

# Options on Discrete Realized Variance – The Convex Order Conjecture

Martin Keller-Ressel  
ETH Zürich

AMaMeF Workshop, TU Berlin  
September 28, 2010

## Part I

# Realized Variance vs. Quadratic Variation

# Realized Variance

$S = S_0 \exp(X)$  ... discounted asset

$T$  ... time horizon in years

$\mathcal{P}$  ... a partition  $0 = t_0 < \dots < t_n = T$  of  $[0, T]$  into  $n$  business days or other increments.

## Realized Variance

The **realized variance** of  $X$  over the partition  $\mathcal{P}$  is defined as

$$RV(X, \mathcal{P}) = \sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})^2.$$

The *annualized* realized variance is given by  $\frac{1}{T} RV(X, \mathcal{P})$ .

# Options on Realized Variance (1)

There exists a considerable number of financial instruments that are based on realized variance as an underlying

- Variance Swap: Pays the difference between a fixed *swap rate* and the annualized realized variance. The swap rate is chosen such that today's fair value of the swap is zero.
- Volatility Swap: Instead of realized variance its square root ('realized volatility') is used.
- Call on realized variance: Pays the difference between annualized realized variance and the strike value, with a floor at zero.
- Puts, Straddles, weighted var swaps, corridor var swaps, ...

For a good overview see Bühler (2006a) or Gatheral (2006, Chapter 11)

## Options on Realized Variance (2)

- If the strike of a call or put option equals the swap rate, the option is called at-the-money (ATM). These are the most frequently traded options.
- All the mentioned instruments allow hedging of – or speculation on – volatility risk, without exposure to other market risk factors, e.g. directional ('delta') risk.

- Given a stochastic model of  $X$ , the standard approach to pricing & hedging of options on variance, is to substitute the *realized variance* (RV) by the *quadratic variation* (QV) of  $X$ :

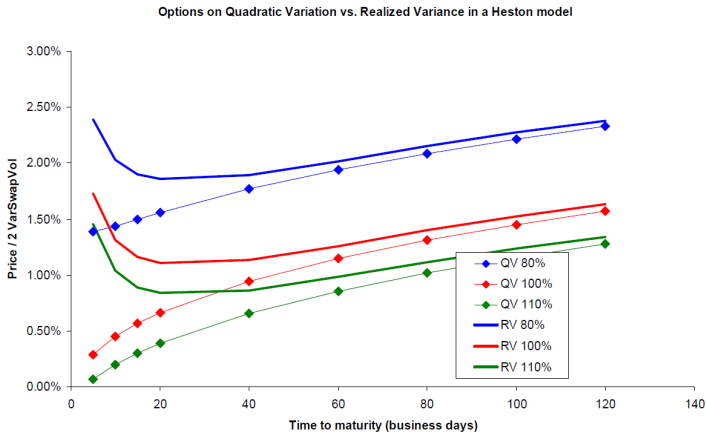
$$RV(X, \mathcal{P}) \approx [X, X]_{\mathcal{T}}$$

- Quadratic Variation is the limit (in probability) of realized variance  $RV(X, \mathcal{P}^n)$ , when the mesh size of  $\mathcal{P}^n$  goes to zero.
- For many stochastic processes quadratic variation is a well-studied quantity. In diffusion models it also yields elegant replication formulas for variance swaps and other contracts (Neuberger 1992).

- The approximation via quadratic variation works well for claims with linear payoffs, such as variance swaps, and claims with close-to-linear payoffs like volatility swaps (cf. Bühler (2006a); Sepp (2008) resp. Broadie and Jain (2008))
- However, citing Bühler (2006a):

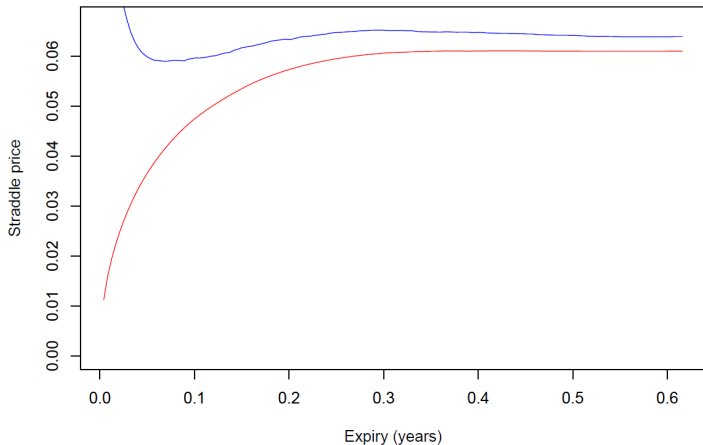
*“while the approximation of realized variance via quadratic variation works very well for variance swaps, it is not sufficient for non-linear payoffs with short maturities. The effect is common to all variance curve models (or stochastic volatility models, for that matter).”*

# Bühler's Examples



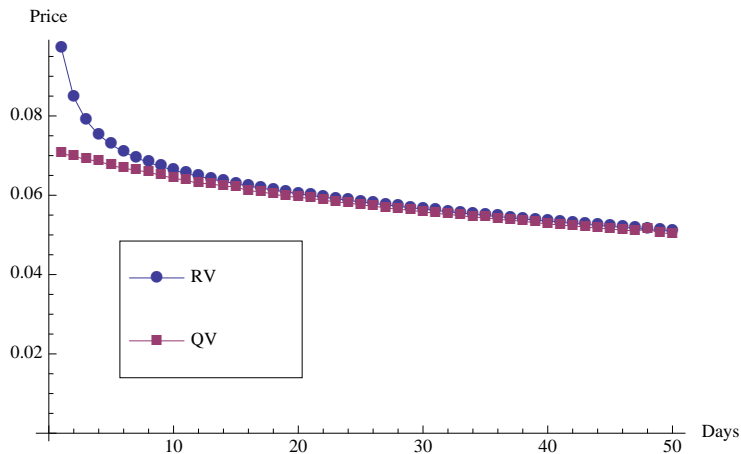
Plot from Bühler (2006b).

# Gatheral's Example



Plot from Gatheral (2008). Shows prices of ATM Variance Straddles in a Double CEV (diffusion) model.

# Own Example



Plot from Keller-Ressel and Muhle-Karbe (2010). Shows ATM call prices in Kou's jump-diffusion model.

Some observations:

- 1 Prices of contracts on RV and QV are significantly different for short ( $\leq 60$  days) maturities.<sup>1</sup>
- 2 Contracts on RV are more expensive than contracts on QV
- 3 For short maturities the term structure is decreasing for RV. In diffusion models it is increasing for QV
- 4 Prices of ATM contracts on QV go to zero for  $T \rightarrow 0$  in diffusion models; prices of contracts on RV do not.

Features seem to be . . .

- parameter-independent,
- largely model-independent (only diffusion vs. jumps matters)
- possibly even payoff independent (for convex payoffs)

---

<sup>1</sup>At least if a diffusive component is present

# Goal: Making things rigorous

General Objective: Making these observations mathematically rigorous (providing proofs)

Observations 1, 3, 4 have been made rigorous in an asymptotic sense ( $T \rightarrow 0$ ) in

KELLER-RESSEL, M. and MUHLE-KARBE, J. (2010). *Asymptotics and exact pricing of options on variance*. arXiv:1003.5514.

In this talk, focus on Observation 2: Are contracts on RV more expensive than contracts on QV?

# The convex order conjecture (1)

## Convex Order conjecture

“The price of a call option on realized variance is bigger than the price of a call option on quadratic variation”

We consider both

- Fixed-strike call  $(x - K)^+$ ,
- (Variable-strike) ATM call  $(x - R)^+$ , where  $R$  is the swap rate.

Note that the swap rate is model-dependent! The second case is a bit harder, but more relevant in practice.

## The convex order conjecture (2)

- We believe that the conjecture is true in a large class of models and for several payoffs; however more numerical evidence is needed.
- The fixed-strike convex order conjecture is equivalent to

$$\mathbb{E}[g([X, X]_T)] \leq \mathbb{E}[g(RV(X, \mathcal{P}))]$$

for all increasing convex  $g$ , i.e.  $RV$  dominates  $QV$  in the convex order  $\leq_c$ .

# The convex order conjecture (3)

## Centering

Define the *centered realized variance* ( $\overline{RV}$ ) and the *centered quadratic variation* ( $\overline{QV}$ ) by

$$\begin{aligned}\overline{RV}(X, \mathcal{P}) &= RV(X, \mathcal{P}) - \mathbb{E}[RV(X, \mathcal{P})], \\ \overline{[X, X]}_T &= [X, X]_T - \mathbb{E}[[X, X]_T].\end{aligned}$$

- For the variable-strike convex order conjecture it is sufficient that

$$\mathbb{E}\left[g\left(\overline{[X, X]}_T\right)\right] \leq \mathbb{E}\left[g\left(\overline{RV}(X, \mathcal{P})\right)\right]$$

for all (even non-increasing) convex  $g$ , i.e.  $\overline{RV}$  dominates  $\overline{QV}$  in convex order. Note that  $\mathbb{E}[\overline{RV}] = \mathbb{E}[\overline{QV}]$ .

## Part II

### Showing the Convex Order Conjecture

**Assumptions**  $X$  has conditionally independent increments & satisfies a symmetry condition.

**Tools** Reverse (Sub)martingales.

## Assumption 1

$X$  is a semi-martingale with  $\mathcal{H}$ -conditionally independent increments, for some  $\mathcal{H} \subset \mathcal{F}_0$ .

- We denote by  $(B, C, \nu)$  the semi-martingale characteristics of  $X$  relative to the truncation function  $h(x) = \mathbf{1}_{\{|x| \leq 1\}}x$ .
- A semi-martingale has  $\mathcal{H}$ -cond. indep. increments if and only if its characteristics have a  $\mathcal{H}$ -measurable version.
- This assumption includes Lévy processes, Sato processes, time-changed Lévy processes and stochastic volatility models without leverage.

Consider a martingale indexed by  $\mathbb{Z}$ :

$$\dots, M_{-2}, M_{-1}, M_0, M_1, M_2, \dots$$

Going backwards

$$Y_n = M_{-n}, \quad (n \in \mathbb{Z})$$

is called a reverse martingale.

Introducing  $\sigma$ -algebras and generalizing a bit, we get...

## Reverse $\mathcal{G}$ -(Sub)martingale

$(\mathcal{G}_n)$ ... *decreasing* sequence of  $\sigma$ -algebras

$Y_n$ ... sequence of integrable random variables such that  $Y_n \in \mathcal{G}_n$ .

$(Y_n)$  is called a **reverse (sub)martingale** with respect to  $\mathcal{G}_n$ , if

$$\mathbb{E}[Y_{n-1} | \mathcal{G}_n] = (\geq) Y_n \quad (1)$$

for all  $n \in \mathbb{N}$ .

- Note that  $(\mathcal{G}_n)$  is *decreasing*, and thus not a filtration.
- Doob's martingale convergence theorem yields that a reverse submartingale converges a.s. to a limit  $Y_\infty$ . (For forward submartingales further conditions are needed!)

# Reverse Submartingales and Convex Order

By Jensen's inequality, a reverse submartingale is decreasing in convex order:

$$\mathbb{E}[g(Y_n)] \geq \mathbb{E}[g(Y_{n+1})] \geq \dots \mathbb{E}[g(Y_\infty)],$$

for all increasing and convex  $g$ .

Conversely, any two random variables  $B \leq_c A$  in convex order, can be connected by a reverse submartingale with  $Y_0 = A, Y_\infty = B$ , by a result of Strassen (1965).

Can we find a reverse submartingale connecting realized variance and quadratic variation?

# Reverse Martingales from Realized Variance (1)

## Theorem

Let  $X$  be a semi-martingale with  $\mathcal{H}$ -conditionally independent, *symmetric* increments, and let  $(\mathcal{P}^n)$  be a sequence of *nested* partitions of  $[0, T]$ . Then the realized variance of  $X$  over the partitions  $(\mathcal{P}^n)$  is a reverse martingale, i.e. it satisfies

$$\mathbb{E} [RV(X, \mathcal{P}^{n-1}) | \mathcal{G}_n] = RV(X, \mathcal{P}^n), \quad (n \in \mathbb{N})$$

where

$$\mathcal{G}_n = \mathcal{H} \vee \sigma \left( RV(X, \mathcal{P}^n), RV(X, \mathcal{P}^{n+1}), \dots \right).$$

Moreover, if the mesh size of  $\mathcal{P}^n$  goes to zero, then  $RV(X, \mathcal{P}^n) \rightarrow [X, X]_T$  a.s.

## Reverse Martingales from Realized Variance (2)

- Used implicitly by Lévy (1940) to show that the realized variance of Brownian motion converges to  $t$  *almost surely* when partitions are nested.
- Shown by Cogburn and Tucker (1961) for processes with independent symmetric increments.
- Here extended to semi-martingales with conditionally independent symmetric increments.
- Proof relies on a reflection principle based on the symmetry of increments.

# Convex Order Relations (1)

Denote by  $\mathcal{G}_\infty = \bigcap_{n \in \mathbb{N}} \mathcal{G}_n$  the tail  $\sigma$ -algebra of  $\mathcal{G}_n$ .

## Convex Order Relation

If  $X$  is a semi-martingale with  $\mathcal{H}$ -conditionally independent, **symmetric** increments, then

$$\mathbb{E} [RV(X, \mathcal{P}) | \mathcal{G}_\infty] = [X, X]_T.$$

This equation implies the convex order relation

$$RV(X, \mathcal{P}) \geq_c [X, X]_T.$$

However, even in the Black-Scholes model, the log-price  $X$  does not have symmetric increments, because of the exponential martingale drift:

$$X_t = B_t - \frac{t}{2}$$

We relax the symmetry condition on the increments, and replace it with a symmetry condition on the jumps

## Assumption 2

The jump measure  $\nu$  of  $X$  is symmetric, i.e. it satisfies

$$\nu(dt, dx) = \nu(dt, -dx).$$

We lose the reverse martingale property over nested partitions, but retain the convex order relation between RV and QV.

### Theorem

*Let  $X$  be a semi-martingale with conditionally independent increments and symmetric jump measure. Suppose that the drift  $B$  of  $X$  is continuous. Then there exists a  $\sigma$ -algebra  $\mathcal{G}_\infty$ , such that*

$$\mathbb{E} [RV(X, \mathcal{P}) | \mathcal{G}_\infty] = [X, X]_T + RV(B, \mathcal{P}).$$

*If  $B$  is deterministic, it also holds that*

$$\mathbb{E} [\overline{RV}(X, \mathcal{P}) | \mathcal{G}_\infty] = \overline{[X, X]}_T,$$

## Convex Order Relations (3)

Consequences for convex order:

### Corrolary

*Under the conditions of the previous Theorem, and if the drift  $B$  of  $X$  is continuous, realized variance dominates quadratic variation in convex order, i.e.*

$$RV(X, \mathcal{P}) \geq_c [X, X]_T.$$

*If  $B$  is deterministic, the result also holds for the centered variances, i.e.*

$$\overline{RV}(X, \mathcal{P}) \geq_c \overline{[X, X]}_T.$$

# Consequences for Option Pricing (1)

## Models with deterministic drift $B$ :

Exponential Lévy models with symmetric jumps

Symmetric Sato Processes A Sato process is a process with indep. increments and the self-similarity property

$$X_{\lambda t} \stackrel{d}{=} \lambda^\gamma X_t, \text{ for a fixed } \gamma > 0.$$

Has been used for variance options e.g. in (Carr et al. 2010).

In these models QV underprices RV for the following payoffs:

- Fixed Strike Calls  $(x - K)^+$
- Relative Strike Calls  $(x - kR)^+$  where  $k \leq 1$  and  $R$  is the swap rate
- ATM Puts and ATM Straddles with payoffs  $(R - x)^+$  and  $|R - x|$  respectively.

### Models with continuous random drift $B$ :

Time-changed Lévy processes with symmetric jumps. The time change is arbitrary, but we assume it to be independent from the Lévy process.

Stochastic Volatility models with jumps We assume that the volatility process is independent from the Brownian motion and jump measure driving the stock price (no leverage effect).

In these models QV underprices RV for:

- Fixed Strike Calls  $(x - K)^+$

i.e. the convex order conjecture holds for  $RV$  and  $QV$  but not necessarily for the centered quantities  $\overline{RV}$  and  $\overline{QV}$ .

- Remove the symmetry condition on jumps, or provide a counterexample to show that it is necessary.
- Incorporate cash dividends and consider options on weighted realized variance.

On a more general note. . .

- Bring together the risk management perspective and the econometric perspective on Realized Variance.

**Thank you for your attention!**

- M. Broadie and A. Jain. The effect of jumps and discrete sampling on volatility and variance swaps. *International Journal of Theoretical and Applied Finance*, 11:761–797, 2008.
- Hans Bühler. *Volatility Markets – Consistent modeling, hedging and practical implementation*. PhD thesis, TU Berlin, 2006a.
- Hans Bühler. Options on variance: Pricing & hedging. Presentation at the IQPC Volatility Trading Conference, London, 2006b.
- Peter Carr, Helyette Geman, Dilip B. Madan, and Marc Yor. Options on realized variance and convex orders. *Quantitative Finance*, iFirst:1–10, 2010.
- Robert Cogburn and Howard G. Tucker. A limit theorem for a function of the increments of a decomposable process. *Transactions of the American Mathematical Society*, 99:278–284, 1961.
- Jim Gatheral. *The Volatility Surface*. Wiley Finance, 2006.
- Jim Gatheral. Consistent modeling of spx and vix options. Presentation at the 5th Bachelier Congress, London, 2008.
- Martin Keller-Ressel and Johannes Muhle-Karbe. Asymptotic behaviour and pricing of options on variance. Preprint, 2010.
- Paul Lévy. Le mouvement brownien plan. *American Journal of Mathematics*, 62(1):487 – 550, 1940.

- A. Neuberger. Volatility trading. London Business School Working Paper, 1992.
- A. Sepp. Analytical pricing of double-barrier options under a double-exponential jump-diffusion process. *International Journal of Theoretical and Applied Finance*, 7:151–175, 2008.
- Volker Strassen. The existence of probability measures with given marginals. *The Annals of Mathematical Statistics*, 36(2):423–439, 1965.