Affine processes and applications to stochastic volatility modelling

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An affine process is a stochastically continuous, time-homogenous Markov process $(X_t)_{t \geq 0}$ with state space $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, such that the cumulant generating function is an **affine** function of the initial state:

$$\Phi_t(u) = \log \mathbb{E}\left[e^{\langle X_t, u \rangle}\right] = \phi(t,u) + \langle X_0, \psi(t,u) \rangle$$

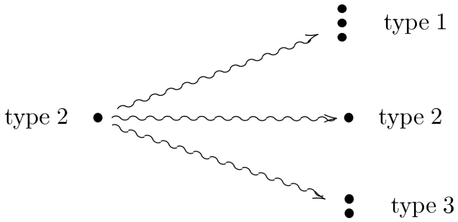
for all $u \in \mathbb{C}^d$ where the expectation is finite.

Background: Branching Processes (1)

Consider a simple branching processes, the Galton-Watson process.

- State Vector $X_t \in \mathbb{N}_0^d$ counts different types of 'particles'.
- At each time step ('generation') each particle splits into a random number of offspring particles of different types.

Here, $d = 3$ and $X_0 = (0, 1, 0)^\top$:



Background: Branching Processes (2)

- Branching probabilities depend on type, but are independent of history and other particles.
- $\Longrightarrow$ Any branching process started at $X_0 \in \mathbb{N}^d$ can be written as the sum of independent branching processes started with a single particle.

In terms of cumulant generating functions:

$$\log \mathbb{E}[e^{\langle X_t, u \rangle}] = \sum_{i=1}^{d} X_0^i \psi_i(t, u) = \langle X_0, \psi(t, u) \rangle$$

where $\psi_i(t, u)$ is the cgf of the process started with a single particle of type i.

Background: Branching Processes (3)

- Introducing random 'immigration' of particles, the linear dependency on X_0 becomes an affine dependency.
- Taking an appropriately scaled continuous-time limit we obtain the class of CBI-processes (*continuously branching with immigration*, see Kawazu and Watanabe [1971])
  The affine dependency of the cumulant generating function on X_0 is preserved.

The class of CBI-processes is *precisely* the class of affine processes with state space $\mathbb{R}^d_{\geqslant 0}$.

Attractiveness for Economics & Finance (1)

Why is this type of process attractive for economics and finance?

- Replace ‘particle type’ by economic factor, e.g.
  - interest rate
  - volatility/variance
  - default intensity of some firm/sector
- The branching property models cross-excitement (contagion) or self-excitement of these factors.
- The class includes many models already used in finance.

Not all economic factors are non-negative, most prominently

- (log-)returns from stocks or other assets

$\Longrightarrow$ Need for state space $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$.

Attractiveness for Economics & Finance (1)

Other aspects:

- Classical models of finance have been diffusion models. Recently, models with jumps seem to have arrived in mainstream finance (Cont and Tankov [2004]). Affine processes feature both diffusive and 'jumpy' dynamics.
- Numerical methods based on Fourier inversion of characteristic functions have gained in popularity in finance (Carr and Madan [1999]). Affine processes directly model the time-evolution of the characteristic function.

Characterization of Affine Processes (1)

These are the main results of Duffie, Filipovic, and Schachermayer [2003]:

An affine process is called *regular*, if the derivatives

$$F(u) := \frac{\partial}{\partial t}\phi(t,u)\Big|_{t=0} \qquad R(u) := \frac{\partial}{\partial t}\psi(t,u)\Big|_{t=0}$$

exist, and are continuous at $u = 0$.

Theorem (Characterization of affine processes)

If $(X_t)_{t\geq 0}$ is a regular affine process, then ϕ and ψ satisfy the generalized Riccati equations

$$\partial_t \phi(t,u) = F(\psi(t,u)), \quad \phi(0,u) = 0$$
$$\partial_t \psi(t,u) = R(\psi(t,u)), \quad \psi(0,u) = u$$

and $\ldots \hookrightarrow$

Characterization of Affine Processes (2)

Theorem (continued)

. . . F, R are of the Levy-Khintchine form:

$$F(u) = \left\langle \frac{a}{2} u, u \right\rangle + \langle b, u \rangle - c + \int_D \left(e^{\langle \xi, u \rangle} - 1 - \langle h_F(\xi), u \rangle \right) \, m(d\xi)$$

$$R_i(u) = \left\langle \frac{\alpha_i}{2} u, u \right\rangle + \langle \beta_i, u \rangle - \gamma_i + \int_D \left(e^{\langle \xi, u \rangle} - 1 - \left\langle h_R^i(\xi), u \right\rangle \right) \, \mu_i(d\xi)$$

where $(a, \alpha_i, b, \beta_i, c, \gamma_i, m, \mu_i)_{i=1,\dots,d}$ is an 'admissible' parameter set.
Moreover $(X_t)_{t \geq 0}$ is a Feller process, and its generator given by. . . $\hookrightarrow$

Characterization of Affine Processes (3)

Theorem (continued)

$$\mathcal{A}f(x) = \frac{1}{2}\sum_{k,l=1}^{d}\left(a_{kl} + \sum_{i=1}^{m}\alpha^i_{kl}x_i\right)\frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle b + \sum_{i=1}^{d}\beta^i x_i, \nabla f(x)\rangle +$$
$$+ \int_{D\setminus\{0\}} \left(f(x+\xi) - f(x) - \langle h_F(\xi), \nabla f(x)\rangle\right) m(d\xi) +$$
$$+ \sum_{i=1}^{m} x_i \int_{D\setminus\{0\}} \left(f(x+\xi) - f(x) - \left\langle h^i_R(\xi), \nabla f(x)\right\rangle\right) \mu^i(d\xi)$$

Conversely, for each admissible parameter set there exists a regular affine process on D with generator $\mathcal{A}$.

Admissibility Conditions(1)

Definition (Admissibility Conditions)

Let $I = \{1, \dots, m\}$, $J = \{m+1, \dots, m+n\}$.
Let a, α^i be positive semi-definite $d \times d$-valued matrices;
$b, \beta^i \in \mathbb{R}^d$; $c, \gamma_i \in \mathbb{R}_{\geqslant 0}$; and $m, \mu^i \dots$ Levy measures on D.
These parameters are called **admissible** if

$$a_{kk} = 0 \text{ for all } k \in I, \qquad \alpha^j = 0 \text{ for all } j \in J\ ,$$
$$\alpha^i_{kl} = 0 \text{ if } k \in I \setminus \{i\} \text{ or } l \in I \setminus \{i\}\ ,$$
$$b \in D, \qquad \beta^i_k \geq 0 \text{ for all } i \in I \text{ and } k \in I \setminus \{i\}\ ,$$
$$\beta^j_k = 0 \text{ for all } j \in J \text{ and } k \in I\ ; \qquad \gamma^j = 0 \text{ for all } j \in J;$$
$$\int_{D \setminus \{0\}} \left\{ \left(|x_I| + |x_J|^2 \right) \wedge 1 \right\} m(d\xi) < \infty; \quad \mu^j = 0 \text{ for all } j \in J;$$
$$\int_{D \setminus \{0\}} \left\{ \left(|x_{I \setminus \{i\}}| + |x_{J \cup \{i\}}|^2 \right) \wedge 1 \right\} \mu_i(d\xi) < \infty \quad \text{for all } i \in I.$$

Admissibility Conditions (2)

$$a = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \geq \end{array}\right) \qquad \underset{(i \in I)}{\alpha^i} = \left(\begin{array}{ccccccc|ccc} & & & 0 & & & & & & \\ & & & \vdots & & & & & & \\ & & & 0 & & & & & & \\ 0 & \cdots & 0 & \alpha^i_{ii} & 0 & \cdots & 0 & \star & \cdots & \star \\ & & & 0 & & & & & & \\ & & & \vdots & & & & & & \\ & & & 0 & & & & & & \\ \hline & & & \star & & & & & & \\ & & & \vdots & & & & & \geq & \\ & & & \star & & & & & & \end{array}\right) \text{ where } \alpha^i_{ii} \geq 0 \qquad \underset{(j \in J)}{\alpha^j} = 0$$

$$b = \left(\begin{array}{c} \geq \\ \vdots \\ \geq \\ \hline \star \\ \vdots \\ \star \end{array}\right) \qquad \underset{(i \in I)}{\beta^i} = \left(\begin{array}{c} \geq \\ \vdots \\ \geq \\ \beta^i_i \\ \geq \\ \vdots \\ \geq \\ \hline \star \\ \vdots \\ \star \end{array}\right) \text{ where } \beta^i_i \in \mathbb{R} \qquad \underset{(j \in J)}{\beta^j} = \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline \star \\ \vdots \\ \star \end{array}\right)$$

Stars denote arbitrary real numbers; the small $\geq$-signs denote non-negative real numbers and the big $\geq$-sign a positive semi-definite matrix.

Outlook

Some open problems related to affine processes:

- If $D = \mathbb{R}^d_{\geqslant 0}$, the regularity condition is not needed. Even on $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ no example of an non-regular affine process is known. Is every affine process regular?
- Duffie et al. [2003] give an example of an affine process, whose maximal state space is the convex hull of a parabola. What other maximal state spaces ($\subseteq \mathbb{R}^d$) are possible?
- How can affine processes be generalized to other state spaces, e.g. matrix-valued or Hilbert-space-valued. Dawson and Li [2006] have provided some insights into the infinite-dimensional case.
- Little is known about properties of the generalized Riccati equations in the (non-decoupling) multi-dimensional case.

…and many, many applications.

Affine stochastic volatility models (ASVMs)

X_t ... (discounted) log-price-process
V_t ... stochastic variance process
$S_t := \exp(rt + X_t)$ ... price-process

Assumptions

- (X_t, V_t) ist an affine process on $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$
- Homogeneity assumption on the log-price process: Shifting X_0 by x, also X_t is simply shifted by x.

This implies for the cumulant generating function, that

$$\Phi_t(u, w) := \log \mathbb{E}[\exp(uX_t + wV_t)] = \phi(t, u, w) + V_0 \psi(t, u, w) + X_0 u$$

Affine Stochastic Volatility Models (2)

We can prove: $(X_t, V_t)_{t \geq 0}$ is automatically regular.

The generalized Riccati equations become:

$$\begin{aligned} \partial_t \phi(t,u,w) &= F(u, \psi(t,u,w)), \quad \phi(0,u,w) = 0 \\ \partial_t \psi(t,u,w) &= R(u, \psi(t,u,w)), \quad \psi(0,u,w) = w \ . \end{aligned}$$

- scalar, autonomous ODEs; u enters as parameter, w as initial condition.
- Most results will follow from a careful quantitative analysis of these equations & convexity properties of F and R.

Affine Stochastic Volatility Models (3)

The following function will appear in many conditions and is related to the tendency of the variance process to revert to its long-term mean:

Definition (Mean-Reversion Function)

For each $u \in \mathbb{R}$ where $R(u, 0) < \infty$, we define the mean reversion function $\lambda(u)$, by

$$\lambda(u) := \frac{\partial R}{\partial w}(u, w)\Big|_{w \uparrow 0} \quad .$$

Martingale & Conservativeness conditions (1)

Theorem (Martingale Property & Conservativeness)

(a) $(S_t)_{t \geq 0}$ *is conservative if and only if* $F(0,0) = R(0,0) = 0$ *and there exists* $\epsilon > 0$ *such that*

$$\int_{-\epsilon}^{0} \frac{d\eta}{R(0,\eta)} = -\infty \,;$$

(b) $(S_t)_{t \geq 0}$ *is a martingale if and only if it is conservative,* $F(1,0) = R(1,0) = 0$ *and there exists* $\epsilon > 0$ *such that*

$$\int_{-\epsilon}^{0} \frac{d\eta}{R(1,\eta)} = -\infty \,.$$

Martingale & Conservativeness conditions (2)

- Results are related to uniqueness of the zero-solution for $(u, w) = (0, 0)$ and $(1, 0)$ respectively.
- A Lipschitz condition would be only sufficient.
- The integral conditions of the theorem are related to a uniqueness condition of Osgood [1898], which is sufficient and necessary in this case.

We will from now on always assume that $(S_t)_{t \geq 0}$ is a martingale.

Long-time behavior of the Variance Process (1)

Theorem (K.-R. and Steiner [2008])

Suppose that $\lambda(0) < 0$ and that the Levy measure m satisfies the logarithmic moment condition

$$\int_{|y|>1} \log y \, m(dx, dy) < \infty.$$

Then $(V_t)_{t \geq 0}$ converges in law to its stationary distribution L, which has the cumulant generating function

$$l(w) = \int_w^0 \frac{F(0,\eta)}{R(0,\eta)} \, d\eta \qquad (w \leq 0) \, .$$

Long-time behavior of the variance process (2)

- This result generalizes Jurek and Vervaat [1983], who study limit laws of (Non-Gaussian) OU-processes (= affine processes where R is linear)
- In the case of OU-processes, the class of limit-distributions is nicely characterized: They are exactly the self-decomposable distributions.

$$X \text{ self-decomp.} \iff \forall\, c \in (0,1)\, \exists\, X_c \text{ indep. } X : \quad X \overset{d}{=} cX + X_c \,.$$

- In the general affine case, no such result is known.
- K.-R. and Steiner [2008] also show that for R a quadratic polynomial, limit distributions are obtained which are not self-decomposable.

Long-time behavior of the log-price process (1)

Lemma

Suppose that $\lambda(0) < 0$ and $\lambda(1) < 0$. Then there exist an open interval I, such that $[0,1] \subseteq I$, and a unique function $w \in C^1(I)$, that cannot be C^1-extended beyond I, such that

$$R(u, w(u)) = 0 \qquad \text{for all} \quad u \in I$$

and $w(0) = w(1) = 0$.
Moreover $w(u) < 0$ for all $u \in (0,1)$ and

$$\frac{\partial R}{\partial w}(u, w(u)) < 0, \quad \text{for all } u \in I.$$

$w(u)$ are exactly the asymptotically stable equilibria of the second generalized Riccati equation.

Long-time behavior of the log-price process (2)

Theorem (Long-term behavior of $(X_t)_{t\geq 0}$)

Suppose that $\lambda(0) < 0$ *and* $\lambda(1) < 0$ *and define*

$$m(u) = F(u, w(u)), \quad J = \{u \in I : m(u) < \infty\} \ .$$

Then $w(u)$ *and* $m(u)$ *are cumulant generating functions of infinitely divisible random variables and*

$$\lim_{t\to\infty} \psi(t, u, 0) = w(u) \quad \textit{for all} \quad u \in I \ ;$$

$$\lim_{t\to\infty} \frac{1}{t}\phi(t, u, 0) = m(u) \quad \textit{for all} \quad u \in J \ .$$

Interpretation: for large t, $(X_t)_{t\geq 0}$ 'looks like' the Levy process with char. exponent $m(u)$.

Example: The Heston Model (1)

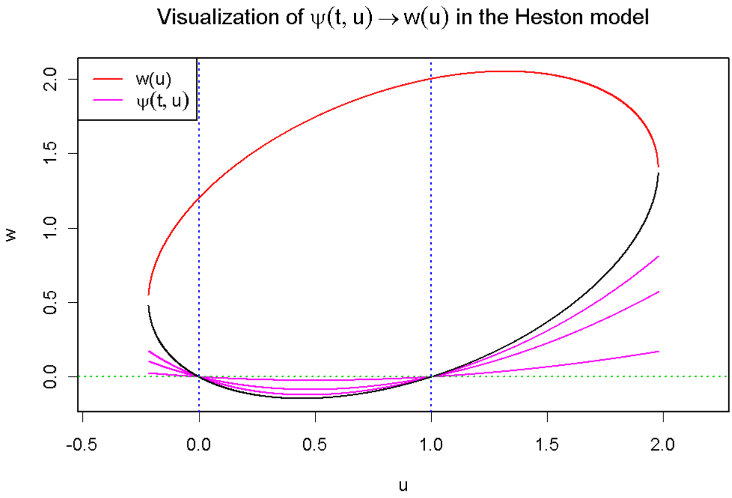
Heston in SDE form

$$
\begin{aligned}
dX_t &= -\frac{1}{2} V_t \, dt + \sqrt{V_t} \, dW_t^1 \\
dV_t &= -\lambda(V_t - \theta) \, dt + \gamma \sqrt{V_t} \, dW_t^2 \\
\langle dW_t^1, dW_t^2 \rangle &= \rho \, dt
\end{aligned}
$$

Heston in dual form

$$
\begin{aligned}
F(u, w) &= \lambda \theta w \\
R(u, w) &= -\frac{1}{2} u - \lambda w + \frac{u^2}{2} + \frac{\gamma^2 w^2}{2} + \rho \gamma w u
\end{aligned}
$$

Example: The Heston Model (2)



Example: The Heston Model (3)

In the Heston model

$$m(u) = \lambda\theta \frac{(\lambda - u\rho\gamma) - \sqrt{(\lambda - u\rho\gamma)^2 - \gamma^2(u^2 - u)}}{\gamma^2} \; .$$

This is the cumulant generating function of a Normal-Inverse-Gaussian distribution.

Application to the implied volatility smile (1)

Write the price of a European call with time-to-maturity T and log-moneyness ξ as Fourier Integral, and use a saddlepoint approximation:

$$\frac{1}{S_0}C(T,\xi) = 1 - \frac{e^{(1-u_*)\xi}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iz\xi}\exp\left(\Phi_T(u_*+iz)\right)}{(z+i(1-u_*))(z-iu_*)}\,dz =$$

$$= 1 - \frac{\exp\left((1-u_*)\xi + Tm(u_*) + C\right)}{2u_*(1-u_*)\pi\sqrt{T}}\sqrt{\frac{1}{2\pi m''(u_*)}} + \mathcal{O}\left(\frac{1}{T}\right)$$

under the condition $m'(u_*) = 0$.

Application to the implied volatility smile (2)

Comparing with a Black-Scholes price yields:

Long-term Asymptotics for the volatility smile

Let u_* be the solution of

$$m'(u_*) = 0 \, .$$

Then

$$\sigma^2_{\text{imp}}(T,\xi)\Big|_{\xi=0} = -8m(u_*) + \mathcal{O}(T^{-1})$$
$$\frac{\partial}{\partial \xi}\sigma^2_{\text{imp}}(T,\xi)\Big|_{\xi=0} = \frac{1}{T}(8u_* - 4) + \mathcal{O}(T^{-2})$$

for $T \to \infty$.

Moment Explosions in ASVMs (1)

Moments of the price process $\mathbb{E}[S_t^u]$ can become infinite in finite time: **Moment Explosions**

Definition

$T_*(u)\ldots$ Time of Moment Explosion

$$T_*(u) := \sup\left\{T \geq 0 : \mathbb{E}[S_T^u] < \infty\right\} \ .$$

For diffusion models, moment explosions have been recently studied in Andersen and Piterbarg [2007].

Moment Explosions in ASVMs (2)

Theorem (Moment Explosions in ASVMs)

Define J as before and

$$f_+(u) := \sup\{w \geq 0 : F(u,w) < \infty\}$$
$$r_+(u) := \sup\{w \geq 0 : R(u,w) < \infty\}\ .$$

Suppose that $F(u,0) < \infty$, $R(u,0) < \infty$ and $\lambda(u) < \infty$.

- *If $u \in J$ then*
$$T_*(u) = +\infty\ .$$
- *If $u \in \mathbb{R} \setminus \overline{J}$, then*
$$T_*(u) = \int_0^{\min(f_+(u), r_+(u))} \frac{1}{R(u,\eta)}\, d\eta\ .$$

↪

Moment Explosions in ASVMs (3)

Theorem (continued)

If $F(u,0) = \infty$, $R(u,0) = \infty$, or $\lambda(u) = \infty$, then

- $$T_*(u) = 0\ .$$

Idea of the proof: Use extension theorems for ODEs.

Define the upper/lower critical moments by

$$u_+(T) = \sup\left\{u \geq 1 : \mathbb{E}[S_T^u] < \infty\right\} = \sup\left\{u \geq 1 :\ T_*(u) < \ T\right\}\ ,$$

$$u_-(T) = \inf\left\{u \leq 0 : \mathbb{E}[S_T^u] < \infty\right\} = \inf\left\{u \leq 0 :\ T_*(u) < \ T\right\}\ .$$

$u_\pm(T)$ are the piecewise inverse functions of $T_*(u)$ on $(1,\infty)$ and $(-\infty,0)$ respectively.

Lee's moment formula

By a result of Lee [2004], $u_\pm(T)$ are related to the large-strike behavior of the volatility smile:

Lee's moment formula

Let $V(T,\xi)$ be the implied Black-Scholes-Variance of a European call with time-to-maturity T and log-moneyness ξ. Then

$$\limsup_{\xi\to-\infty} \frac{V(T,\xi)}{|\xi|} = \frac{\varsigma(-u_-(T))}{T}$$

and

$$\limsup_{\xi\to\infty} \frac{V(T,\xi)}{|\xi|} = \frac{\varsigma(u_+(T)-1)}{T}$$

where $\varsigma(x) = 2 - 4\left(\sqrt{x^2+x} - x\right)$ and $u_\pm(T)$ are the critical moment functions.

The stationary variance regime

Define $(\widetilde{X}_t, \widetilde{V})_t$ as the Markov process with the same transitional probabilities as (X_t, V_t), but started at $X_0 = 0$ a.s. and V_0 distributed according to its stationary distribution L.

It is easy to see that

$$\log \mathbb{E}[\exp(u\widetilde{X}_t + w\widetilde{V}_t)] = \phi(t,u,w) + l(\psi(t,u,w))$$

As before, define

Definition

$T_*^S(u)\ldots$ Time of Moment Explosion under stationary variance

$$T_*^S(u) := \sup\left\{T \geq 0 : \mathbb{E}[\widetilde{S}_T^u] < \infty\right\} .$$

Motivation: Forward-starting Options (1)

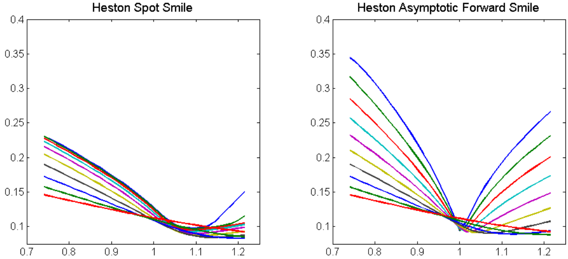
- At time $t = 0$, fix start date τ, strike date $T + \tau$, moneyness ratio M. Define moneyness $\xi = \log M + rT$.
- The payoff at time $T + \tau$ of a **forward-starting option** is

$$\left(\frac{S_{T+\tau}}{S_\tau} - M\right)_+$$

- Value does not depend on marginal distributions of $(X_t)_{t \geq 0}$, but on transitional distributions.
- Even models that are perfectly calibrated to the plain vanilla smile, will yield different prices for forward-starting options.
- Forward-starting options are often used as building blocks of more complex derivatives, such as Cliquet options (see Gatheral [2006]).

Motivation: Forward-starting Options (2)

- Define **implied forward volatility** $\sigma(\tau, T, \xi)$, by comparing with a Black-Scholes model.
- We expect $\sigma(\tau, T, \xi)$ to increase with τ, since the uncertainty of V_τ has to be priced in.
- Under some conditions, $\sigma(\tau, T, \xi)$ will converge to a limit $\widetilde{\sigma}(T, \xi)$ for $\tau \to \infty$.



Motivation: Forward-starting Options (3)

Asymptotic Forward Volatilities

Suppose that $(X_t, V_t)_{t \geq 0}$ is an ASVM, satisfying the conditions for existence of a stationary variance distribution. Let $\sigma(\tau, T, \xi)$ be the implied forward volatility in this model. Then

$$\lim_{\tau \to \infty} \sigma(\tau, T, \xi) = \widetilde{\sigma}(T, \xi) \, ,$$

where $\widetilde{\sigma}(T, \xi)$ is the implied volatility of a European call with payoff $\left(e^{\widetilde{X}_T} - e^{\xi}\right)_+$, and $\widetilde{X}_T$ is the log-price process of the model under stationary variance.

To apply Lee's moment formula to the asymptotic forward smile, we consider moment explosions in the stationary variance regime.

Moment explosions under stationary variance (1)

Moment Explosions under stationary variance

Define J as before and

$$f_+(u) := \sup\{w \geq 0 : F(u,w) < \infty\}$$
$$r_+(u) := \sup\{w \geq 0 : R(u,w) < \infty\}$$
$$l_+ := \sup\{w \geq 0 : l(w) < \infty\}$$

Suppose that $F(\omega,0) < \infty$, $R(\omega,0) < \infty$ and $\lambda(u) < \infty$.

- If $u \in J$ and $w(u) \leq l_+$, then

$$T_*^S(u) = +\infty\ ;$$

- If $u \in \mathbb{R} \setminus \overline{J}$ or $w(u) > l_+$, then

$$T_*^S(u) = \int_0^{\min(f_+(u), r_+(u), l_+)} \frac{1}{R(u,\eta)}\, d\eta\ .$$

Moment explosions under stationary variance (2)

(continued)

If $F(\omega, 0) = \infty$, $R(\omega, 0) = \infty$ or $\lambda(u) = \infty$ then

- $$T_*^S(u) = 0 \; .$$

It holds that

$$T_*^S(u) \le T_*(u) \quad \text{for all } u \in \mathbb{R} \; .$$

Summary and Outlook

- In the area of finance, the class of affine processes is frequently applied to the modelling of stochastic volatility, interest rates and default risk.
- For an affine process, the time-evolution of the cumulant generating function can be described by an autonomous ODE (the generalized Riccati equations).
- In the case of stochastic volatility models, interesting results can be obtained by a qualitative analysis of these ODEs.
- It would be nice to generalize these results to a multidimensional setting.

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