volatility factor and one dependent factor in each case. Due to conditions (C1)–(C4), the model parameters are of the form $A = \left(\begin{smallmatrix} p & q \\ 0 & r \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & s \\ 0 & 0 \end{smallmatrix}\right)$, and $\Theta = (\theta_1, 0)$, with $p < 0$, $q \geq 0$, $r < 0$, $s \geq 0$, and $\theta_1 \geq 0$.

- $q = s = 0$: This specification fully decouples the system (10), which can then easily be solved explicitly. In this case the moment explosion time is given by

$$T_*(u_1, u_2) = \begin{cases} +\infty, & u_1 \leq -p \\ \frac{1}{p}\log\left(1 + \frac{p}{u_1}\right), & u_1 > -p \, . \end{cases}$$

  Note that the moment explosion time does not depend on u_2.
- $s > 0$, $q = 0$, $r = p < 0$: In this case the system (10) decouples only partially: The equation for the second component becomes $\dot{x}_2 = px_2$, with the solution $x_2(t) = u_2 e^{pt}$. Substituting into the equation for the first component yields $\dot{x}_2(t) = px_1 + x_1^2 + su_2^2 e^{2pt}$; a non-autonomous Riccati equation. After the transformation $\xi(t) = e^{-pt}x(t)$ it can be solved explicitly, and the

moment explosion time is determined as

$$T_*(u_1, u_2) = \frac{1}{p} \log \max \left\{ 0, \left(\frac{p}{|u_2|\sqrt{s}} \operatorname{arccot} \left(\frac{u_1}{|u_2|\sqrt{s}} \right) + 1 \right) \right\} .$$

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