Limit Distributions of Continuous-State Branching Processes with Immigration

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Part I

Introduction: Ornstein-Uhlenbeck-type processes and CBI-processes on $\mathbb{R}_{\geqslant 0}$

OU-type processes on $\mathbb{R}_{\geqslant 0}$

Definition (OU-type process)

$\mathbb{R}_{\geqslant 0}$-valued Ornstein-Uhlenbeck-type process satisfies SDE

$$\mathrm{d}X_t = -\lambda X_t \mathrm{d}t + \mathrm{d}Z_t, \qquad \lambda > 0,$$

Z a Lévy subordinator with drift $b \in \mathbb{R}_{\geqslant 0}$ and Lévy measure $m(\mathrm{d}\xi)$.

- This is the classical Ornstein-Uhlenbeck SDE with BM replaced by an increasing Lévy process.
- It follows easily that if $X_0 \in \mathbb{R}_{\geqslant 0}$, then X is $\mathbb{R}_{\geqslant 0}$-valued.

Limit distributions of OU-type processes have been studied extensively.

Self-decomposable distributions on $\mathbb{R}_{\geqslant 0}$: SD_+

Y has *self-decomposable distribution* if $\forall c \in [0,1]$ there exists a random variable Y_c, independent of Y, s.t.

$$Y \overset{d}{=} cY + Y_c.$$

We write $Y \in \mathrm{SD}_+$.

(i) Self-decomposable dist. are infinitely divisible.

(ii) Self-decomposable dist. have increased degree of regularity:
  - any non-degenerate $Y \in \mathrm{SD}_+$ has a density (i.e. abs. cont.);
  - every distribution in SD_+ is unimodal.

Theorem (Paul Lévy)

Distribution is in $\mathrm{SD}_+ \iff$ *Lévy triplet is* $(\gamma, 0, (k(x)/x)\mathrm{d}x)$ *and*

$$\gamma \geq 0 \quad \textit{and} \qquad k : (0,\infty) \to [0,\infty) \quad \textit{non-increasing.}$$

Self-Decomposable distributions and OU-type processes

Theorem (Jurek and Vervaat [1983], Sato and Yamazato [1984])

- *Let X be an OU-type process on $\mathbb{R}_{\geqslant 0}$ and suppose that $m(\mathrm{d}\xi)$ satisfies the log-moment condition $\int_{\xi>1} \log \xi \, m(\mathrm{d}\xi) < \infty$. Then X converges to a self-decomposable limit distribution L.*
- *Conversely, for every self-decomposable distribution L on $\mathbb{R}_{\geqslant 0}$ there exists a unique OU-type process, such that the Lévy measure $m(\mathrm{d}\xi)$ of its driver satisfies $\int_{\xi>1} \log \xi \, m(\mathrm{d}\xi) < \infty$ and L is the limit distribution of the process.*

One-to-one-correspondence: OU-type processes with log-moment-condition $\longleftrightarrow$ self-decomposable distributions.

CBI-processes (1)

Definition (CBI-process)

A process $X = (X_t)_{t \geq 0}$ is a *continuous-state branching process with immigration (CBI-process)* if:

- X stochastically continuous conservative Markov pr. on $\mathbb{R}_{\geqslant 0}$
- with affine Laplace exponent.:
  $-\log \mathbb{E}^x \left[\mathrm{e}^{-uX_t} \right] = \phi(t, u) + x\psi(t, u)$,

- Can be obtained as a scaling limit of Galton-Watson processes with immigration.
- The class of CBI-processes coincides with the class of affine processes on $\mathbb{R}_{\geqslant 0}$ in the sense of Duffie, Filipovic, and Schachermayer [2003].
- Contains all OU-type processes on $\mathbb{R}_{\geqslant 0}$.

CBI-processes (2)

Theorem (Kawazu and Watanabe [1971])

Let $(X_t)_{t\geq 0}$ be a CBI-process. Then the functions $\phi(t,u)$ and $\psi(t,u)$ are t-differentiable with derivatives

$$F(u) = \frac{\partial}{\partial t}\phi(t,u)\Big|_{t=0}, \qquad R(u) = \frac{\partial}{\partial t}\psi(t,u)\Big|_{t=0}$$

which are of Levy-Khintchine form

$$F(u) = bu - \int_{(0,\infty)} \left(\mathrm{e}^{-u\xi} - 1\right) m(\mathrm{d}\xi),$$

$$R(u) = -\alpha u^2 + \beta u - \int_{(0,\infty)} \left(\mathrm{e}^{-u\xi} - 1 + u h_R(\xi)\right) \mu(\mathrm{d}\xi),$$

where $\alpha, b \in \mathbb{R}_{\geqslant 0}$, $\beta \in \mathbb{R}$ and m, μ are Lévy measures on $(0,\infty)$, with m satisfying $\int_{(0,\infty)} (x \wedge 1)\, m(\mathrm{d}x) < \infty$.

CBI-processes (3)

- F is the Laplace exponent of a Lévy subordinator X^F.
- R is the Laplace exponent of a spectrally positive Lévy process X^R.
- One-to-one Correspondence:
  CBI-process $X \longleftrightarrow$ Lévy processes (X^F, X^R).
- $\phi(t,u), \psi(t,u)$ satisfy the generalised Riccati equations:

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(t,u) &= F(\psi(t,u)), \qquad \phi(0,u) = 0\ , \\
\frac{\partial}{\partial t}\psi(t,u) &= R(\psi(t,u)), \qquad \psi(0,u) = u\ .
\end{aligned}$$

- Special cases:
  (i) X CB-process: $F \equiv 0$.
  (ii) X OU-type process: $R(u) = \beta u$ for $\beta \in \mathbb{R}$.
  (iii) X Feller diffusion: $F(u) = bu$ and $R(u) = \beta u + \frac{\alpha^2}{2}u^2$.

Pathwise description of CBI-processes

L-K triplet of X^R is $(\beta, 2\alpha, \mu(\mathrm{d}x))$ and W its BM component.

$N_1(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\xi)$ Poisson random measure with comp. $\mathrm{d}s\,\mathrm{d}u\,\mu(\mathrm{d}\xi)$;
Denote the compensated Poisson random measure

$\widetilde{N}_1(\mathrm{d}t, \mathrm{d}u, \mathrm{d}\xi) = N_1(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\xi) - \mathrm{d}s\,\mathrm{d}u\,\mu(\mathrm{d}\xi)$.

Dawson and Li 2006 show that CBI-process satisfies SDE

$$\mathrm{d}X_t = \mathrm{d}X_t^F + \beta X_t \mathrm{d}t + \sqrt{2\alpha X_t}\,\mathrm{d}W_t + \int_0^{X_{t-}} \int_0^\infty \xi \widetilde{N}_1(\mathrm{d}t, \mathrm{d}u, \mathrm{d}\xi)$$

under certain moment conditions on the Lévy measures m, μ.

Some applications of CBI-processes

- Natural candidates for modelling of positive, mean-reverting quantities: volatility, interest rates, default intensities.
- Diffusive behaviour and jumps can be combined.
- Processes can show self-exciting behaviour through state-dependent jump intensity.
- Tractable because the Laplace transform of the transition density is known up to a scalar ODE (gen. Riccati equation).

Part II

Results on the limit distrutions of CBI-processes

First representation formula

Theorem (Pinsky (1972), K.-R., Mijatovic (2011))

X CBI-process on $\mathbb{R}_{\geqslant 0}$ with $R'_+(0) < 0$. TFAE:

(a) *X converges to a limit distribution L as $t \to \infty$;*

(b) *X has the unique invariant distribution L;*

(c) *the measure $m(\mathrm{d}\xi)$ satisfies the log-moment condition*

$$\int_{\xi > 1} \log \xi \, m(\mathrm{d}\xi) < \infty.$$

Moreover the limit L has the following properties:

(i) *L is infinitely divisible;*

(ii) *the Laplace exponent $l(u) = -\log \int_{[0,\infty)} e^{-ux} \mathrm{d}L(x)$ of L is given by*

$$l(u) = -\int_0^u \frac{F(s)}{R(s)} \mathrm{d}s \qquad (u \geq 0) .$$

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What is the Lévy-Khintchine triplet for L?

Definition (Scale Function)

The *scale function* of the dual $\widehat{X}^R = -X^R$, which is spectrally negative, is the unique increasing continuous function $W : (0, \infty) \to [0, \infty)$ that satisfies

$$\int_0^\infty e^{-ux} W(x) dx = -\frac{1}{R(u)} \quad \text{for all} \quad u > 0.$$

- Under the assumption that $R'_+(0) < 0$, we have

$$W(x) = -\frac{1}{R'_+(0)} \mathbb{P}\left[\sup_{t \geq 0} X_t^R \leq x\right] \quad \text{for} \quad x > 0.$$

Scale functions of spectrally negative Lévy processes

Some properties of the scale function W of the dual $\widehat{X}^R$:

- W has non-degenerate right limit at zero:

$$\lim_{x \downarrow 0} W(x) = W_+(0) = \mathbb{P}\left[\sup_{t \geq 0} X_t^R = 0\right].$$

Excursion theory for X^R implies further:

- W is log-concave and hence continuous on $[0, \infty)$;
- the scale function has left- and right-derivative on $(0, \infty)$;
- if $\alpha > 0$ (recall $(\beta, \alpha, \mu(\mathrm{d}x))$ is the L-K triplet of X^R), then $W \in C^2(0, \infty)$.

Second representation formula

Theorem (K.-R., Mijatovic (2011))

If CBI-process X converges to the limit dist. L, then L is infinitely divisible with Laplace exponent

$$l(u) = u\gamma - \int_{(0,\infty)} \left(\mathrm{e}^{-xu} - 1\right) \frac{k(x)}{x} \,\mathrm{d}x,$$

where $\gamma \geq 0$ and $k : (0,\infty) \to \mathbb{R}_{\geqslant 0}$ are given by

$$\gamma = bW(0),$$

$$k(x) = b\,W'_+(x) + W(x)m(x,\infty) + \int_{(0,x]} \left[W(x) - W(x-\xi)\right] m(\mathrm{d}\xi).$$

Second representation formula (2)

- For an OU-type process $\widehat{X}^R = \lambda t$, $W(x) = \frac{1}{\lambda}\mathbf{1}_{\{[0,\infty)\}}(x)$ and we re-obtain $k(x) = \frac{1}{\lambda}m(x,\infty)$.
- Last integral is always finite since W is left-differentiable at $x > 0$.
- Analogous representations of function $k(x)$, based on the law of the ultimate supremum $\sup_{t\geq 0} X_t^R$ or on the Itô excursion measure for the Poisson point process of excursions from supremum of $\widehat{X}^R$, are also available.
- Formula for $k(x)$ looks like a Lévy generator applied to the scale function $W(x)$. Rigorous result possible?

Second representation formula: $k = -\mathcal{A}_{X^F} W$

Pick $h : (0, \infty) \to (0, \infty)$ such that

- h is continuous and bounded;
- $\lim_{x \downarrow 0} h(x) = 0$;
- $h(x) \sim e^{-cx}$ as $x \to \infty$ for some $c > 0$.

Define the weighted Banach space

$$L_1^h(0, \infty) := \left\{ f \in L_1^{\text{loc}}(0, \infty) : \int_0^\infty |f(x)| h(x) \mathrm{d}x < \infty \right\}.$$

Semigroup of X^F acts on $L_1^h(0, \infty)$ and W is in the domain of its generator $\mathcal{A}_F$. Theorem implies that the following holds

$$k = -\mathcal{A}_{X^F} W.$$

Properties of the limit distribution L

If X is $\mathbb{R}_{\geqslant 0}$-valued OU-type processes, then L is self-decomposable.

Sato 1999 shows that in this case we have:

(i) the support of L is $[b/\lambda, \infty)$;

(ii) the distribution of L is absolutely continuous;

(iii) the asymptotic behaviour of the density of L at b/λ is determined by $c = \lim_{x \downarrow 0} k(x)$.

What are the properties of a general limit L of a CBI-process?

Properties of the limit distribution L: the degenerate case

Definition

A CBI-process X is *degenerate of the*

(i) *first kind*, if it is deterministic for all $X_0 = x \in \mathbb{R}_{\geqslant 0}$;

(ii) *second kind*, if it is deterministic when started at $X_0 = 0$.

Proposition (K.-R., Mijatovic (2011))

X a CBI-process with limit distribution L.

- *X deg. of the first kind* $\Longrightarrow \operatorname{supp} L = \{-b/\beta\}$;
- *X deg. of the second (not first) kind* $\Longrightarrow \operatorname{supp} L = \{0\}$.

Support of L: the non-degenerate case

The effective drift $\lambda_0 > 0$ of $\widehat{X}^R$ is defined as

$$\lambda_0 = \begin{cases} \int_{(0,\infty)} h_R(\xi)\mu(\mathrm{d}\xi) - \beta, & \text{if} \quad X^R \text{ has bounded variation,} \\ +\infty, & \text{if} \quad X^R \text{ has unbounded variation,} \end{cases}$$

where $h_R(\xi) = \xi/(1+\xi^2)$ is truncation function in R.

Proposition (K.-R., Mijatovic (2011))

X a non-degenerate CBI-process with limit distribution L. Then

$$\operatorname{supp} L = [b/\lambda_0, \infty),$$

and hence

$$\operatorname{supp} L = \mathbb{R}_{\geqslant 0} \quad \Longleftrightarrow \quad b = 0 \text{ or paths of } X^R \text{ have inf. variation.}$$

Absolute continuity of the limit distribution L

Proposition (K.-R., Mijatovic (2011))

X CBI-process with limit distrib. L and λ_0 effective drift of X^R. Then the following holds:

- *L is absolutely continuous on $\mathbb{R}_{\geqslant 0}$ if and only if*

$$\int_0^1 \frac{k(x)}{x} \mathrm{d}x = \infty;$$

- *L is absolutely continuous on $\mathbb{R}_{\geqslant 0} \setminus \{b/\lambda_0\}$ with an atom at $\{b/\lambda_0\}$ if and only if*

$$\int_0^1 \frac{k(x)}{x} \mathrm{d}x < \infty.$$

$\mathrm{SD}_+ \subsetneq \mathrm{CLIM} \subsetneq \mathrm{ID}_+$

- ID_+ infinitely divisible distributions on $\mathbb{R}_{\geqslant 0}$;
- SD_+ self-decomposable distributions on $\mathbb{R}_{\geqslant 0}$.

Definition

CLIM: limit distribs on $\mathbb{R}_{\geqslant 0}$ of CBI-processes with $R'_+(0) < 0$.

Proposition (K.-R., Mijatovic (2011))

$$\mathrm{SD}_+ \subsetneq \mathrm{CLIM} \subsetneq \mathrm{ID}_+.$$

$\mathrm{SD}_+ \subsetneq \mathrm{CLIM} \subsetneq \mathrm{ID}_+$: proof

Proof.

- all distributions in SD_+ are either degenerate or absolutely continuous;
- all distributions in CLIM are absolutely continuous on $\mathbb{R}_{\geqslant 0} \setminus \{b/\lambda_0\}$, but some concentrate non-zero mass at $\{b/\lambda_0\}$;
- the class ID_+ contains singular distributions.

□

There are also concrete examples of non-selfdecomposable limit distributions of CBI-processes...

Example: non-selfdecomposable limit distribution

- X^F: generating triplet $(b, 0, m(\mathrm{d}x))$;
- X^R: characteristic exponent $R(u) = -\alpha u^2 + \beta u$ (and $\beta < 0$).

Scale function of dual process $\widehat{X}^R$:

$$W(x) = \left[\exp\left(x\beta/\alpha\right) - 1\right]/\beta \quad \text{and} \quad W'(x) = \exp\left(x\beta/\alpha\right)/\alpha.$$

***k*-function of limit distribution L:**

$$k(x) = \mathrm{e}^{x\beta/\alpha}\left[\frac{b}{\alpha} + \frac{1}{\beta}\left(m(x,\infty) + \int_{(0,x]}\left(1 - \mathrm{e}^{-\xi\beta/\alpha}\right) m(\mathrm{d}\xi)\right)\right] - \frac{m(x,\infty)}{\beta}$$

Choose X^F as compound Poisson pr. with exp. jumps

$$m(x,\infty) = \mathrm{e}^{-x}, b = 0, \alpha = 1/2, \beta = -1 \Longrightarrow k(x) = 2(\mathrm{e}^{-x} - \mathrm{e}^{-2x}).$$

$\Rightarrow$ k non-mononote $\Rightarrow$ L not self-decomposable.

Sufficient conditions for L to be self-decomposable

- X^F: generating triplet $(b, 0, m(\mathrm{d}x))$;
- X^R: generating triplet $(\beta, 2\alpha, \mu(\mathrm{d}x))$.

Proposition

L limit distribution of CBI-process X. L self-decomposable if:

(a) $\mu = 0$ *and* $\alpha = 0$,

(b) $\mu = 0$ *and* $m = 0$,

(c) $m = 0$ *and* W *is concave on* $(0, \infty)$.

Conversely, if $m = 0$ *and* L *is self-decomposable, then* W *is concave on* $(0, \infty)$.

Open problems

(i) Characterise limit of CBI-process X in the boundary case that $R'_+(0) = 0$.

(ii) Find structural characterisation of the possible limit distributions using formula

$$k(x) = b\, W'_+(x) + W(x) m(x, \infty) + \int_{(0,x]} \left[W(x) - W(x - \xi) \right] \, m(\mathrm{d}\xi).$$

- for OU-type processes the limit class has an structural description as self-decomposable distributions.

Thank you for your attention!

Preprint:

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