Non-parametric Calibration of the Barndorff-Nielsen-Shephard Model

Martin Keller-Ressel

Vienna University of Technology
Research Group for Financial and Actuarial Mathematics (FAM)
http://www.fam.tuwien.ac.at

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The Barndorff-Nielsen-Shephard Model

Definition

In the Barndorff-Nielsen-Shephard model the asset price is given by $S_t = S_0 \exp(rt + X_t)$ where X_t follows the SDE

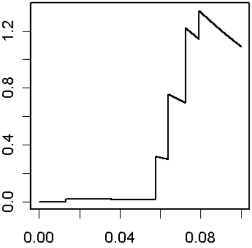
$$\begin{aligned} dX_t &= (\mu + \beta\sigma_t^2)dt + \sigma_t\, dW_t + \rho\, dZ_{\lambda t} \\ d\sigma_t^2 &= -\lambda\sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 \geq 0 \end{aligned} \tag{1}$$

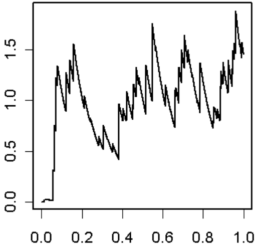
with $\lambda > 0$, $\rho \leq 0$. W_t is a Brownian motion and Z_t a Lévy subordinator with Lévy measure ν.

This model was introduced by [Barndorff-Nielsen and Shephard(2001)].

The Volatility Process σ_t^2

The volatility process σ_t^2 is a Non-Gaussian Ornstein-Uhlenbeck process which, under some mild conditions on ν, has an invariant distribution. The invariant distribution is self-decomposable and depends only on ν.





The BNS Model - Martingale Properties

The process S_t is a martingale if and only if

$$\beta = -\frac{1}{2} \quad \text{and} \quad \mu = -\int_{\mathbb{R}_+} (e^{\rho x} - 1)\, \nu(dx) \tag{2}$$

The remaining parameters σ_0^2, λ, ρ and ν can be interpreted as follows:

- σ_0^2 is the volatility at time 0.
- λ is the decay rate of the volatility process.
- ρ is the leverage effect that links (upward) jumps in volatility with (downward) jumps in price.
- ν is the Lévy measure that determines size and intensity of jumps in both volatility and price.

The BNS Model - Moment Generating Function

The mgf of X_t is given by

Moment Generating Function

$$\phi(u,T) = \exp\left\{-u\left(\frac{\sigma_0^2}{2}\alpha(\lambda) + \lambda T\int_{\mathbb{R}_+}\left(e^{\rho x}-1\right)\nu(dx)\right) + \right.$$
$$\left. + u^2\frac{\sigma_0^2}{2}\alpha(\lambda) + \int_{\mathbb{R}_+}\left[e^{\rho x u} I\left(\frac{x}{\lambda}\frac{u^2-u}{2}; e^{-\lambda T}, 1\right) - \lambda T\right]\nu(dx)\right\} \tag{3}$$

where

$$\alpha(\lambda) = \frac{1-e^{-\lambda T}}{\lambda} \quad \text{and} \quad I(y;a,b) = \int_a^b \frac{1}{t}\exp(y(1-t))\,dt \tag{4}$$

The Parameter Space $\mathcal{P}$

Define $\mathcal{L}^B$ as the space of all Lévy measures ν on $(0, B]$ satisfying $\int x\,\nu(dx) < \infty$. and denote by

$$p = (\sigma_0^2, \lambda, \rho, \nu)$$

the elements of the parameter space

$$\mathcal{P} = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_- \times \mathcal{L}^B$$

The BNS-model together with the option pricing formula maps each $p \in \mathcal{P}$ to a family of European option prices

$$\left\{ V_{(T,K)}(p) : T \in \{T_1, \dots, T_M\}, K \in [K_{\min}, K_{\max}] \right\}$$

The Calibration Problem

We are given a finite set

$$V^M = \left\{ V_i^M = V^M(T_i, K_i) : i = 1, \ldots, N \right\}$$

of observed market prices for European options.
The first problem we consider is the exact calibration problem:

Definition (exact calibration problem)

Find a $p \in \mathcal{P}$ such that

$$V_{(T_i,K_i)}(p) = V_i^M \qquad \forall\, i = 1, \ldots, N$$

The existence of a solution to this problem is not guaranteed in general.

Define the error functional

$$\left\| V(p) - V^M \right\|_w^2 := \sum_{i=1}^{N} w_i \left(V_{(T_i,K_i)}(p) - V_i^M \right)^2 \qquad (5)$$

where $w = (w_1, \ldots, w_N)$ is a vector of non-negative weights summing to one.
We can now formulate the least-squares calibration problem:

Definition (Least-Squares Calibration problem)

$$\left\| V(p) - V^M \right\|_w^2 \quad \rightarrow \quad \text{min!} \qquad p \in \mathcal{P} \qquad (6)$$

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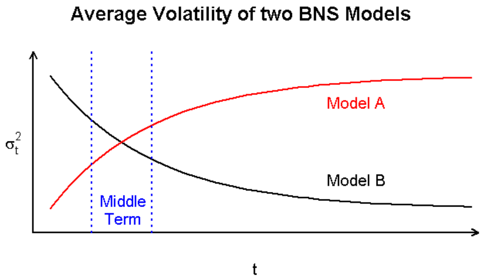
But: The least-squares calibration problem is ill-posed.

An Example for Ill-posedness

The basic idea: Construct 2 BNS volatility processes that are different in the short and long term, but similar in the 'middle term'.



If our option price data is from the middle term only, it might be hard to distinguish if it is coming from model A or model B.

Making it precise: Define $p_n = (0, \lambda, -1, n\delta_{1/n}(x))$.
The mgf of $X_T(p_n)$ can be written as

$$\phi_n(u, T) = \exp\left(u \cdot D + \frac{u^2}{2}\frac{e^{-\lambda T} + \lambda T - 1}{\lambda} + O(n^{-1})\right)$$

which shows that the prices implied by p_n converge to *Black-Scholes* prices, i.e.

$$\lim_{n\to\infty} V_{(T,K)}(p_n) = V_{\mathrm{BS}}\left(\sigma_{\mathrm{BS}}^2 = \frac{e^{-\lambda T} + \lambda T - 1}{\lambda}, T, K\right)$$

Define $p' = (\sigma_0^2 > 0, \lambda, \rho, 0)$. Then by choosing

$$\sigma_0^2 = \frac{\lambda T}{1 - e^{-\lambda T}} - 1$$

we obtain that for a fixed time-to-maturity T

$$V_{(T,K)}(p') = \lim_{n \to \infty} V_{(T,K)}(p_n) \qquad \forall K \in [K_{\min}, K_{\max}]$$

even though

$$p' \neq \lim_{n \to \infty} p_n$$

The Remedy: Regularization

The remedy to the problem of ill-posedness: Regularization

Definition (Regularized Calibration Problem)

Given the regularization parameter $\alpha \in (0, \infty)$ and a 'prior' Lévy measure $\pi \in \mathcal{L}^B \setminus \{0\}$ the regularized calibration problem is the problem of minimizing the functional

$$J_\alpha(p) = \| V_M - V(p) \|_w^2 + \alpha P_\pi(p) \tag{7}$$

where

$$P_\pi(p) = I_f(\nu, \pi) + g(\sigma_0^2, \lambda, \rho) \tag{8}$$

and $\curvearrowright$

Definition (Regularized Calibration Problem (cont.))

the f-divergence $I_f(\nu, \pi)$ is given by

$$I_f(\nu, \pi) := \begin{cases} \int_{(0,\infty)} f\left(\frac{d\nu}{d\pi}\right) \, d\pi & \text{if} \quad \nu \ll \pi \\ +\infty & \text{else} \end{cases} \tag{9}$$

where f is a strictly convex, lower semi-continuous function from $[0, \infty)$ to $[0, \infty)$ satisfying $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$ and assuming the value 0 somewhere.
The function g is assumed to be non-negative, lower semi-continuous and coercive.

Relation to Existing Methods

[Cont and Tankov(2004)] consider the non-parametric calibration problem for an exponential Lévy model, i.e.

$$S_t = S_0 \exp(rt + Z_t)$$

where Z_t is a Lévy process.
They use a relative-entropy regularization approach which essentially corresponds to a choice of

$$f(x) = x \log(x) - x + 1$$

in the penalization term.

Defining a Topology on $\mathcal{P}$

Choice 1 Work with $\mathcal{L}^B_{\text{fin}}$ and the $\sigma(\mathcal{L}^B_{\text{fin}}, C_b(0,B])$-topology.

Choice 2 Work with $\mathcal{L}^B$ and the $\sigma(\mathcal{L}^B, C_x(0,B])$-topology, where

$$C_x(0,B] = \left\{ f \in C_b(0,B] : \sup_{x \in (0,B]} \left| \frac{f(x)}{x} \right| < \infty \right\}$$

Existence of a Solution

Theorem (Existence of Solution)

The regularized calibration problem always has a solution.

Proof (Sketch): The functional

$$\left\| V(p) - V^M \right\|_w^2 + \alpha \left(I_f(\nu, \pi) + g(\sigma_0^2, \lambda, \rho) \right)$$

has a non-empty compact level set L_c for some $c \geq 0$. As a lower semicontinuous function it must take a minimum on the compact set L_c. This minimum is the solution of the calibration problem.

Corollary

If π is of finite activity then any solution ν^* of the calibration problem will have finite activity.

Stability Results

Theorem (Stability w.r.t. perturbations of the market price)

Let V_k^M be a sequence of sets of market prices such that $\left\| V_k^M - V^M \right\|_w^2 \to 0$. For each $k \in \mathbb{N}$ let p_k^* be a solution of

$$\left\| V(p) - V_k^M \right\|_w^2 + \alpha P_\pi(p) \quad \to \quad \text{min!}$$

Then

(i) *The sequence $\{p_k^*\}_{k \in \mathbb{N}}$ has a convergent subsequence.*

(ii) *The limit of every convergent subsequence of $\{p_k^*\}_{k \in \mathbb{N}}$ is a solution of the calibration problem*

$$\left\| V(p) - V^M \right\|_w^2 + \alpha P_\pi(p) \quad \to \quad \text{min!}$$

Theorem (Stability under approximation of the prior)

Let $\{\pi_k\}_{k\in\mathbb{N}}$ *and* π *be Lévy measures in* $\mathcal{L}^B_{fin} \setminus \{0\}$ *such that* $\pi_k \to \pi$ *in the* $\sigma(\mathcal{L}^B_{fin}, C_b)$*-topology. For each* $k \in \mathbb{N}$ *let* p^*_k *be a solution of the regularized calibration problem with prior* π_k*, i.e.,*

$$\left\| V(p) - V^M \right\|^2_w + \alpha P_{\pi_k}(p) \to \text{min!}$$

Then

(i) *The sequence* $(p^*_k)_{k\in\mathbb{N}}$ *has a convergent subsequence.*

(ii) *The limit of every convergent subsequence of* $(p^*_k)_{k\in\mathbb{N}}$ *is a solution of the calibration problem with prior* π*.*

Convergence to a Solution of the Exact Problem

Theorem (Convergence to an exact solution)

Let $(\delta_k)_{k\in\mathbb{N}}$ *and* $(\alpha_k)_{k\in\mathbb{N}}$ *be sequences in* $\mathbb{R}_+$ *such that*

$$\delta_k \to 0, \quad \alpha_k \to 0 \quad \text{and} \quad \frac{\delta_k^2}{\alpha_k} \to 0 \qquad \text{for} \quad k \to \infty$$

Let $(V_k^M)_{k\in\mathbb{N}}, V^M$ *be sets of market prices such that*

$$\left\| V_k^M - V^M \right\|_w \leq \delta_k \qquad \forall\, k \in \mathbb{N}$$

For each $k \in \mathbb{N}$ *let* $p_k^* \in \mathcal{P}$ *be a solution of*

$$\left\| V(p) - V_k^M \right\|_w^2 + \alpha_k P_\pi(p) \to \min! \qquad \curvearrowright$$

Theorem (Convergence to an exact solution (cont.))

If the unregularized calibration problem has an exact solution, i.e. if

$$\left\| V(p) - V^M \right\|_w^2 = 0 \qquad p \in \mathcal{P} \tag{10}$$

has solutions such that the set

$$S = \{p \text{ solves } (10) : P_\pi(p) < \infty\}$$

is non-empty, then

(i) *The sequence $\{p_k^*\}_{k\in\mathbb{N}}$ has a convergent subsequence.*

(ii) *The limit of every convergent subsequence of $\{p_k^*\}_{k\in\mathbb{N}}$ is an element of S, i.e. an exact solution of the unregularized problem.*

Uniqueness

Theorem (Uniqueness)

Suppose that g is strictly convex and that f is uniformly convex, i.e there is a $c > 0$ such that

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f\left(\frac{x+y}{2}\right) + c|x-y|^2 \qquad \forall x, y \in \mathbb{R}$$

then there exists a $\alpha_* > 0$ such that for all $\alpha \geq \alpha_*$ the regularized calibration problem

$$\left\| V(p) - V_k^M \right\|_w^2 + \alpha P_\pi(p) \to \min!$$

has a unique solution.

The Fourier Transform Method of Carr and Madan

To simplify calculations parameterize each option in terms of time-to-maturity T and log-forward-moneyness

Definition (Log-Forward-Moneyness)

$$\xi = \log\left(\frac{e^{-rT}K}{S_0}\right) = \log K - \log S_0 - rT$$

Define the dampened option price as

$$W(T,\xi) = e^{\beta\xi}V(T,\xi)$$

where $\beta > 0$ is used for Calls and $\beta < -1$ for Puts. Then calculating the Fourier Transform of $W(T,\xi)$ one obtains a formula for option pricing, which was introduced by [Carr and Madan(1999)].

Theorem (Carr-Madan Option Pricing Formula)

Let $\phi(u, T)$ be the moment generating function of X_T and let $\beta \in \mathbb{R} \setminus [-1, 0]$. Then the price $V(\xi, T)$ of a European option with maturity T and log-forward-moneyness ξ is given by

$$V(\xi, T) = \frac{S_0 e^{-\beta\xi}}{2\pi i} \int_{-\infty}^{\infty} e^{-iz\xi} \frac{\phi(iz + \beta + 1, T)}{(iz + \beta)(iz + \beta + 1)} \, dz \qquad (11)$$

where choosing $\beta > 0$ gives the price of a **call** *option and $\beta < -1$ gives the price of a* **put** *option*

Evaluating the 'Ugly Integral'

The bottleneck in calculating option prices is the evaluation of the moment generating function $\phi(u, T)$, in particular of the integral

$$\int_{e^{-\lambda T}}^{1} \frac{1}{t} \exp\left(\frac{x}{\lambda}\frac{u^2-u}{2}(1-t)\right) \, dt \tag{12}$$

which has to be evaluated for many different values of T,x and u. Writing

$$I(z; a, b) = \int_a^b \frac{1}{t} \exp\left(z(1-t)\right) \, dt \qquad a, b \in (0, 1] \tag{13}$$

there are several approaches to evaluating $I(z; a, b)$

- Write $I(z;a,b) = e^z\left(E_1(az) - E_1(bz)\right)$ and use the exponential integral function $E_1(z)$ (cf. Abramowitz and Stegun)

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt \qquad |\arg z| < \pi$$

  which is implemented in MATLAB as `expint`
- Use the continued fraction expansion

$$e^z E_1(z) = \frac{1}{z+}\,\frac{1}{1+}\,\frac{1}{z+}\,\frac{2}{1+}\,\frac{2}{z+}\,\frac{3}{1+}\,\frac{3}{z+}\,\frac{4}{1+}\cdots$$

- Numerical Integration

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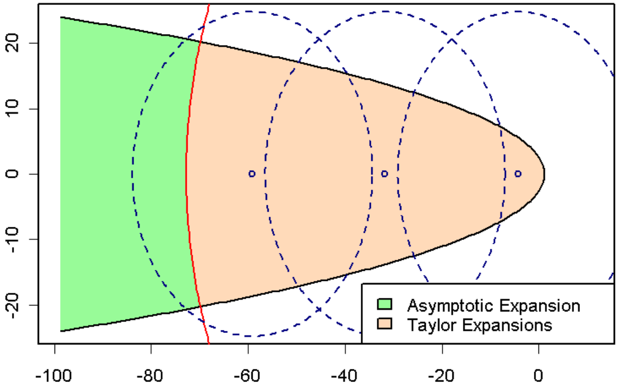
- Numerical Integration

However all these approaches have drawbacks either in terms of speed or in terms of accuracy

Our approach is based on the observation that $I(z; a, b)$ has to be evaluated only on a subset of $\mathbb{C}$, which is delimited by a parabola. On this subset we use an asymptotic expansion for large values of $|z|$ and Taylor expansions of low order for all other z.



The Gradient of $V(p)$

The partial derivatives of $V(p)$ w.r.t the parameters $(\sigma_0^2, \lambda, \rho)$ and the derivative in direction of a Lévy measure $h \in \mathcal{L}^B$ can be written in the form

$$\frac{S_0 e^{-\beta\xi}}{2\pi} \int_{-\infty}^{\infty} e^{-iz\xi} \frac{\phi(iz+\beta+1, T) k(iz+\beta+1)}{(iz+\beta)(iz+\beta+1)}\, dz$$

where $k(u)$ is explicitly known in all cases.
Applying results on Schwartz space it can be shown that the derivatives are continuous in the topology of uniform convergence on compacts.

Implementation Details

The MATLAB implementation uses

- 30 discretization points for the Lévy measure ν.
- a grid of $\approx$ 16000 points for the Fast Fourier Transform
- a Black-Scholes approximation to obtain starting values for (σ_0^2, λ).
- the L-BFGS (Limited Memory Quasi-Newton with Broyden-Fletcher-Goldfarb-Shanno update) optimization algorithm.

A Calibration Example

Data: 5×18 European call option prices with

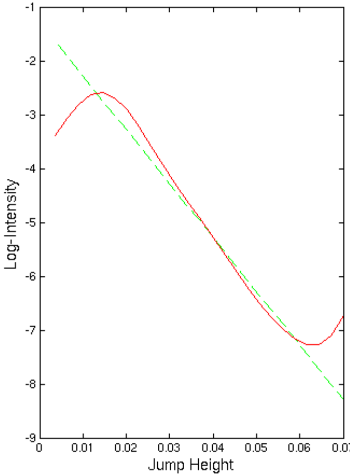
$$T \in \{0.1, 0.2, 0.5, 1.0, 2.0\} \quad \text{and} \quad e^{-rT}K \in [0.65, 1.4]$$

from a BNS model with $(\sigma_0^2, \lambda, \rho) = (0.065, 1.7, -4.5)$ driven by a $\Gamma(1, 100)$-subordinator. A $\Gamma(a, b)$-subordinator is a Lévy process with density

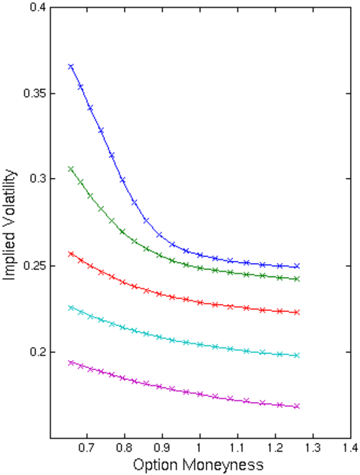
$$\nu(dx) = abe^{-bx}\mathbf{1}_{\{x>0\}}\, dx$$

Calibration Results:

	true value	starting v.	calibrated v.
σ_0^2	0.065	0.1	0.0650
λ	1.7	1.2	1.6705
ρ	-4.5	-2.5	-4.2421



True Lévy density (dashed green) and calibrated Lévy density (solid red)



Simulated market data (crosses) and calibrated model (solid lines)

Ole E. Barndorff-Nielsen and Neil Shephard.
Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics.
Journal of the Royal Statistical Society B, 63:167–241, 2001.

P. Carr and D. B. Madan.
Option valuation using the Fast Fourier Transform.
Journal of Computational Finance, 2(4), 1999.

R. Cont and P. Tankov.
Non-parametric calibration of jump-diffusion option pricing models.
Journal of Computational Finance, 7(3), 2004.

R. Cont and P. Tankov.
Retrieving Lévy processes from option prices: Regularization of an ill-posed problem.
SIAM Journal of Control and Optimization, 45(1), 2006.

Ken-Iti Sato.
Lévy processes and infinitely divisible distributions.
Cambridge University Press, 1999.