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Dimension of posets with cover graphs in minor-closed classes

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Introduction

Partially ordered sets, also known as posets, are ubiquitous objects in combinatorics. Every finite poset is isomorphic to a subposet of the product of a number of linear orders (equipped with the product order), and the least number of linear orders for which such an isomorphic subposet can be found is called the dimension of the poset. The notion of poset dimension was introduced in 1941 by Dushnik and Miller [6], and it is an important measure of poset complexity with many applications in theoretical computer science. For instance, posets of small dimension can be efficiently stored in memory, requiring much less space than when storing the matrix of poset comparabilities. Dimension is also intriguing from the perspective of computational complexity. Already the problem of determining whether a poset has dimension 3 is NP-complete [36], and no polynomial time algorithm exists that approximates the dimension within a factor of $\mathcal{O}(n^{1-\varepsilon})$ for any $\varepsilon > 0$ [3]. We do not know any nontrivial poset classes for which dimension can be effectively computed.

The theory of dimension for partial orders is a rich part of combinatorics which has many deep connections with graph theory. For instance, poset dimension can be used to characterize planar graphs [27] and nowhere dense classes of graphs [15]. Recent research explores dim-boundedness, which is a poset-theoretic counterpart of χ-boundedness from the realm of graphs. Classes of posets known to be dim-bounded include posets with cover graphs of bounded pathwidth [13] or treewidth [16], and some of the recent results [20] provide a promising approach to solving a more than 40 years old conjecture that posets with planar cover graphs are dim-bounded.

In this thesis I explore links between dimension of posets and properties of the graphs associated with them. My goal is to address the following question.

Which minor-closed graph classes $\mathcal{C}$ have the property that posets with cover graphs in $\mathcal{C}$ have dimension bounded by a constant?

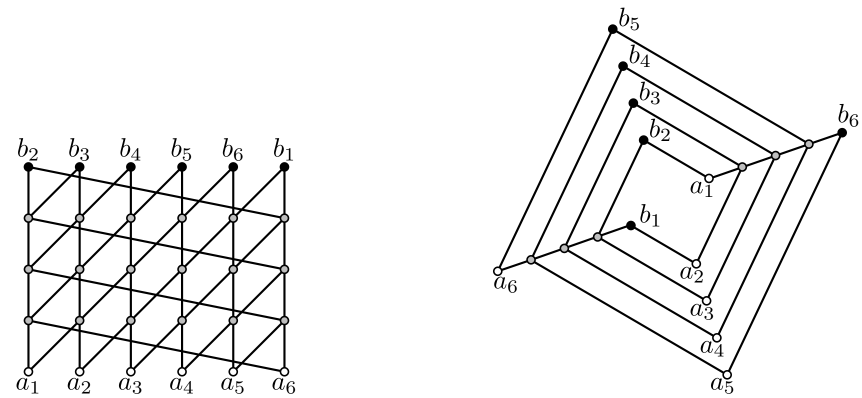
This question deserves providing some context.

There are several ways to associate a graph with a poset. In the simplest one, the vertices are the elements of the poset, and two distinct vertices are adjacent when they are comparable in the poset. That graph is called the comparability graph of the poset. Intuitively, posets with "sparse" comparability graph should have small dimension. The simplest way to formalize sparsity is to consider graphs of bounded degree, and the dimension of posets with comparability graph of bounded degree has been studied extensively [10, 7, 28]. Scott and Wood [28] proved that posets with comparability graphs of maximum degree Δ have dimension $\Delta \log^{1+o(1)} \Delta$, which, by a result of Erdős, Kierstead and Trotter [7] is within a $\log^{o(1)} \Delta$ factor of optimal.

Every chain in a poset forms a clique in the comparability graph, so any poset with a comparability graph of maximum degree Δ has height at most $\Delta + 1$. It turns out that in the bounded height setting, for the dimension to be bounded it suffices to assume sparsity of its cover graph.

The cover graph of a poset is the subgraph of its comparability graph consisting of only those edges which are not implied by transitivity of the order relation. In other words, the cover graph of a poset is its Hasse diagram seen as an abstract undirected graph. In 2014, Streib and Trotter [30] proved that posets with planar cover graphs have dimension bounded in terms of height. This discovery initiated a line of research aiming to understand, for which graph classes $\mathcal{C}$ it is true that all posets with cover graphs from $\mathcal{C}$ have dimension bounded by a function of height. The aforementioned bound on the dimension for posets with comparability graph of bounded maximum degree implies that this holds when $\mathcal{C}$ has bounded maximum degree. A sequence of results revealed that this also holds when $\mathcal{C}$ is a class of bounded treewidth [14], a class excluding a fixed graph as a minor or as a topological minor [34, 24], or a class of bounded expansion [18]. Note that graphs excluding a fixed graph as a minor generalize planar graphs and graphs of bounded treewidth, and graphs excluding a fixed graph as a topological minor generalize graph of bounded degree and graphs excluding a fixed graph as a minor. Classes of bounded expansion generalize all classes mentioned before.

This brings us back to our initial question. When does there exist a



(a) Trotter's construction (b) Kelly's construction

Figure 1: Two posets with planar cover graph and dimension 6. Both constructions contain a standard example of dimension 6, which is a poset consisting of the elements $a_1, \ldots, a_6, b_1, \ldots, b_6$ such that each a_i is incomparable with b_i and $a_i < b_j$ for $i \neq j$.

bound on the dimension which does not depend on height? In 1977, Trotter and Moore [33] showed that every poset whose cover graph is a forest, has dimension at most 3. Soon after, Trotter [32] found a construction of posets with planar cover graphs and arbitrarily large dimension, see Figure 1a. This shows that only for some proper minor-closed classes there exists a constant bound. Furthermore, in 1981, Kelly [19] constructed posets with arbitrarily large dimension and planar cover graphs of treewidth (and pathwidth) 3, see Figure 1b.

Nevertheless, a constant bound is known for several examples of minor-closed classes other than the class of forests. Felsner, Trotter and Wiechert proved that the dimension is at most 4 for the class of outerplanar graphs. For the class of graphs of pathwidth at most 2, Biró, Keller and Young [1] showed that the dimension is at most 17, which was later improved to 6 by Wiechert [35]. Joret, Micek, Trotter, Wang and Wiechert [17], showed that for the class of graphs of treewidth at most 2 (which are exactly the graphs which exclude K_4 as a minor) the dimension is at most 1276. Finally, it is an easy consequence of folklore results that for any class of bounded treedepth the dimension is bounded as well.

Where exactly is the boundary between the minor-closed classes for which the dimension is bounded and those for which it is unbounded? The necessary condition for a class to have bounded dimension is to exclude the cover graph of some poset from the Kelly's construction. It is conjectured, that this condition is also sufficient because the cover graphs of posets from Kelly's construction can be found as minors in all known constructions of posets of large dimension. Although the conjecture remains open, the results presented in this thesis make a substantial progress in finding the answer to the question. Moreover, the work on this question has led to new discoveries in other areas: a qualitative structure theorem for graphs excluding long ladders and an improved bound on the dimension in terms of height for posets with planar cover graphs.

In this thesis, I present four major results. The first result, is an improved bound on the dimension for posets with cover graphs of treewidth at most 2, published in [29]. The new proof not only gives a substantially better bound (12 in place of 1276), but also is much simpler than the original proof by Joret et al. [17].

The second result is my unpublished result that for a fixed n, posets excluding $K_{2,n}$-minors in their cover graphs have bounded dimension. The proof relies on a characterization of graphs without large $K_{2,n}$-minors by Ding [5].

The third result shows that posets excluding a $2 \times n$ grid (a ladder) as a minor for a fixed n have bounded dimension. This is a joint work with Huynh, Joret, Micek and Wollan [12]. In our work, we developed a new structure theorem for graphs without long ladders, which is of independent interest. We present some applications of this structure theorem outside poset theory.

The final main result is a theorem, which I proved together with Gorsky [11], that posets with k-outerplanar cover graphs have bounded dimension. Our bound is $\mathcal{O}(k^3)$. This generalizes the fact that posets with outerplanar (that is 1-outerplanar) cover graphs have bounded dimension. An important consequence of this is that height-h posets with planar cover graph have dimension $\mathcal{O}(h^3)$. Previously, the best known bound was $\mathcal{O}(h^6)$.

Chapter 1

Preliminaries

Graphs

In order to make the thesis self-contained, we introduce the standard definitions and notation from graph theory in this and the following section. For a broader introduction to graphs, we refer the reader to the excellent textbook on graph theory by Diestel [4].

A *graph* is a pair $G = (V, E)$ where V is a set whose elements are called *vertices* and E is a set whose elements are 2-element subsets of V called *edges*. We always assume the sets V and E to be disjoint and finite. A graph with vertex set V is said to be *on* V. The vertex set and the edge set of a graph G are referred to as $V(G)$ and $E(G)$, regardless of any actual names of these sets. For instance, the vertex set of a graph $H = (W, F)$ is referred to as $V(H)$, not as $W(H)$. The *empty* graph is $(\varnothing, \varnothing)$.

An edge $e = \{x, y\}$ is usually written as xy or yx, and the vertices x and y are called *ends* of e. We also say that the vertices x and y are *incident* with the edge e. We mainly use the notation $\{x, y\}$ for pairs which may or may not be edges of the graph. When a graph G has an edge xy, the vertices x and y are called *adjacent* or *neighbors*. The set of neighbors of a vertex x in a graph G is denoted by $N_G(x)$, and the number of neighbors of x is the *degree* of x. A graph is *complete* if all its vertices are pairwise adjacent. We denote by K_n a complete graph with n vertices. A graph G is *bipartite* if its vertex set admits a partition into two sets A and B such that each edge of G has ends in A and B. If additionally G contains all possible edges with ends in A and B, we call G *complete bipartite*. We denote by $K_{n,m}$ a complete bipartite graph with a corresponding partition $\{A, B\}$ satisfying $|A| = n$ and $|B| = m$.

Two graphs G_1 and G_2 are *isomorphic* if there exists an *isomorphism* between them, that is a bijection $\varphi\colon V(G_1) \to V(G_2)$ such that for any pair of distinct vertices x and y of G_1 we have $\{x, y\} \in E(G_1)$ if and only if $\{\varphi(x), \varphi(y)\} \in E(G_2)$. The *isomorphism class* of a graph G is the collection of all graphs isomorphic to G (If G is nonempty, then this collection does not form a set, as the vertices can be arbitrary sets and there is no set of all sets). Since we only consider finite graphs, there are only countably many distinct isomorphism classes of graphs. A *class of graphs* (or a *graph class*) is any collection $\mathcal{C}$ of graphs such that whenever a graph belongs to it, so do all graphs isomorphic to it. Hence, every class of graphs is the union of (at most countably many) isomorphism classes of some graphs.

The *union* and *intersection* of two graphs G_1 and G_2 are defined as

$$G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$$

and

$$G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$$

respectively. The graphs G_1 and G_2 are *disjoint* when $G_1 \cap G_2$ is the empty graph (which is equivalent to $V(G_1) \cap V(G_2) = \varnothing$).

If H and G are two graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, G is a *supergraph* of H, and we write $H \subseteq G$. If additionally H contains all edges $xy \in E(G)$ with $\{x, y\} \subseteq V(H)$, then H is called an *induced subgraph*. For a subset of vertices $U \subseteq V(G)$, the induced subgraph of G with the vertex set U is called the subgraph *induced by* U and denoted by $G[U]$. We denote by $G - U$ the subgraph induced by $V(G)\backslash U$, that is the graph obtained by *deleting* all vertices in U and all edges incident with them. For a set F of 2-element subsets of $V(G)$ which may or may not be edges of G, we define $G - F = (V(G), E(G)\backslash F)$ and $G + F = (V(G), E(G) \cup F)$

A *path* is a graph W which consists of distinct vertices $x_0, \ldots, x_k$ such that $E(W) = \{x_i x_{i+1} : i \in \{0, \ldots, k-1\}\}$. When there is no ambiguity with the notation xy for edges, we denote such a path by $x_0 \cdots x_k$. The *length* of a path is the number of its edges. A path of length 0 is *trivial*. The vertices x_0 and x_k are the *ends* of the path while $x_1, \ldots, x_{k-1}$ are the *inner* vertices. Two or more paths are *internally disjoint* if none of them contains an inner vertex of another.

When A and B are sets of vertices and W is a path $x_0 \cdots x_k$ such that $V(W) \cap A = \{x_0\}$ and $V(W) \cap B = \{x_k\}$, we call W an *A–B path*. We simplify notation when either of the sets A and B is a singleton, so that for instance we write a–B path rather than $\{a\}$–B path.

A graph is *connected* if it is nonempty and for any two vertices x and y, the graph contains an x–y path. Equivalently, a nonempty graph G is connected if and only if and only if for every partition of $V(G)$ into two nonempty sets A and B there exists an edge with ends in both sets A and B. A subset of vertices $U \subseteq V(G)$ is *connected* if the induced subgraph $G[U]$ is connected. Every graph can be uniquely represented as the union of disjoint connected graphs, called the *components* of the graph.

For vertex subsets A, B and X in a graph G, we say that X *separates* A and B if every A–B path contains a vertex from X. If for a vertex $x \in V(G)$ there exists vertices $a, b \in V(G)\backslash\{x\}$ lying in one component of G such that $\{x\}$ separates $\{a\}$ and $\{b\}$, we call x a *cutvertex*. Thus, x is a cutvertex in G if the graph $G - \{x\}$ has more components than G.

A graph G is *k-connected* if $|V(G)| > k$ and $G - X$ is connected whenever $X \subseteq V(G)$ and $|X| < k$. A *block* of a graph G is a maximal connected subgraph of G without a cutvertex. A block can be a vertex of degree 0, an edge e with its ends, or a 2-connected subgraph of G.

A *cycle* is a graph of the form $W + \{e\}$, where W is a path $x_0 \cdots x_k$ with $k \geqslant 2$, and $e = x_0 x_k$. A *Hamiltonian cycle* in a graph G is a cycle in G (that is a cycle which is a subgraph of G) which contains all vertices of G.

A graph which does not contain a cycle is a *forest*, and a connected forest is a *tree*. In a tree T, there is a unique x–y path between each pair of vertices x and y, and we denote it by xTy. A *rooted tree* is a tree with a distinguished vertex called a *root*. We sometimes refer to the vertices of trees as *nodes*. If T is a rooted tree with a root u_0 and u and v are two nodes such that $u \in V(u_0 T v)$, then u is an *ancestor* of v, and v is a *descendant* of u. If additionally u and v are adjacent, then u is the *parent* of u, and v is a *child* of u. The *lowest common ancestor* of nodes u and v is the unique node w which is an ancestor of u and v but does not have a child which is an ancestor of u and v. A *leaf* is a node without a child, and a node with at least one child is *inner*. The *height* of a rooted tree is the maximum length of a path between the root and a leaf.

Minors, planarity and tree-decompositions

The simplest way in which a graph can contain another graph, is as a subgraph. Another way in which a graph can be contained is as "minor". This section introduces the basics of the graph minors and some important examples of minor-closed classes of graphs.

When G is a graph and y is a vertex of G with exactly two neighbors x and z, we say the graph $(G-\{y\})+\{xz\}$ is obtained from G by *suppressing* y. The operation inverse to suppressing is subdividing. *Subdividing* an edge $e = xz$ in a graph G yields the graph $(V(G) \cup \{y\}, (E(G)\backslash\{e\}) \cup \{xy, yz\})$ where y is a new vertex not appearing in G. A graph obtained from G by repeatedly subdividing edges is called a *subdivision* of G. If G does not have vertices of degree 0, then every subdivision of G is the union of a family of internally disjoint paths $\{W_e\}_{e\in E(G)}$ such that each path W_e has the same ends as the edge e. A graph H is a *topological minor* of a graph G when G has a subgraph isomorphic to a subdivision of G.

Suppressing vertices of degree 2 is generalized by edge contraction. When G is a graph with an edge $e = xy$, one can *contract* the edge e *to* the vertex x to obtain the graph $(G - \{y\}) + \{xz : z \in N_G(y)\backslash\{x\}\}$. Note that if the degree of y is 2, then contracting e to x is equivalent to suppressing y. We say that a graph H is a *minor* of a graph G (or *G contains H as a minor*) if a graph isomorphic to H can be obtained from a subgraph of G by repeatedly contracting edges. Equivalently, H is a minor of G if and only if there exists an indexed family $\{U_x\}_{x\in V(H)}$ of pairwise-disjoint connected subsets of vertices in G such that for every edge $xy \in E(H)$, the graph G contains an edge with ends in U_x and U_y in G. Every topological minor of a graph G is a minor of G, and every minor of a graph G in which every vertex has degree at most 3 is a topological minor of G. If H is not a minor of G, we say that G is *H-minor-free*.

A class of graphs $\mathcal{C}$ is *minor-closed* if for every $G \in \mathcal{C}$, all minors of G belong to $\mathcal{C}$. A seminal result by Robertson and Seymour [26] states that for every minor-closed class $\mathcal{C}$ there exists a finite set $\{H_1, \ldots, H_k\}$ of graphs such that $\mathcal{C}$ consists of exactly those graphs which do not contain any of the graphs $H_1, \ldots, H_k$ as a minor.

A *planar drawing* of a graph G is a drawing where the vertices are represented by points on a plane and the edges are represented by non-crossing curves between the vertices. More formally, in a planar drawing of G, each vertex $x \in V(G)$ is represented by a point $p_x \in \mathbb{R}^2$ and each edge $xy \in E(G)$

is represented by a simple curve $\gamma_{xy} \subseteq \mathbb{R}^2$ with endpoints in p_x and p_y so that (1) $p_x \neq p_y$ for distinct $x, y \in V(G)$, (2) $p_z \notin \gamma_{xy}$ for $xy \in E(G)$ and $z \in V(G) \backslash \{x, y\}$, and (3) $\gamma_{xy} \cap \gamma_{x'y'} \subseteq \{p_x, p_y\}$ for distinct $xy, x'y' \in E(G)$. A graph is *planar* if it admits a planar drawing. Planar graphs form a minor-closed class of graphs consisting of exactly those graphs which do not contain K_5 nor $K_{3,3}$ as minors.

Let G be a graph with a planar drawing $(\{p_x\}_{x \in V(G)}, \{\gamma_e\}_{e \in E(G)})$. For every subgraph $H \subseteq G$, we define an *inherited* planar drawing of H as $(\{p_x\}_{x \in V(H)}, \{\gamma_e\}_{e \in E(H)})$. The components (in the topological sense) of $\mathbb{R}^2 \backslash \{p_x : x \in V(G)\} \cup \bigcup_{e \in E(G)} \gamma_e$ are called *faces* of the drawing. Exactly one face in a drawing is unbounded, and we call it the *outer face*. A graph is *outerplanar* if it admits a planar drawing such that every vertex lies on the boundary of the outer face. A graph is outerplanar if and only if it does not contain K_4 nor $K_{2,3}$ as a minor. The $m \times n$ grid is a planar graph on $\{1, \ldots, m\} \times \{1, \ldots, n\}$ where two vertices (i, j) and (i', j') are adjacent when $|i - i'| + |j - j'| = 1$. Every planar graph is a minor of the $n \times n$ grid for some n.

A *k-tree* is any graph obtained from a complete graph on $k + 1$ vertices by repeatedly adding vertices in such a way that the neighbors of every added vertex form a k-clique (for instance, 1-trees are exactly trees on at least 2 vertices). A *partial k-tree* is any subgraph of a k-tree. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$ is the least k such that G is a partial k-tree.

For every nonnegative integer k, graphs of treewidth at most k form a minor-closed class of graphs. Graphs of treewidth 0 are graphs without edges, and graphs of treewidth 1 are forests which contain at least one edge. For $k \in \{0, 1, 2\}$, graphs of treewidth at most k are exactly K_{k+2}-minor-free graphs. Graphs of treewidth at most 3 can be characterized by a list of 4 forbidden minors, and for $k \geqslant 4$ the complete list of forbidden minors for graphs of treewidth at most k is not known.

A more complex, but also more useful definition of treewidth involves tree-decompositions. A pair $(T, \{V_u\}_{u \in V(T)})$ is called a *tree-decomposition* of a graph G when T is a tree and $\{V_u\}_{u \in V(T)}$ is a family of subsets of $V(G)$ such that

(T1) $\bigcup_{u \in V(T)} V_u = V(G)$,

(T2) for each $xy \in E(G)$ there exists $u \in V(T)$ such that $\{x, y\} \subseteq V_u$, and

(T3) for any nodes u_1, u_2 and u of T, if $u \in V(u_1 T u_2)$, then $V_{u_1} \cap V_{u_2} \subseteq V_u$.

The *width* of $(T, \{V_u\}_{u \in V(T)})$ is $\max\{|V_u| : u \in V(T)\} - 1$. The treewidth of a graph can be equivalently defined as the minimum width of its tree-decomposition. The *pathwidth* of a graph is the minimum width of its tree-decomposition $(T, \{V_u\}_{u \in V(T)})$ such that T is a path.

Lemma 1.1. *Let $(T, \{V_u\}_{u \in V(T)})$ be a tree-decomposition of a graph G, let v_1 and v_2 be two nodes of T, and let $e = u_1u_2 \in E(v_1Tv_2)$. If W is a path in G with ends in V_{v_1} and V_{v_2}, then W contains a vertex from $V_{u_1} \cap V_{u_2}$.*

Proof. For $i \in \{1, 2\}$, let T_i denote the component of $T - \{e\}$ containing v_i, and let $G_i = G[\bigcup_{v \in V(T_i)} V_v]$. By (T1) and (T2), we have $G = G_1 \cup G_2$, so there are no edges between $V(G_1)\backslash V(G_2)$ and $V(G_2)\backslash V(G_1)$ in G. Since W is a connected subgraph of G intersecting both $V(G_1)$ and $V(G_2)$, this implies that W intersects $V(G_1) \cap V(G_2)$. By (T3), we have $V(G_1) \cap V(G_2) \subseteq V_{u_1} \cap V_{u_2}$, which implies the lemma. $\square$

For each $n \geqslant 1$, the treewidth of the $n \times n$ grid is n. Furthermore, the Grid-Minor Theorem by Robertson and Seymour [25] states that the treewidth of a graph is bounded in terms of the size of its largest $n \times n$ grid minor. These results imply a deep connection between planarity and treewidth: A graph H is planar if and only if there exists an integer c such that any H-minor-free graph has treewidth at most c.

Posets

A *partial order* on a set V is a binary relation $\leqslant$ on V such that for any elements $x, y, z \in V$, the following hold:

(1) $x \leqslant x$ (reflexivity),

(2) if $x \leqslant y$ and $y \leqslant x$, then $x = y$ (antisymmetry), and

(3) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$ (transitivity).

A partial order $\leqslant$ is called a *linear* order if for any $x, y \in V$ we have $x \leqslant y$ or $y \leqslant x$. When a linear order is named with a letter, say L, we usually write "$x \leqslant y$ in L" rather than "xLy".

A *strict partial order* on a set V is a binary relation $<$ on V such that for any elements $x, y, z \in V$, the following hold:

(1) not $x < x$ (irreflexivity),

(2) if $x < y$ then not $y < x$ (asymmetry), and

(3) if $x < y$ and $y < z$, then $x < z$ (transitivity).

A *poset* is a pair $P = (V, \leqslant_P)$, where V is a set and $\leqslant_P$ is a partial order on V. The set V is called the *ground set* of P, and its elements are called the *elements* of P. In this thesis, we always assume the ground set to be finite. Often, we do not give an explicit name to the ground set and the partial order of a poset, but instead we write $x \in P$ when $x \in V_P$, and we use the same relation symbol $\leqslant$ for all posets. We avoid ambiguity by always explicitly specifying to which poset the symbol refers, for example we write "$x \leqslant y$ in P" when $x \leqslant_P y$. When we do not have $x \leqslant y$, we write $x \nleqslant y$.

In a poset P, elements x and y are *comparable* when $x \leqslant y$ or $y \leqslant x$. When $x \leqslant y$, we also write $y \geqslant x$, and we write $x < y$ or $y > x$ when $x \leqslant y$ and $x \neq y$. When x and y are not comparable, they are *incomparable* and we write $x \parallel y$. For a linear order L, we analogously define the notation $y \geqslant x$, $x < y$ and $y > x$. An element x in a poset P is *minimal* if there does not exist an element z with $z < x$ in P, and *maximal* if there does not exist an element z with $x < z$ in P. In a poset P, we denote by $\mathrm{Min}(P)$ and $\mathrm{Max}(P)$ respectively the set of minimal elements and the set of maximal elements of P.

For two elements x and y of a poset P, we say that x is *covered* by y if $x < y$ in P and there does not exist an element z such that $x < z < y$ in P. The *cover graph* of P is a graph on the ground set of P in which two elements are adjacent if one of them is covered by the other. For two elements x and y of P, we have $x \leqslant y$ in P if and only if there is a path $x_0 \cdots x_k$ in the cover graph such that $x_0 = x$, $x_k = y$ and x_{i-1} is covered by x_i for each $i \in \{1, \ldots, k\}$. Such a path is called a *witnessing path from x to y*.

Posets are usually visualized with diagrams. A *diagram* of a poset is obtained by identifying each element of the poset with a distinct point on the plane and drawing an upward curve from x to y for each pair of elements such that x is covered by y in the poset. The curves in a diagram may intersect arbitrarily.

When U is a subset of elements of a poset P, we denote by $P[U]$ the poset with U as the ground set such that for any $x, y \in Y$ we have $x \leqslant y$ in $P[U]$ if and only if $x \leqslant y$ in P. The poset $P[U]$ is the *subposet of P induced* *by* U. If V is the ground set of P, we denote by $P - U$ the subposet induced by $V \backslash U$.

In general, the cover graph of a subposet of a poset P is not a subgraph of the cover graph of P. A subset of elements U in a poset P is *convex* if whenever $x \leqslant y \leqslant z$ in P and $\{x, z\} \subseteq U$, we have $y \in U$. If a set U is convex in P, then $P[U]$ is a *convex* subposet of P, and the cover graph of $P[U]$ is an induced subgraph of the cover graph of P.

In a poset, a subset of pairwise comparable elements is called a *chain*. The *height* of a poset is the size of a largest chain in it. The height of a poset is the maximum number of vertices in a witnessing path in the poset. We note that unlike in the height of a tree, we count vertices, not edges.

For an element x of a poset P, we define its *upset* $\mathrm{U}_P(x)$ and *downset* $\mathrm{D}_P(x)$ as

$$\mathrm{U}_P(x) = \{y \in P : y \geqslant x \text{ in } P\} \quad \text{and} \quad \mathrm{D}_P(x) = \{y \in P : y \leqslant x \text{ in } P\}$$

Similarly, for a subset U of elements of elements of a poset P, the *upset* and the *downset* of U are defined as $\mathrm{U}_P(U) = \bigcup_{x \in U} \mathrm{U}_P(x)$ and $\mathrm{D}_P(U) = \bigcup_{x \in U} \mathrm{D}_P(x)$, respectively.

A *realizer* of a poset P is a nonempty set of linear orders $\{L_1, \ldots, L_d\}$ of the ground set of P such that for any pair of elements $x, y \in P$ we have

$$x \leqslant y \text{ in } P \quad \text{ if and only if } \quad x \leqslant y \text{ in } L_i \text{ for each } i \in \{1, \ldots, d\}.$$

The *dimension* of a poset P, denoted $\dim(P)$, is the size of its smallest realizer. Note that according to this definition, every poset has positive dimension, and a poset with at most one element has dimension 1.

A *linear extension* of a poset P is a linear order L on the ground set of P such that $x \leqslant y$ in L whenever $x \leqslant y$ in P. The linear orders in any realizer of P are linear extensions of P, and the set of all linear extensions of P is a realizer of P. We denote by $\mathrm{Inc}(P)$ the set of all ordered pairs of incomparable elements in P. A linear extension L *reverses* a pair $(a, b) \in \mathrm{Inc}(P)$ if $b < a$ in L, and a subset $I \subseteq \mathrm{Inc}(P)$ is *reversible* if there exists a linear extension L of P which reverses all pairs from I.

The definition of a poset can be reformulated in terms of partitions of $\mathrm{Inc}(P)$ into reversible sets. Commonly, the sets in a partition are required to be nonempty, but for convenience we allow empty sets in partitions. Hence, a *partition* of a set I is a family $\{I_1, \ldots, I_d\}$ of pairwise disjoint sets whose union is I, and zero or more of these sets may be empty. For a subset $I \subseteq \mathrm{Inc}(P)$, we denote by $\dim_P(I)$, the least integer $d \geqslant 1$ such that I can be partitioned into d reversible sets. Note that just like in the case of the dimension of a poset, $\dim_P(I)$ is always a positive integer (even if I is empty).

Proposition 1.2. *For every poset P, we have*

$$\dim_P(\mathrm{Inc}(P)) = \dim(P).$$

Proof. Let $d = \dim_P(\mathrm{Inc}(P))$. If $\mathrm{Inc}(P) = \varnothing$, then the partial order of P is linear and $\dim(P) = 1 = d$, so let us assume that $\mathrm{Inc}(P) \neq \varnothing$.

Let $\{I_1, \ldots, I_d\}$ be a partition of $\mathrm{Inc}(P)$ into the smallest possible number of reversible sets. For each $i \in \{1, \ldots, d\}$, let L_i be a linear extension of P reversing all pairs from I_i. We show that $\{L_1, \ldots, L_d\}$ is a realizer of P. For any elements $x, y \in P$, if $x \leqslant y$ in P, then we have $x \leqslant y$ in every linear extension of P, in particular in each of the linear extensions L_1, ..., L_d. Now suppose that we have $x \leqslant y$ in each of L_1, ..., L_d. If x and y were incomparable in P, we would have $(x, y) \in I_i$ for some i, and thus $y < x$ in L_i, so x and y must be comparable in P. Since we have $x \leqslant y$ in the linear extension L_1, we have $x \leqslant y$ in P. Therefore $\{L_1, \ldots, L_d\}$ is a realizer of P, so $\dim(P) \leqslant d = \dim_P(\mathrm{Inc}(P))$.

Now, let $\{L_1, \ldots, L_{\dim(P)}\}$ be a smallest realizer of P, and for each $i \in \{1, \ldots, \dim(P)\}$, let I_i denote the set of pairs from $\mathrm{Inc}(P)$ which are reversed in L_i but not in any of L_1, ..., L_{i-1}. By definition of realizer, every $(x, y) \in \mathrm{Inc}(P)$ is reversed in some L_i, and therefore $\{I_1, \ldots, I_{\dim(P)}\}$ is a partition of $\mathrm{Inc}(P)$ into reversible sets. This proves $\dim_P(\mathrm{Inc}(P)) \leqslant \dim(P)$. □

A sequence $((a_1, b_1), \ldots, (a_k, b_k))$ of pairs from $\mathrm{Inc}(P)$ with $k \geqslant 2$ is an *alternating cycle* if $a_i \leqslant b_{i+1}$ in P for each $i \in \{1, \ldots, k\}$ (in alternating cycles we always interpret the indices cyclically, so that $b_{k+1} = b_1$). An alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ is *strict* if $a_i \parallel b_j$ whenever $j \neq i + 1$.

It is well-known that alternating cycles can be used to characterize reversible sets.

Lemma 1.3. *Let P be a poset, and let $I \subseteq \mathrm{Inc}(P)$. The following are equivalent:*

(1) *I is reversible;*

(2) *there does not exist an alternating cycle in I;*

(3) *there does not exist a strict alternating cycle in I.*

Proof. We start with a proof of the implication (1)⇒(2). Let L be a linear extension of P which reverses all pairs from I. Towards a contradiction, suppose that there is an alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ in I. In L we have

$$a_1 \leqslant b_2 < a_2 \leqslant b_3 < \cdots < a_k \leqslant b_1 < a_1,$$

which is a contradiction.

Next, we prove (2)⇒(1). Suppose that there does not exist an alternating cycle in I. Let $\leqslant_I$ denote a binary relation on the same ground set of P such that for any elements x and y, we have $x \leqslant_I y$ if either $x \leqslant y$ in P, or there exists an *alternating path* $((a_1, b_1), \ldots, (a_k, b_k))$ with $k \geqslant 2$ consisting of elements of I such that in the poset P we have $x \leqslant b_k$, $a_1 \leqslant y$, and $a_i \leqslant b_{i+1}$ for $i \in \{1, \ldots, k-1\}$. The reflexivity and transitivity of $\leqslant_I$ is obvious. If there existed distinct elements x and y such that $x \leqslant_I y$ and $y \leqslant_I x$, then combining the corresponding alternating paths we would obtain an alternating cycle, contradicting our assumption. Hence $\leqslant_I$ is a partial order such that $x \leqslant_I y$ when $x \leqslant y$ in P and $b \leqslant_I a$ for $(a, b) \in I$. Therefore any linear extension of $\leqslant_I$ is a linear extension of P reversing all pairs from I.

The implication (2)⇒(3) is trivial, so it remains to prove (3)⇒(2). By contraposition, it suffices to show that if I contains an alternating cycle then it contains a strict alternating cycle. We claim that an alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ in I with the smallest possible value of k is strict. Suppose to the contrary that there exist indices i and j such that $j \neq i + 1$ and a_i is comparable with b_j in P. If $b_j \leqslant a_i$, then for $i' = j - 1$ and $j' = i + 1$ we have $a_{i'} \leqslant b_j \leqslant a_i \leqslant b_{j'}$ in P and $j' = i + 1 \neq j = i' + 1$. Hence, possibly after replacing i and j with i' and j', respectively, we assume that $a_i \leqslant b_j$. After shifting the cycle, we may also assume that $j < i$, so that $((a_j, b_j), \ldots, (a_i, b_i))$ is an alternating cycle of length less than k (because $j \neq i + 1$). The contradiction proves that there is a strict alternating cycle in I. □

For subsets A and B of the ground set of a poset P, we define

$$\mathrm{Inc}_P(A, B) = \mathrm{Inc}(P) \cap (A \times B) \quad \text{and} \quad \dim_P(A, B) = \dim_P(\mathrm{Inc}_P(A, B)).$$

The *dual* of a poset P is the poset P^d with the same ground set as P such that $x \leqslant y$ in P^d if an only if $y \leqslant x$ in P. Note that P and P^d have the same cover graph and the same dimension. More generally, for any $I \subseteq \mathrm{Inc}(P)$, we have $\dim_P(I) = \dim_{P^d}(I^{-1})$, where $I^{-1} = \{(b,a) : (a,b) \in I\}$.

The following lemma appeared first implicitly in [34]. We include a proof for completeness.

Lemma 1.4. *Let P be a poset with a cover graph G, and let $X \subseteq V(G)$ be a vertex subset such that every poset whose cover graph is a subgraph of $G - X$ has dimension at most d. Then $\dim(P) \leqslant 2^{|X|} \cdot d$.*

Proof. By a simple induction, it suffices to prove the case $|X| = 1$. Suppose that X consists of a single element x. Let $V = V(G)$. Every element of $\mathrm{D}_P(x)$ is comparable with every element of $\mathrm{U}_P(x)$, so for every $(a,b) \in \mathrm{Inc}(P)$ we have $a \notin \mathrm{D}_P(x)$ or $b \notin \mathrm{U}_P(x)$, that is

$$\mathrm{Inc}(P) = \mathrm{Inc}(V \setminus \mathrm{D}_P(x), V) \cup \mathrm{Inc}_P(V, V \setminus \mathrm{U}_P(x)).$$

We need to prove that $\dim(P) \leqslant 2d$, so because of duality, it suffices to show that $\dim_P(V \setminus \mathrm{D}_P(x), V) \leqslant d$.

The poset $P - \mathrm{D}_P(x)$ is a convex subposet of P whose cover graph is a subgraph of $G - \{x\}$, so its dimension is at most d. Let $\{I_1, \ldots, I_d\}$ be a partition of $\mathrm{Inc}(P - \mathrm{D}_P(x))$ into reversible sets. Suppose that the set $I_1 \cup \mathrm{Inc}(V \setminus \mathrm{D}_P(x), \mathrm{D}_P(x))$ is not reversible in P. By Lemma 1.3, we can find in it an alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$. For each $i \in \{1, \ldots, k\}$ we have $a_{i-1} \notin \mathrm{D}_P(x)$ and therefore $b_i \notin \mathrm{D}_P(x)$, so the alternating cycle is contained in I_1, contradicting its reversibility. Hence the set $I_1 \cup \mathrm{Inc}_P(V \setminus \mathrm{D}_P(x), \mathrm{D}_P(x))$ must be reversible. A partition of $\mathrm{Inc}_P(V \setminus \mathrm{D}_P(x), V)$ into d reversible sets can be obtained as

$$\{I_1 \cup \mathrm{Inc}_P(V \setminus \mathrm{D}_P(x), \mathrm{D}_P(x)), I_2, \ldots, I_d\}.$$

□

A *connected* poset is a poset with a connected cover graph, and a *component* of a poset is a subposet induced by the vertex set of a component of the cover graph.

Lemma 1.5. *Let P be a poset, and let $I \subseteq \mathrm{Inc}(P)$. If $\dim_P(I) \geqslant 3$, then P has a component Q such that $\dim_Q(I \cap \mathrm{Inc}(Q)) = \dim_P(I)$. In particular, if $\dim(P) \geqslant 3$, then P has a component Q such that $\dim(Q) = \dim(P)$.*

Proof. Let $Q_1, \ldots, Q_k$ be the components of P, and for each $i \in \{1, \ldots, k\}$, let $d_i = \dim_{Q_i}(I \cap \mathrm{Inc}(Q_i))$. Since $I \cap \mathrm{Inc}(Q_i) \subseteq I$, we have $d_i \leqslant \dim_P(I)$. Let $d = \max\{d_1, \ldots, d_k, 2\}$. To complete the proof it suffices to show that $\dim_P(I) \leqslant d$: In such case we have $3 \leqslant \dim_P(I) = d$, so there exists $i \in \{1, \ldots, k\}$ such that $d = d_i = \dim_{Q_i}(I \cap \mathrm{Inc}(Q_i))$.

For each $i \in \{1, \ldots, k\}$, let V_i be the ground set of Q_i, and let $\{I_i^1, \ldots, I_i^d\}$ be a partition of $I \cap \mathrm{Inc}(Q_i)$ into reversible sets. For each $j \in \{1, \ldots, d\}$, let $I^j = I_1^j \cup \cdots \cup I_k^j$. Define

$$I_< = I \cap \bigcup_{i_1 < i_2} \mathrm{Inc}_P(V_{i_1}, V_{i_2}) \quad \text{and} \quad I_> = I \cap \bigcup_{i_1 > i_2} \mathrm{Inc}_P(V_{i_1}, V_{i_2}).$$

We claim that for each $j \in \{1, \ldots, d\}$, the set $I^j \cup I_<$ is reversible in P. Suppose to the contrary that there is an alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ in $I^j \cup I_<$. For each $i \in \{1, \ldots, k\}$, if $a_i \in V_{i_1}$ and $b_i \in V_{i_2}$, then $i_1 \leqslant i_2$. However, a_i and b_{i+1} are comparable and thus lie in one component. Therefore all a_i and b_i belong to the same component, so all pairs of the cycle are in one I_i^j, contradicting its reversibility. Hence $I^j \cup I_<$ (and in particular I^j) is reversible. A symmetric argument shows that $I^j \cup I_>$ is reversible. Hence the inequality $\dim_P(I) \leqslant d$ is witnessed by the partition

$$\{I^1 \cup I_<, I^2 \cup I_>, I^3, \ldots I^d\}.$$

□

When bounding the dimension of a poset, we may restrict our attention not only to a component, but actually to a block. A *block* of a poset is a subposet induced by the vertex set of a block of the cover graph. A block of a poset P is a convex subposet, so its cover graph is an induced subgraph which either has at most 2 elements, or is 2-connected.

Lemma 1.6 (Trotter, Walczak, Wang [31]). *If every block of a poset P has dimension at most d, then the dimension of P is at most $d + 2$.*

Chapter 2

Tree-width at most 2

Already in 1977, Trotter and Moore [33] showed that posets whose cover graphs are forests have dimension at most 3. Forests are exactly graphs of treewidth at most 1, and it is natural to ask whether posets with cover graphs of bounded treewidth have bounded dimension. The answer to this question is negative: Kelly [19] constructed posets of arbitrarily large dimension with cover graphs of treewidth (and pathwidth) at most 3.

Do posets with cover graphs of treewidth 2 have bounded dimension? There are several special cases for which an affirmative answer has a simple proof. Felsner, Trotter and Wiechert [9] showed that posets with outerplanar cover graphs have dimension at most 4. Biró, Keller and Young [1] showed that posets with cover graphs of pathwidth 2 have dimension at most 17. Wiechert [35] generalized this result by showing that the dimension of a poset is at most 6 if its cover graph can be obtained from an outerplanar graph by subdividing each edge at most once.

The general case was eventually settled by Joret, Micek, Trotter, Wang and Wiechert [17], who showed that posets with cover graphs of treewidth 2 have dimension at most 1276. The proof introduces many techniques which prove themselves useful in subsequent work, but as the authors admit, it is "lengthy and technical", and they "believe there is still room for improvements". In this chapter we present a simple proof with a significantly better bound.

Theorem 2.1 (Seweryn [29])**.** *Every poset with a cover graph of treewidth at most 2 has dimension at most 12.*

The key idea of our proof is to facilitate the characterization of graphs of treewidth at most 2 as subgraphs of series-parallel graphs. Working with a series-parallel supergraph of the cover graph introduces more structure than just an arbitrary tree-decomposition of width 2.

Felsner, Trotter and Wiechert [9] showed that there exists a poset with an outerplanar cover graph and dimension 4, so the largest dimension of a poset with a cover graph of treewidth 2 is at least 4 and at most 12. We do not know the exact value, but it seems that it should be greater than 4. However, proving that is not an easy task, because, in general, lower bounds for dimension are more difficult to prove than upper bounds.

2.1 Series-parallel graphs

A *two-terminal graph* (*TTG*) is a triple (G, s, t) where G is a graph, and s and t are distinct vertices of G called *source* and *sink*, respectively. If G consists only of the vertices s and t and an edge between them, we call (G, s, t) a *single edge*.

Let $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ be two TTGs with $\boldsymbol{G}_i = (G_i, s_i, t_i)$ for $i \in \{1, 2\}$. If $t_1 = s_2$ and $V(G_1) \cap V(G_2) = \{t_1\}$, we define the *series composition* of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ as the TTG

$$\boldsymbol{S}(\boldsymbol{G}_1, \boldsymbol{G}_2) = (G_1 \cup G_2, s_1, t_2).$$

If $t_1 = t_2$, $s_1 = s_2$, $V(G_1) \cap V(G_2) = \{s_1, t_1\}$ and $E(G_1) \cap E(G_2) = \varnothing$, we define the *parallel composition* of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ as the TTG

$$\boldsymbol{P}(\boldsymbol{G}_1, \boldsymbol{G}_2) = (G_1 \cup G_2, s_1, t_1).$$

(Note that $\boldsymbol{P}(\boldsymbol{G}_1, \boldsymbol{G}_2) = \boldsymbol{P}(\boldsymbol{G}_2, \boldsymbol{G}_1)$.)

A TTG is *series-parallel* if it can be produced by a sequence of series and parallel compositions from single edges, and a graph G is *series-parallel* if it contains vertices s and t such that the TTG (G, s, t) is series-parallel.

A recursive construction of a series-parallel TTG $\boldsymbol{G}$ can be represented by a binary tree. Here, a *binary tree* is a rooted tree in which every inner node u has exactly two children: a *left child* $\ell(u)$ and a *right child* $r(u)$. A *series-parallel decomposition* of a TTG $\boldsymbol{G}$ is a pair $(T, \{\boldsymbol{G}_u\}_{u \in V(T)})$ where T is a binary tree with a root u_0 and $\{\boldsymbol{G}_u\}_{u \in V(T)}$ is a family of TTGs such that $\boldsymbol{G}_{u_0} = \boldsymbol{G}$ and for each $u \in V(T)$, one of the following holds:

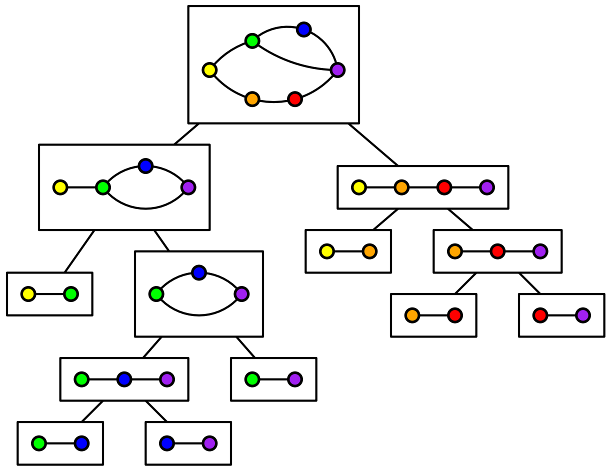


Figure 2.1: A series-parallel decomposition of a TTG. In each of the graphs, the leftmost vertex is its source, and the rightmost vertex is its sink.

(1) u is an inner node such that $\boldsymbol{G}_u = \boldsymbol{S}(\boldsymbol{G}_{\ell(u)}, \boldsymbol{G}_{r(u)})$,

(2) u is an inner node such that $\boldsymbol{G}_u = \boldsymbol{P}(\boldsymbol{G}_{\ell(u)}, \boldsymbol{G}_{r(u)})$, or

(3) u is a leaf and $\boldsymbol{G}_u$ is a single edge.

(See Figure 2.1.)

An inner node u is called an $\mathcal{S}$*-node* if it satisfies (1), or a $\mathcal{P}$*-node* if it satisfies (2). For every $\mathcal{S}$-node u, the vertices $t_{\ell(u)}$ and $s_{r(u)}$ are the same vertex which we denote by m_u when the decomposition is clear from the context. Clearly, a TTG admits a series-parallel decomposition if and only if it is series-parallel.

Swapping the source and the sink in the series-parallel TTG (G, s, t) yields the TTG (G, t, s) which is also series-parallel, and its decomposition can be obtained by *reversing* the decomposition of (G, s, t). The *reversed* series-parallel decomposition is $(T', \{(G_u, t_u, s_u)\}_{u \in V(T')})$, where T' is obtained from T by swapping left and right children of every inner node.

Observe that if u' is a child of a node u in a series-parallel decomposition $(T, \{(G_u, s_u, t_u)\}_{u \in V(T)})$, then $V(G_{u'}) \subseteq V(G_u)$ and $V(G_{u'}) \setminus \{s_{u'}, t_{u'}\} \subseteq V(G_u) \setminus \{s_u, t_u\}$. By a simple induction, it suffices that u' is a descendant of u for these inclusions to hold.

It is well-known that a graph has treewidth at most 2 if and only if it is a subgraph of a series-parallel graph. This is a consequence of the following two lemmas.

Lemma 2.2. *Every* 2*-tree is a series-parallel graph.*

Proof. A complete graph on 3 vertices is series-parallel, so it suffices to show that if (G, s, t) is a series-parallel TTG and G' is obtained from G by adding a vertex x adjacent to the ends of an edge of G, then the TTG (G', s, t) is still series-parallel. We prove this by induction on $|V(G)|$. In the base case, (G, s, t) is a single edge. The graph G' is the complete graph on $\{s, t, x\}$ and (G', s, t) is indeed series-parallel.

For the inductive step, assume that $|V(G)| \geqslant 3$,. The TTG (G, s, t) is a series or parallel composition of two series-parallel TTGs (G_1, s_1, t_1) and (G_2, s_2, t_2). The neighbors of x in G' are adjacent in G and the edge between them lies in G_i for some $i \in \{1, 2\}$. Fix that i, let $G'_i = G'[V(G_i) \cup \{x\}]$ and let $G'_{3-i} = G_{3-i}$. By the induction hypothesis, the TTG (G'_i, s_i, t_i) is series-parallel. Since (G', s, t) can be obtained as a series or parallel composition of (G'_1, s_1, t_1) and (G'_2, s_2, t_2), we conclude that (G', s, t) is series-parallel. This completes the inductive proof.

□

Lemma 2.3. *Let* $(T, \{(G_u, s_u, t_u)\}_{u \in V(T)})$ *be a series-parallel decomposition of a TTG* (G, s, t)*, and for each* $u \in V(T)$ *let*

$$V_u = \begin{cases} \{s_u, m_u, t_u\} & \text{if } u \text{ is an } \mathcal{S}\text{-node,} \\ \{s_u, t_u\} & \text{if } u \text{ is a } \mathcal{P}\text{-node or a leaf.} \end{cases}$$

Then $(T, \{V_u\}_{u \in V(T)})$ *is a tree-decomposition of* G*.*

Proof. For every edge $xy \in E(G)$ there exists a leaf u of T such that $s_u t_u = xy$, and thus $\{x, y\} \subseteq V_u$, so the condition (T2) of a tree-decomposition holds. Furthermore, since (G, s, t) is series-parallel, every vertex of G is an end of an edge, so

$$V(G) = \bigcup_{xy \in E(G)} \{x, y\} \subseteq \bigcup_{u \in V(T)} V_u \subseteq V(G),$$

which means that $\bigcup_{u \in V(T)} V_u = V(G)$, so the condition (T1) of a tree-decomposition holds.

For the proof of the condition (T3), fix nodes u_1, u_2 and u of T such that $u \in V(u_1 T u_2)$. Suppose first that u_1 is an ancestor of u_2. The inclusion

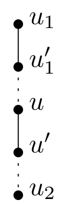


Figure 2.2: The order of nodes on the path $u_1 T u_2$. A solid line represents an edge and a dashed line represents a path of length 0 or more.

$V_{u_1} \cap V_{u_2} \subseteq V_u$ holds true if $u \in \{u_1, u_2\}$, so let us assume that $u_1 \neq u \neq u_2$. Let u'_1 denote the child of u_1 which is an ancestor of u, and let u' denote the child of u which is an ancestor of u_2 (see Figure 2.2). We have

$$\begin{aligned} V_{u_2} \subseteq V(G_{u_2}) &\subseteq V(G_{u'}) \\ &\subseteq (V(G_{u'}) \backslash \{s_{u'}, t_{u'}\}) \cup V_u \subseteq (V(G_{u'_1}) \backslash \{s_{u'_1}, t_{u'_1}\}) \cup V_u. \end{aligned}$$

Since $V_{u_1} \cap (V(G_{u'_1}) \backslash \{s_{u'_1}, t_{u'_1}\}) = \varnothing$, we conclude that indeed $V_{u_1} \cap V_{u_2} \subseteq V_u$. The case when u_2 is an ancestor of u_1 follows from symmetric arguments.

It remains to consider the case when neither of u_1 and u_2 is an ancestor of the other. Let v be the lowest common ancestor of u_1 and u_2. One of u_1 and u_2 is a descendant of $\ell(v)$ and one is a descendant of $r(v)$, so

$$V_{u_1} \cap V_{u_2} \subseteq V(G_{u_1}) \cap V(G_{u_2}) \subseteq V(G_{\ell(v)}) \cap V(G_{r(v)}) \subseteq V_v.$$

The node u lies on the path vTu_i for some $i \in \{1, 2\}$. As v is an ancestor of u_i, we already know that $V_v \cap V_{u_i} \subseteq V_u$, and thus $V_{u_1} \cap V_{u_2} \subseteq V_v \cap V_{u_i} \subseteq V_u$. $\square$

In order to prove some properties of series-parallel decompositions, let us fix a series-parallel TTG (G, s, t) with a series-parallel decomposition $(T, \{(G_u, s_u, t_u)\}_{u \in V(T)})$, and let $(T, \{V_u\}_{u \in V(T)})$ be the tree-decomposition of G as in the statement of Lemma 2.3.

Lemma 2.4. *For each $x \in V(G) \backslash \{s, t\}$, there exists a unique $\mathcal{S}$-node v such that $m_v = x$.*

Proof. If u_1 and u_2 are two nodes of T such that u_1 is an ancestor of u_2 and $x \in V(G_{u_2}) \backslash \{s_{u_2}, t_{u_2}\}$, then $x \in V(G_{u_1}) \backslash \{s_{u_1}, t_{u_1}\}$. Moreover, every inner node $u \in V(T)$ with $x \in V(G_u) \backslash \{s_u, t_u\}$ has at most one child u' such that $x \in V(G_{u'}) \backslash \{s_{u'}, t_{u'}\}$. Since $x \in V(G) \backslash \{s, t\}$, this implies that there exists a unique node v such that for every $u \in V(T)$ we have $x \in V(G_u) \backslash \{s_u, t_u\}$ if and only if u is an ancestor of v. In particular, v is the only node such that $x \in V(G_v) \backslash \{s_v, t_v\}$ and for each child v' of v we have $x \notin V(G_{v'}) \backslash \{s_{v'}, t_{v'}\}$. Hence v is the only $\mathcal{S}$-node such that $m_v = x$. □

Lemma 2.5. *Let u_1 and u_2 be nodes of T, and let W be a path with ends in V_{u_1} and V_{u_2}. Then for every node u which is an ancestor of exactly one of the nodes u_1 and u_2, the path W contains s_u or t_u.*

Proof. Without loss of generality, assume that u is an ancestor of u_1 but not of u_2. Hence the parent v of u satisfies $uv \in E(u_1 T u_2)$ and $V_u \cap V_v = \{s_u, t_u\}$, so by Lemma 1.1, W contains s_u or t_u. □

Lemma 2.6. *Let u_1 and u_2 be two nodes of T such that one of them is an ancestor of the other and let W be an s_{u_1}–t_{u_2} path in G. Then there exists $v \in V(u_1 T u_2)$ such that $\{s_v, t_v\} \subseteq V(W)$.*

Proof. Every inner node u on the path $u_1 T u_2$ is an ancestor of exactly one of the nodes u_1 and u_2, so by Lemma 2.5, the path W contains s_u or t_u. Since W contains s_{u_1} and t_{u_2}, there must exist an edge $v_1 v_2 \in E(u_1 T u_2)$ such that W contains s_{v_1} and t_{v_2}. We have $s_{v_1} = s_{v_2}$ or $t_{v_1} = t_{v_2}$, so for some $v \in \{v_1, v_2\}$ the path W contains s_v and t_v, as claimed.

□

2.2 The proof

Let P be a poset whose cover graph has treewidth at most 2. By Lemma 2.2, there exists a series-parallel TTG (G, s, t) such that the cover graph of P is a subgraph of G. Let us fix such (G, s, t). After replacing (G, s, t) with its series composition with two single edges, we assume that neither s nor t is an element of P. Let $(T, \{(G_u, s_u, t_u)\}_{u \in V(T)})$ be a series-parallel decomposition of (G, s, t), and let $(T, \{V_u\}_{u \in V(T)})$ be the corresponding tree-decomposition as in Lemma 2.3. Recall that for every $\mathcal{S}$-node u, the vertices $t_{\ell(u)}$ and $s_{r(u)}$ are the same vertex, which we denote by m_u. As s and t are not elements of P, by Lemma 2.4, for each element $x \in P$ there exists a unique $\mathcal{S}$-node $v \in V(T)$ such that $m_v = x$, and we denote that node by $v(x)$.

Let L_{in} denote the linear order in which the nodes of T are visited in the in-order traversal of T. In other words, for two nodes u_1 and u_2 with a lowest common ancestor v we have $u_1 \leqslant u_2$ in L_{in} if and only if u_1 is v or a descendant of $\ell(v)$ and u_2 is v or a descendant of $r(v)$. Let us partition $\mathrm{Inc}(P)$ into two sets $I_<$ and $I_>$ defined as

$$I_< = \{(a,b) \in \mathrm{Inc}(P) : v(a) < v(b) \text{ in } L_{\mathrm{in}}\}, \text{ and}$$
$$I_> = \{(a,b) \in \mathrm{Inc}(P) : v(a) > v(b) \text{ in } L_{\mathrm{in}}\}.$$

Swapping the source s with the sink t and reversing the series-parallel decomposition swaps the sets $I_<$ and $I_>$. Hence, without loss of generality we assume that $\dim(I_>) \leqslant \dim(I_<)$, so that

$$\dim(P) = \dim_P(\mathrm{Inc}(P)) \leqslant \dim(I_<) + \dim(I_>) \leqslant 2 \cdot \dim(I_<).$$

Therefore, to complete the proof of the theorem it remains to show that $I_<$ can be partitioned into six reversible sets.

For every $(a,b) \in I_<$, let $v(a,b)$ denote the lowest common ancestor of $v(a)$ and $v(b)$ in T. Note that for every $(a,b) \in I_<$, the node $v(a,b)$ is inner and we have $v(a) \leqslant v(a,b) \leqslant v(b)$ in L_{in}. Let

$$I_0^1 = \{(a,b) \in I_< : a \nleqslant s_{\ell(v(a,b))} \text{ and } a \nleqslant t_{\ell(v(a,b))} \text{ in } P\}.$$

Claim 2.7. *The set I_0^1 is reversible.*

Proof. Towards a contradiction, suppose that the set I_0^1 is not reversible. Let $((a_1,b_1),\ldots,(a_k,b_k))$ be an alternating cycle in I_0^1. Let $i \in \{1,\ldots,k\}$, and let $v = v(a_i,b_i)$. Since $(a_i,b_i) \in I_0^1$, we have $a_i \nleqslant t_{\ell(v)}$, so in particular $v(a_i) \neq v$. As $(a_i,b_i) \in I_<$, this means that $v(a_i)$ is a descendant of $\ell(v)$. Let W be a witnessing path from a_i to b_{i+1}. We have $a_i \nleqslant s_{\ell(v)}$ and $a_i \nleqslant t_{\ell(v)}$ in P because $(a_i,b_i) \in I_0^1$, so the path W does not contain $s_{\ell(v)}$ nor $t_{\ell(v)}$. Hence, Lemma 2.5 implies that $v(b_{i+1})$ is a descendant of $\ell(v)$. In particular, $v(b_{i+1}) < v$ in L_{in}. This means that $v(b_{i+1}) < v = v(a_i,b_i) \leqslant v(b_i)$ in L_{in}. Since i was chosen arbitrarily, this holds for each $i \in \{1,\ldots,k\}$, so we have $v(b_k) < \cdots < v(b_1) < v(b_k)$ in L_{in}, a contradiction. □

A symmetric argument shows that the set I_0^2 defined as

$$I_0^2 = \{(a,b) \in I_< : s_{r(v(a,b))} \nleqslant b \text{ and } t_{r(v(a,b))} \nleqslant b \text{ in } P\}$$

is reversible as well. Therefore

$$\dim(I_0^1 \cup I_0^2) \leqslant 2.$$

Let $I_1 = I_< \backslash (I_0^1 \cup I_0^2)$. It remains to show that I_1 can be partitioned into four reversible sets.

Let us partition the pairs (a, b) from I_1 into two sets $I_{\mathcal{S}}$ and $I_{\mathcal{P}}$ depending on the type of the node $v(a, b)$:

$$\begin{aligned} I_{\mathcal{S}} &= \{(a, b) \in I_1 : v(a, b) \text{ is an } \mathcal{S}\text{-node}\}, \\ I_{\mathcal{P}} &= \{(a, b) \in I_1 : v(a, b) \text{ is a } \mathcal{P}\text{-node}\}. \end{aligned}$$

Let $(a, b) \in I_{\mathcal{P}}$. For each $v' \in \{\ell(v(a, b)), r(v(a, b))\}$ we have $s_{v'} = s_{v(a,b)}$ and $t_{v'} = t_{v(a,b)}$. Since $(a, b) \notin I_0^1$, we have $a \leqslant s_{v(a,b)}$ or $a \leqslant t_{v(a,b)}$ in P, and since $(a, b) \notin I_0^2$, we have $b \geqslant s_{v(a,b)}$ or $b \geqslant t_{v(a,b)}$ in P. As $a \parallel b$ in P, there does not exist $c \in \{s_{v(a,b)}, t_{v(a,b)}\}$ such that $a \leqslant c$ and $c \leqslant b$ in P. Hence we can partition $I_{\mathcal{P}}$ into two sets $I_{\mathcal{P}}^1$ and $I_{\mathcal{P}}^2$ defined as follows:

$$\begin{aligned} I_{\mathcal{P}}^1 &= \{(a, b) \in I_{\mathcal{P}} : a \leqslant s_{v(a,b)} \nleqslant b \text{ and } a \nleqslant t_{v(a,b)} \leqslant b \text{ in } P\}, \\ I_{\mathcal{P}}^2 &= \{(a, b) \in I_{\mathcal{P}} : a \leqslant t_{v(a,b)} \nleqslant b \text{ and } a \nleqslant s_{v(a,b)} \leqslant b \text{ in } P\}. \end{aligned}$$

Let $J = I_{\mathcal{S}} \cup I_{\mathcal{P}}^1$. We aim to show that $\dim(J) \leqslant 2$. The key property shared by the pairs from the sets $I_{\mathcal{S}}$ and $I_{\mathcal{P}}^1$ is captured by the following claim.

Claim 2.8. *Let $(a, b) \in J$ and let $x \in P$ be such that $v(x)$ is not a descendant of $v(a, b)$.*

(1) *If $a \leqslant x$ in P, then $a \leqslant s_{v(a,b)} \leqslant x$ in P.*

(2) *If $b \geqslant x$ in P, then $b \geqslant t_{v(a,b)} \geqslant x$ in P.*

Proof. We only show (1), as the proof for (2) is dual. Let $v = v(a, b)$. If $(a, b) \in I_{\mathcal{S}}$, then $s_{r(v)} = t_{\ell(v)}$ and $t_{r(v)} = t_v$, so the fact that $(a, b) \notin I_0^2$ means that $t_{\ell(v)} \leqslant b$ or $t_v \leqslant b$ in P. On the other hand, if $(a, b) \in I_{\mathcal{P}}^1$, then $t_{\ell(v)} = t_v \leqslant b$ in P. Hence, in both cases we conclude that $t_{\ell(v)} \leqslant b$ or $t_v \leqslant b$ in P. As $a \parallel b$ in P, this implies

$$a \nleqslant t_{\ell(v)} \text{ or } a \nleqslant t_v \text{ in } P.$$

Let W be a witnessing path from a to x. If $v(a) = v$, then v is an $\mathcal{S}$-node and $a = t_{\ell(v)} \in V_{\ell(v)}$, and if $v(a) \neq v$, then $v(a)$ is a descendant of $\ell(v)$. Hence, there always exists a descendant v_1 of $\ell(v)$ (and of v) such that $a \in V_{v_1}$. The node $v(x)$ is not a descendant of v (or $\ell(v)$), so by Lemma 2.5 applied to W we have

$$V(W) \cap \{s_{\ell(v)}, t_{\ell(v)}\} \neq \emptyset \quad \text{and} \quad V(W) \cap \{s_v, t_v\} \neq \emptyset.$$

As $a \nleqslant t_{\ell(v)}$ or $a \nleqslant t_v$ in P, this implies that $s_{\ell(v)} \in V(W)$ or $s_v \in V(W)$. But $s_{\ell(v)} = s_v$, so $V(W)$ must contain s_v, and therefore we have $a \leqslant s_v \leqslant x$ in P, as claimed. □

Let us partition J into two sets J_1 and J_2 defined as

$$J_1 = \{(a, b) \in J : a \leqslant s_u \text{ and } a \leqslant t_u \text{ in } P \text{ for some ancestor } u \text{ of } v(a, b)\},$$
$$J_2 = J \backslash J_1.$$

We prove that J_1 and J_2 are reversible with a sequence of claims.

Claim 2.9. *Let $(a, b) \in \mathrm{Inc}(P)$ and suppose that there exists an ancestor v of $v(a, b)$ such that $s_v \leqslant b$ and $t_v \leqslant b$ in P. Then there does not exist an ancestor u of $v(a, b)$ such that $a \leqslant s_u$ and $a \leqslant t_u$ in P. In particular, $(a, b) \notin J_1$.*

Proof. Towards a contradiction, suppose that there exists such an ancestor u of $v(a, b)$. If u is a descendant of v, then by Lemma 2.5 applied to a witnessing path from s_v to b, we have $s_u \leqslant b$ or $t_u \leqslant b$, and thus $a \leqslant b$ in P, contradicting $a \parallel b$ in P. Similarly, if u is an ancestor of v, then by Lemma 2.5 applied to a witnessing path from a to s_u, we have $a \leqslant s_v$ or $a \leqslant t_v$ in P, and thus $a \leqslant b$ in P, again contradicting $a \parallel b$ in P. □

Claim 2.10. *Let $v \in V(T)$ and let $((a_1, b_1), \ldots, (a_k, b_k))$ be an alternating cycle in J such that $v(a_i, b_i) = v$ for each $i \in \{1, \ldots, k\}$. Then the cycle contains a pair from J_1 and a pair from J_2.*

Proof. Let $W_{k,1}$ be a witnessing path from a_k to b_1, and let $W_{1,2}$ be a witnessing path from a_1 to b_2. Since $a_1 \parallel b_1$ in P, the witnessing paths $W_{k,1}$ and $W_{1,2}$ are disjoint. For each $i \in \{1, \ldots, k\}$, we have $(a_i, b_i) \in I_<$, and thus there exists a descendant u_1 of $\ell(v)$ such that $a_i \in V_{u_1}$ and a descendant u_2 of $r(v)$ such that $b_i \in V_{u_2}$. Hence, by Lemma 2.5, each of the witnessing paths $W_{k,1}$ and $W_{1,2}$ has nonempty intersections with $\{s_{\ell(v)}, t_{\ell(v)}\}$ and $\{s_{r(v)}, t_{r(v)}\}$.

Towards a contradiction, suppose that v is a $\mathcal{P}$-node. We have $(a_i, b_i) \in I^1_{\mathcal{P}}$ for each $i \in \{1, \ldots, k\}$, so $a_1 \nleqslant t_v = t_{\ell(v)}$ and $s_{\ell(v)} = s_v \nleqslant b_2$ in P. This contradicts $W_{1,2}$ having a nonempty intersection with $\{s_{\ell(v)}, t_{\ell(v)}\}$. It follows that v must be an $\mathcal{S}$-node.

We have $s_{\ell(v)} = s_v$, $t_{\ell(v)} = s_{r(v)} = m_v$, $t_{r(v)} = t_v$, and each of the witnessing paths $W_{k,1}$ and $W_{1,2}$ has nonempty intersections with the sets $\{s_v, m_v\}$ and $\{m_v, t_v\}$. Since the paths $W_{k,1}$ and $W_{1,2}$ are disjoint, one of them does not contain m_v and thus contains s_v and t_v. If $W_{k,1}$ contains s_v and t_v, then $a_k \leqslant s_v \leqslant b_1$ and $a_k \leqslant t_v \leqslant b_1$ in P, so $(a_k, b_k) \in J_1$, and, by Claim 2.9, $(a_1, b_1) \in J_2$. Similarly, if $W_{1,2}$ contains s_v and t_v, then $(a_1, b_1) \in J_1$ and $(a_2, b_2) \in J_2$. □

Claim 2.11. *Let $((a_1, b_1), \ldots, (a_k, b_k))$ be a strict alternating cycle in I_1, let $j \in \{1, \ldots, k\}$, and let u be a node such that $a_j \leqslant s_u \leqslant b_{j+1}$ and $a_j \leqslant t_u \leqslant b_{j+1}$ in P. If at least one of the nodes $v(a_j, b_j)$ and $v(a_{j+1}, b_{j+1})$ is a descendant of u, then for each $i \in \{1, \ldots, k\}$, $v(a_i, b_i)$ is a descendant of u.*

Proof. We only prove the case when $v(a_{j+1}, b_{j+1})$ is a descendant of u as the proof for the case when $v(a_j, b_j)$ is a descendant of u is symmetric.

Without loss of generality, we assume that $j = k$, that is u is an ancestor of $v(a_1, b_1)$ such that

$$a_k \leqslant s_u \leqslant b_1 \quad \text{and} \quad a_k \leqslant t_u \leqslant b_1 \quad \text{in } P.$$

We prove the claim by induction on i. The base case $i = 1$ holds true. For the inductive step, let $i \in \{2, \ldots, k\}$ and suppose that $v(a_{i-1}, b_{i-1})$ is a descendant of u. Since the alternating cycle is strict, $a_{i-1} \nleqslant b_1$ in P, and therefore any witnessing path from a_{i-1} to b_i is disjoint from $\{s_u, t_u\}$. Hence, by Lemma 2.5, $v(b_i)$ is a descendant of u. Since $(a_i, b_i) \notin I_0^2$, there exists a witnessing path from an element of $V_{v(a_i,b_i)}$ to b_i. Since the alternating cycle is strict, we have $a_k \nleqslant b_i$ in P, and therefore a witnessing path from an element of $V_{v(a_i,b_i)}$ to b_i is disjoint from $\{s_u, t_u\}$. Hence, by Lemma 2.5, $v(a_i, b_i)$ is a descendant of u. The inductive proof is complete. □

Claim 2.12. *The sets J_1 and J_2 are reversible.*

Proof. We prove the claim by showing that every strict alternating cycle in J contains a pair from J_1 and a pair from J_2. Fix a strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ in J. For each $i \in \{1, \ldots, k\}$, let $v_i = v(a_i, b_i)$. If all nodes v_i are equal, the claim follows from Claim 2.10. Let us hence assume that not all v_i are equal. There must exist $i \in \{1, \ldots, k\}$ such that $v_i < v_{i+1}$ in L_{in}. Without loss of generality, we assume that this holds for $i = 1$, that is $v_1 < v_2$ in L_{in}. Let v denote the lowest common ancestor of v_1 and v_2. To complete the proof, it suffices to show that

$$a_1 \leqslant s_v \leqslant b_2 \quad \text{and} \quad a_1 \leqslant t_v \leqslant b_2 \quad \text{in } P.$$

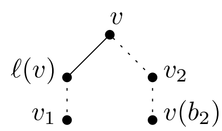


Figure 2.3: The order of nodes on the path $v_1 T v(b_2)$. A solid line represents an edge and a dashed line represents a path of length 0 or more.

as in such case we have $(a_1, b_1) \in J_1$, and, by Claim 2.9, $(a_2, b_2) \in J_2$. Since $v_1 \neq v_2$, we have $v_1 \neq v$ or $v_2 \neq v$. The reasoning for both cases is symmetric, and therefore we assume without loss of generality that $v_1 \neq v$. Since v_1 is a descendant of v such that $v_1 \leqslant v$ in L_{in}, this means that v_1 is a descendant of $\ell(v)$, see Figure 2.3.

Since $v \leqslant v_2 \leqslant v(b_2)$ in L_{in}, the node $v(b_2)$ is not a descendant of $\ell(v)$. Hence, by Lemma 2.5 applied to a witnessing path from a_1 to b_2, there exists $c \in \{s_{\ell(v)}, t_{\ell(v)}\}$ such that $a_1 \leqslant c \leqslant b_2$ in P. We claim that $c = s_{\ell(v)}$. Suppose to the contrary that $c = t_{\ell(v)}$. By Claim 2.8, we have $a_1 \leqslant s_{v_1} \leqslant t_{\ell(v)}$ in P, and by Lemma 2.6 applied to a witnessing path from s_{v_1} to $t_{\ell(v)}$, there exists $u \in V(\ell(v) T v_1)$ such that $a_1 \leqslant s_u \leqslant b_2$ and $a_1 \leqslant t_u \leqslant b_2$ in P. Hence, by Claim 2.11, u is an ancestor of v_2, which contradicts v being the lowest common ancestor of v_1 and v_2. Hence, $c = s_{\ell(v)} = s_v$ and $a_1 \leqslant s_v \leqslant b_2$ in P.

By Claim 2.8, we have $s_v \leqslant t_{v_2} \leqslant b_2$ in P, and by Lemma 2.6 applied to a witnessing path from s_v to t_{v_2}, there exists $u \in V(v T v_2)$ such that $a_1 \leqslant s_u \leqslant b_2$ and $a_1 \leqslant t_u \leqslant b_2$ in P. By Claim 2.11, the node v_1 is a descendant of u. This is possible only if $u = v$, and therefore we have $a_1 \leqslant s_v \leqslant b_2$ and $a_1 \leqslant t_v \leqslant b_2$ in P as desired. $\square$

Claim 2.12 shows that $\dim(I_{\mathcal{S}} \cup I^1_{\mathcal{P}}) \leqslant 2$. If in the above reasoning we ignore the pairs from $I_{\mathcal{S}}$, we obtain a proof that $\dim(I^1_{\mathcal{P}}) \leqslant 2$. Since the sets $I^1_{\mathcal{P}}$ and $I^2_{\mathcal{P}}$ are defined symmetrically, we can use symmetric arguments to show that

$$\dim(I^2_{\mathcal{P}}) \leqslant 2.$$

We now are equipped with all parts needed to complete the proof:

$$\begin{aligned}
\dim(P) &= \dim(\mathrm{Inc}(P)) \\
&\leqslant \dim(I_<) + \dim(I_>) \\
&\leqslant 2 \cdot \dim(I_<) \\
&\leqslant 2 \cdot (\dim(I_0^1) + \dim(I_0^2) + \dim(I_1)) \\
&\leqslant 2 \cdot (2 + \dim(I_1)) \\
&\leqslant 2 \cdot (2 + \dim(I_{\mathcal{S}} \cup I_{\mathcal{P}}^1) + \dim(I_{\mathcal{P}}^2)) \\
&\leqslant 2 \cdot (2 + 2 + 2) = 12.
\end{aligned}$$

Chapter 3

Excluding a $K_{2,n}$-minor

In this chapter, we study the case when the cover graph of a poset excludes the complete bipartite graph $K_{2,n}$ as a minor for some fixed n. We prove that in such case the dimension is bounded.

Theorem 3.1 (Seweryn)**.** *For every $n \geqslant 1$ there exists $d \geqslant 1$ such that every poset with a cover graph excluding a $K_{2,n}$-minor has dimension at most d.*

The proof of Theorem 3.1 relies on a characterization of graphs without large $K_{2,n}$-minors by Ding [5]. Ding's result does not give a precise characterization of $K_{2,n}$-minor-free graphs for every n. Instead, it describes the approximate structure of graphs excluding a $K_{2,n}$-minor similarly as the Grid-Minor Theorem describes the structure of graphs without large $n \times n$ grid-minors. Namely, Ding constructed an infinite sequence $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots$ of graph classes with the property that any class of graphs $\mathcal{H}$ excludes a $K_{2,n}$-minor for some n if and only if there exists a nonnegative integer m such that $\mathcal{H} \subseteq \mathcal{G}_m$.

3.1 Graphs without large $K_{2,n}$-minors

Ding's characterization of graphs without large of $K_{2,n}$ is a bit complicated, but, roughly speaking, it states that every 2-connected graph without a large $K_{2,n}$-minor can be obtained from parts of a simple structure in a small number of iterations, where in each iteration we attach any number of parts to already constructed graph. There is a minor technical detail in this theorem which makes it difficult to apply the original formulation. Without going into too much detail, the problem arises when in some iteration we attach one new part to parts which were constructed in different iterations. However, a closer inspection of the proof reveals, that the described problematic case never occurs. Therefore, we state a more low-level variant of the characterization of graphs without large $K_{2,n}$-minors which is implicit in Ding's manuscript.

Let G be a graph with a specified Hamiltonian cycle C. The edges in $E(G)\backslash E(C)$ are called *chords*. For two chords ac and bd without a common end, we say that ac *crosses* bd if $\{b, d\}$ separates $\{a\}$ from $\{c\}$ in C. Clearly, ac crosses bd if and only if bd crosses ac. Using this terminology, the 2-connected outerplanar graphs can be characterized as those graphs for which one can choose the Hamiltionian cycle C so that no two chords cross. At the base of the characterization of $K_{2,n}$-minor-free graphs is a graph class $\mathcal{P}$ which generalizes 2-connected outerplanar graphs so that some pairs of edges may cross, but only in a very specific case. The class $\mathcal{P}$ consists of all graphs G which admit a Hamiltonian cycle C such that each chord crosses at most one other chord and for every pair of crossing chords ac and bd we have $\{ab, cd\} \subseteq E(C)$ or $\{bc, da\} \subseteq E(C)$. A Hamiltonian cycle C with these properties is called a *reference cycle* for G. (See Figure 3.1.)

A *labeled graph* is a pair (G, L) consisting of a graph G and a set L of pairwise nonadjacent vertices of degree 2 in G. If (G_1, L_1) and (G_2, L_2) are two labeled graphs and there exist vertices x, y and z such that $V(G_1) \cap V(G_2) =$

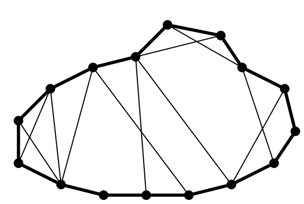


Figure 3.1: A graph from $\mathcal{P}$.

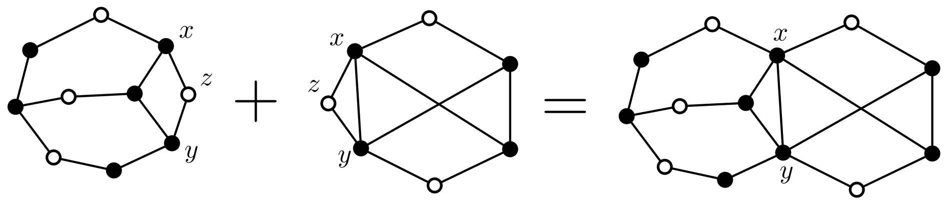


Figure 3.2: Two labeled graphs (G_1, L_1) and (G_2, L_2) and their 2-sum (G, L). The white vertices are the elements of the sets L_1, L_2 and L.

$\{x, y, z\}$, $z \in L_1 \cap L_2$ and $N_{G_1}(z) = N_{G_2}(z) = \{x, y\}$, then (G_1, L_1) and (G_2, L_2) are 2-*summable*, and we define the 2-*sum* of (G_1, L_1) and (G_2, L_2) as the labeled graph (G, L) where $G = (G_1 \cup G_2) - z$ and $L = (L_1 \cup L_2) \backslash \{z\}$. See Figure 3.2.

For a graph G we recursively define a *decomposition into labeled graphs* as either the singleton $\{(G, \varnothing)\}$, or a family obtained from another decomposition by replacing its element (G', L') with two 2-summable labeled graphs (G'_1, L'_1) and (G'_2, L'_2) whose 2-sum is (G', L'). We assume that whenever we replace a labeled graph (G', L') with (G'_1, L'_1) and (G'_2, L'_2), the vertex in $L'_1 \cap L'_2$ is a fresh vertex which does not appear in any labeled graph in the decomposition. This way the labeled graph (G, L) can be restored by 2-summing the elements of the decomposition in any order.

With a decomposition of a graph G into labeled graphs we can associate a tree T such that each node u corresponds to one element (G_u, L_u) of the decomposition and two nodes u and v are adjacent if L_u and L_v share a vertex (in which case (G_u, L_u) and (G_v, L_v) are 2-summable). The pair $(T, \{(G_u, L_u)\}_{u \in V(T)})$ is called a *tree structure* and G is called its *2-sum*. Observe that $(T, \{V(G_u) \backslash L_u\}_{u \in V(T)})$ is a tree-decomposition of G.

A graph is *internally* 3-*connected* if it is obtained from a 3-connected graph by subdividing each edge at most once. The following result is implicit in the proof of [5, Lemma 2.1].

Lemma 3.2. *Every* 2*-connected* $K_{2,n}$*-minor free graph* G *is the* 2*-sum of a tree structure* $(T, \{(G_u, L_u)\}_{u \in V(T)})$ *such that* T *is a tree of height at most* n *and for each* $u \in V(T)$*, the graph* G_u *belongs to* $\mathcal{P}$ *or is an internally* 3*-connected* $K_{2,n}$*-minor-free graph.*

This lemma combined with a result describing the structure of internally 3-connected $K_{2,n}$-minor-free graph will yield a characterization of graphs without large $K_{2,n}$-minors.

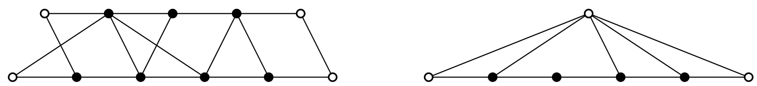


Figure 3.3: Two strips. The white vertices are the corners. The strip on the right is a fan.

A *strip* is a graph of the form $G - F$ where G is a graph from $\mathcal{P}$ for which there exists a reference cycle C with edges e_1 and e_2 such that all chords in G are between the two components of $C - \{e_1, e_2\}$, and $F \subseteq \{e_1, e_2\}$ is such that the minimum degree of $G - F$ is at least 2. The ends of the edges e_1 and e_2 are the *corners* of the strip. When one component of $C - \{e_1, e_2\}$ consists of a single vertex x, we call the strip a *fan* with a *center* in x. See Figure 3.3. A fan has exactly 3 corners, and a strip which is not a fan has exactly 4 corners. An *augmentation* of a graph G^0 is a graph of the form $G^0 \cup \bigcup_{i=1}^{k} H_i$ where each H_i is a strip which intersects G_0 exactly in its corners, and if two strips H_i and H_j intersect, then H_i and H_j are two fans which intersect in only one vertex which is their common center. We denote by $\mathcal{A}_m$ the class of all graphs which can be obtained as an augmentation of a graph on at most m vertices.

Ding proved the following [5, Theorem 5.1].

Lemma 3.3. *For every positive integer n there exists an integer m such that all internally 3-connected $K_{2,n}$-minor free graphs belong to $\mathcal{A}_m$.*

Let $\mathcal{G}$ be a class of graphs. We denote by $\mathcal{B}(\mathcal{G})$ the class of all graphs which have all their blocks in $\mathcal{G}$, and for a positive integer m, let us denote by $\mathcal{G}^{(m)}$ the class of all graphs which can be obtained as the 2-sum of a tree structure $(T, \{(G_u, L_u)\}_{u \in V(T)})$ such that T is a tree of height at most m and each G_u belongs to $\mathcal{G}$. Lemmas 3.2 and 3.3 imply the following.

Theorem 3.4. *If a graph class $\mathcal{H}$ excludes a $K_{2,n}$-minor for some n, then there exists an integer m such that $\mathcal{H} \subseteq \mathcal{B}((\mathcal{P} \cup \mathcal{A}_m)^{(m)})$.*

Although this does not play a role in our proof, we note that the existence of m as in the statement of Theorem 3.4 is not only necessary, but also sufficient for the class $\mathcal{H}$ to exclude a $K_{2,n}$-minor for some n.

3.2 Posets with cover graphs in $\mathcal{P}$

The goal of this section is to prove the following lemma.

Lemma 3.5. *Every poset whose cover graph is a subgraph of a graph from $\mathcal{P}$ has dimension at most* 6*.*

The following lemma shows that it suffices to consider induced subgraphs.

Lemma 3.6. *Every subgraph of a graph from $\mathcal{P}$ is an induced subgraph of some graph from $\mathcal{P}$.*

Proof. Every subgraph of a graph can be obtained by a sequence of vertex and edge deletions. Since removing a vertex preserves being an induced subgraph, we only need to argue that if H is an induced subgraph of $G \in \mathcal{P}$ and $e \in E(H)$, then $H - \{e\}$ is an induced subgraph of some graph $G' \in \mathcal{P}$. Let C be a reference cycle for G. If e is a chord of C, then we can take $G' = G - \{e\}$. Let us hence assume that $e \in E(C)$. If there exists crossing chords ac and bd such that $e = ab$ and $cd \in E(C)$, then $G - \{e\}$ belongs to $\mathcal{P}$ as witnessed by the reference cycle $(C - \{ab, cd\}) + \{ac, bd\}$ and we can again take $G' = G - \{e\}$. Finally if such chords ac and bd do not exist, then we can take as G' the graph obtained from G by subdividing e once. $\square$

We proceed to the proof of Lemma 3.5.

Let P be a poset whose cover graph is a subgraph of a graph $G \in \mathcal{P}$. By Lemma 3.6 we may assume that the cover graph of P is an induced subgraph of G. Fix a reference cycle C for G.

We claim that there exists an edge $e_0 \in E(C)$ such that for every pair of crossing chords ac and bd we have $\{ab, cd\} \subseteq E(C)\backslash\{e_0\}$ or $\{bc, da\} \subseteq E(C)\backslash\{e_0\}$. When there are no crossing chords, we can take any edge of C as e_0. Otherwise, consider a pair of crossing chords ac and bd and an a–d path W in C such that $\{ab, cd\} \subseteq E(C)$ and the length of W is smallest possible. In particular, there does not exist a pair of crossing chords with all ends on W. Since C is a reference cycle for G, ac is the only chord crossed by bd and bd is the only chord crossed by ac. Hence ac and bd are the only chords with exactly one end on W. Therefore, no pair of crossing chords distinct from (ac, bd) contains a vertex of W, and we can take any edge of W as e_0.

Let $\pi = (z_1, \ldots, z_N)$ denote the sequence consisting of all vertices of G in the order in which they appear on the path $C - \{e_0\}$ (starting from any end of e_0). A tuple of indices $(\alpha, \beta, \gamma, \delta)$ with $1 \leqslant \alpha < \beta < \gamma < \delta \leqslant N$ is called a *cross* if $\{z_\alpha z_\gamma, z_\beta z_\delta\} \subseteq E(G)$ (in which case $z_\alpha z_\gamma$ crosses $z_\beta z_\delta$). By our choice of e_0, for every cross $(\alpha, \beta, \gamma, \delta)$ we have $\beta - \alpha = \delta - \gamma = 1$. For two vertices z_α and z_β of G we write $z_\alpha \leqslant_\pi z_\beta$ when $\alpha \leqslant \beta$, and $z_\alpha \prec_\pi z_\beta$ when $\alpha < \beta$.

We need to show that $\mathrm{Inc}(P)$ can be partitioned into six reversible sets. For every $(a, b) \in \mathrm{Inc}(P)$ we have $a \neq b$, so in particular either $a \prec_\pi b$ or $b \prec_\pi a$. Let us partition $\mathrm{Inc}(P)$ into sets $I_\prec$ and $I_\succ$ defined as

$$I_\prec = \{(a, b) \in \mathrm{Inc}(P) : a \prec_\pi b\}, \text{ and}$$
$$I_\succ = \{(a, b) \in \mathrm{Inc}(P) : b \prec_\pi a\}$$

After possibly reversing the ordering of the vertices, we may assume that $\dim(I_\succ) \leqslant \dim(I_\prec)$, so that

$$\dim(P) \leqslant \dim(I_\prec) + \dim(I_\succ) \leqslant 2 \cdot \dim(I_\prec).$$

Therefore, it suffices to show that $\dim(I_\prec) \leqslant 3$. Let I_0^1 denote the subset of $I_\prec$ defined as

$$I_0^1 = \{(a, b) \in I_\prec : y \leqslant_\pi b \text{ for every } y \in P \text{ such that } y \geqslant a \text{ in } P\}.$$

Claim 3.7. *The set I_0^1 is reversible.*

Proof. Towards a contradiction, suppose that the set I_0^1 is not reversible. Let $((a_1, b_1), \ldots, (a_k, b_k))$ be an alternating cycle in I_0^1. For each $i \in \{1, \ldots, k\}$, we have $b_{i+1} \geqslant a_i$ in P and $(a_i, b_i) \in I_0^1$, so $b_{i+1} \leqslant_\pi b_i$. This implies that all b_i are equal, which is impossible in an alternating cycle. □

A symmetric argument shows that the set I_0^2 defined as

$$I_0^2 = \{(a, b) \in I_\prec : a \leqslant_\pi x \text{ for every } x \in P \text{ such that } x \leqslant b \text{ in } P\},$$

is reversible as well. Let $I_1 = I_\prec \backslash (I_0^1 \cup I_0^2)$. We need to show that the set I_1 is reversible.

Claim 3.8. *Let x, a, b, y be elements of P such that $x \prec_\pi a \prec_\pi b \prec_\pi y$ and we have $a \leqslant y$, $x \leqslant b$ and $a \parallel b$ in P. Then there exists at a cross $(\alpha, \beta, \gamma, \delta)$ such that*

$$x \leqslant_\pi z_\alpha \prec_\pi z_\beta \leqslant_\pi a \quad \text{and} \quad b \leqslant_\pi z_\gamma \prec_\pi z_\delta \leqslant_\pi y,$$

and we have

$$a \leqslant z_\beta < z_\delta \leqslant y \quad \text{and} \quad x \leqslant z_\alpha < z_\gamma \leqslant b \text{ in } P.$$

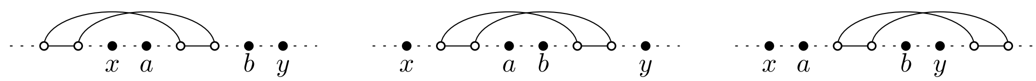


Figure 3.4: Three possibilities for the position of the cross relative to the vertices x, a, b, y.

Proof. Let W_{ay} be a witnessing path from a to y in P, and let W_{xb} be a witnessing path from x to b in P. The paths W_{ay} and W_{xb} must be disjoint because $a \parallel b$ in P. Since $x \prec_\pi a \prec_\pi b \prec_\pi y$, some edge of W_{ay} must cross some edge of W_{xb}, that is, there is a cross $(\alpha, \beta, \gamma, \delta)$ such that one of the edges $z_\alpha z_\gamma$ and $z_\beta z_\delta$ belongs to W_{ay} and the other to W_{xb}. Among all such crosses $(\alpha, \beta, \gamma, \delta)$ choose one with the smallest difference $\delta - \alpha$. The only edges between $\{z_\beta, \ldots, z_\gamma\}$ and $V(G) \backslash \{z_\beta, \ldots, z_\gamma\}$ are the edges between $\{z_\beta, z_\gamma\}$ and $\{z_\alpha, z_\delta\}$, so each of the paths W_{ay} and W_{xb} has exactly one end in $\{z_\beta, \ldots, z_\gamma\}$. Therefore we have three options, illustrated in Figure 3.4:

(a) $z_\alpha \prec_\pi z_\beta \preccurlyeq_\pi x$ and $a \preccurlyeq_\pi z_\gamma \prec_\pi z_\delta \preccurlyeq_\pi b$,

(b) $x \preccurlyeq_\pi z_\alpha \prec_\pi z_\beta \preccurlyeq_\pi a$ and $b \preccurlyeq_\pi z_\gamma \prec_\pi z_\delta \preccurlyeq_\pi y$, or

(c) $a \preccurlyeq_\pi z_\alpha \prec_\pi z_\beta \preccurlyeq_\pi b$ and $y \preccurlyeq_\pi z_\gamma \prec_\pi z_\delta$.

We claim that only the option (b) is possible.

Towards a contradiction, suppose that the vertices are ordered as in (a), that is $z_\beta \preccurlyeq_\pi x \prec_\pi a \preccurlyeq_\pi z_\gamma$, It is impossible that $z_\alpha z_\gamma \in E(W_{xb})$ and $z_\beta z_\delta \in E(W_{ay})$ as then some edge of $xW_{xb}z_\gamma$ would have to cross some edge of $aW_{ay}z_\beta$, contradicting minimality of the cross $(\alpha, \beta, \gamma, \delta)$. Thus, $z_\alpha z_\gamma \in E(W_{ay})$ and $z_\beta z_\delta \in E(W_{xb})$, so we have $z_\beta < z_\delta \leqslant b$ and $a \leqslant z_\gamma < z_\alpha$ in P. Since the cover graph of P is an induced subgraph of G, it contains the edges $z_\alpha z_\beta$ and $z_\gamma z_\delta$. It is impossible that z_γ is covered by z_δ in P as that would imply $a \leqslant z_\gamma < z_\delta \leqslant b$ in P. Hence z_δ is covered by z_γ in P, so we have $z_\beta < z_\delta < z_\gamma < z_\alpha$ in P, which contradicts $z_\alpha z_\beta$ being an edge of the cover graph. This contradiction excludes the option (a), and dual arguments exclude the option (c), so the vertices must indeed ordered as in (b).

We have $z_\beta \preccurlyeq_\pi a \prec_\pi b \preccurlyeq_\pi z_\gamma$. It is impossible that $z_\alpha z_\gamma \in E(W_{ay})$ and $z_\beta z_\delta \in E(W_{xb})$ as then some edge of $aW_{ay}z_\gamma$ would have to cross some edge

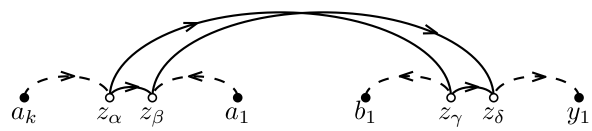


Figure 3.5: A solid arrow from x to y means that x is covered by y and a dashed arrow from x to y represent a witnessing path from x to y in P (possibly of length 0).

of $z_\beta W_{xb} b$ contradicting minimality of the cross $(\alpha, \beta, \gamma, \delta)$. Hence $z_\alpha z_\gamma \in E(W_{xb})$ and $z_\beta z_\delta \in E(W_{ay})$, which implies that we have $a \leqslant z_\beta < z_\delta \leqslant y$ and $x \leqslant z_\alpha < z_\gamma \leqslant b$ in P. $\square$

Claim 3.9. *The set I_1 is reversible.*

Proof. Suppose to the contrary that I_1 is not reversible. Fix a strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ in I_1. There must exist an index i such that $a_i \prec_\pi a_{i+1}$ (cyclically). Without loss of generality we assume that $a_k \prec_\pi a_1$. Since $(a_1, b_1) \in I_1$, we have $(a_1, b_1) \notin I_0^1$, so there exists an element $y_1 \in P$ such that $b_1 \prec_\pi y_1$ and $y_1 \geqslant a_1$ in P. Let us fix any such y_1.

By Claim 3.8 applied to $x = a_k$, $a = a_1$, $b = b_1$ and $y = y_1$, there exists a cross $(\alpha, \beta, \gamma, \delta)$ in (G, π) with

$$a_k \leqslant_\pi z_\alpha \prec_\pi z_\beta \leqslant_\pi a_1 \quad \text{and} \quad b_1 \leqslant_\pi z_\gamma \prec_\pi z_\delta \leqslant_\pi y_1$$

such that $a_1 \leqslant z_\beta < z_\delta \leqslant y_1$ and $a_k \leqslant z_\alpha < z_\gamma \leqslant b_1$ hold in P. Since the cover graph of P is an induced subgraph of G, it contains the edges $z_\alpha z_\beta$ and $z_\gamma z_\delta$. It is impossible that z_β is covered by z_α as that would imply $a_1 \leqslant z_\beta < z_\alpha < b_1$ in P. Therefore, z_α is covered by z_β in P. Similarly, z_γ must be covered by z_δ as otherwise we would have $a_1 < z_\delta < z_\gamma \leqslant b_1$ in P. See Figure 3.5.

We prove inductively that for each $i \in \{1, \ldots, k\}$, we have

$$z_\beta \leqslant_\pi a_i \prec_\pi b_i \leqslant_\pi z_\gamma.$$

This is true for the base case $i = 1$. For the inductive step, let $i \in \{2, \ldots, k\}$, and suppose that $z_\beta \leqslant_\pi a_{i-1} \prec_\pi b_{i-1} \leqslant_\pi z_\gamma$.

We first show that $z_\beta \leqslant_\pi b_i \leqslant_\pi z_\gamma$. Suppose to the contrary that $b_i \notin \{z_\beta, \ldots, z_\gamma\}$, and let W be a witnessing path from a_{i-1} to b_i in P. Every edge between $\{z_\beta, \ldots, z_\gamma\}$ and $V(G) \backslash \{z_\beta, \ldots, z_\gamma\}$ in G has an end in z_β or

z_γ, so W contains z_β or z_γ. Since $a_k < z_\beta$ and $a_k < z_\gamma$ in P, we have $a_k < b_i$ in P, which contradicts strictness of the cycle. Hence indeed $z_\beta \leqslant_\pi b_i \leqslant_\pi z_\gamma$.

Since $(a_i, b_i) \in I_<$, we have $a_i \prec_\pi b_i$. It remains to show that $z_\beta \leqslant_\pi a_i$. Suppose to the contrary that $a_i \prec_\pi z_\beta$ (and thus $a_i \leqslant_\pi z_\alpha$). Since $(a_i, b_i) \notin I_0^2$, there exists an element x such that $x \leqslant b_i$ in P and $x \prec_\pi a_i$. Similarly as earlier, a witnessing from x to b_i has to contain z_β or z_γ, so $a_k \leqslant b_i$ holds in P, contradicting strictness of the cycle again. This completes the inductive proof.

We have just shown that $z_\beta \leqslant_\pi a_i$ for all $i \in \{1, \ldots, k\}$. But we also have $a_k \leqslant_\pi z_\alpha \prec_\pi z_\beta$, so we reach a contradiction. The proof follows.

□

The sets I_0^1, I_0^2 and I_1 partition $I_<$ into reversible sets, so $\dim(I_<) \leqslant 3$. Therefore,

$$\dim(P) \leqslant 2 \cdot \dim(I_<) \leqslant 6.$$

This completes the proof of Lemma 3.5.

3.3 Gadget extensions

Let P be a poset with a cover graph G, and let $(T, \{V_u\}_{u \in V(T)})$ be a tree-decomposition of G such that $|V_u \cap V_v| = 2$ for each $uv \in E(T)$. In general, even if each bag V_u induces a subposet of small dimension, the dimension of P can be large. However, Walczak [34] showed that when the height of P is bounded, the dimension of P can be bounded in terms of the maximum dimension of a 'gadget extension' of a subposet induced by a bag V_u. In this section, we present a simple variant of gadget extensions suited for tree-decompositions corresponding to tree structures. We will prove that whenever all gadget extensions have bounded dimension and the height of the tree T in the tree-decomposition is bounded, the dimension of P can be bounded.

For a fixed poset P with a cover graph G and a tree-decomposition $(T, \{V_u\}_{u \in V(T)})$ of G, we define gadget extensions as follows. Let $u \in V(T)$, and let $\mathcal{X} = \{X_v\}_{v \in N_T(u)}$ and $\mathcal{Y} = \{Y_v\}_{v \in N_T(u)}$ be two indexed families of subsets of V_u such that $X_v \cup Y_v \subseteq V_u \cap V_v$ for each $v \in N_T(u)$. For such $\mathcal{X}$ and $\mathcal{Y}$, we define two superposets of $P[V_u]$: the *weak gadget extension* $Q_u(\mathcal{X}, \mathcal{Y})$ and the *strong gadget extension* $Q'_u(\mathcal{X}, \mathcal{Y})$. The poset $Q_u(\mathcal{X}, \mathcal{Y})$ is

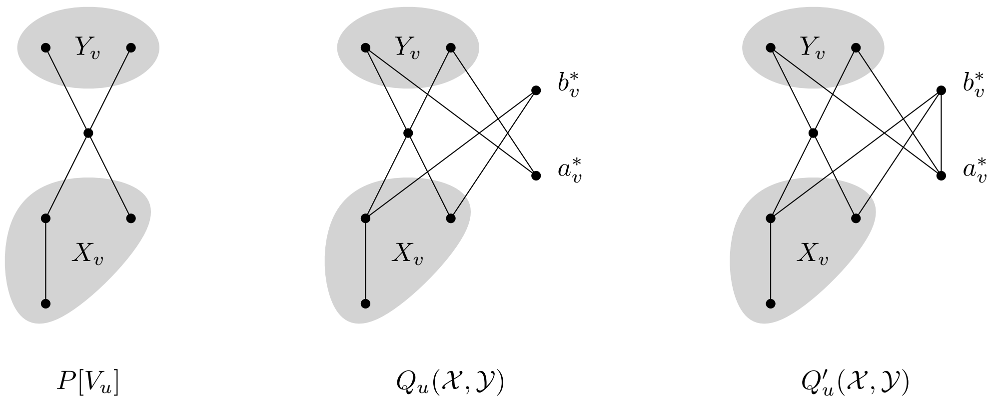


Figure 3.6: Gadget extensions of $P[V_u]$ in the simple case where u is adjacent to only one node v in T.

obtained from $P[V_u]$ by adding a minimal element a_v^* and a maximal element b_v^* for each $v \in N_T(u)$, where a_v^* is covered by the elements from $\mathrm{Min}(P[Y_v])$ and b_v^* covers the elements from $\mathrm{Max}(P[X_v])$. In particular, we have $Y_v \subseteq \mathrm{U}_{Q_u(\mathcal{X},\mathcal{Y})}(a_v^*)$ and $X_v \subseteq \mathrm{D}_{Q_u(\mathcal{X},\mathcal{Y})}(b_v^*)$, and we have $a_v^* < b_v^*$ in $Q_u(\mathcal{X},\mathcal{Y})$ if and only if there exist $x \in X_v$ and $y \in Y_v$ such that $y \leqslant x$ in P. The poset $Q'_u(\mathcal{X},\mathcal{Y})$ has the same ground set as $Q_u(\mathcal{X},\mathcal{Y})$, and we have $a \leqslant b$ in $Q'_u(\mathcal{X},\mathcal{Y})$ if $a \leqslant b$ in $Q'_u(\mathcal{X},\mathcal{Y})$ or there exists $v \in N_T(u)$ such that $a = a_v^*$ and $b = b_v^*$. Note that the cover graph of $Q'_u(\mathcal{X},\mathcal{Y})$ can be obtained from the cover graph of $Q_u(\mathcal{X},\mathcal{Y})$ by adding an edge $a_v^*b_v^*$ whenever $a_v^* \parallel b_v^*$ in $Q_u(\mathcal{X},\mathcal{Y})$. See Figure 3.6.

Lemma 3.10. *Let P be a poset with a cover graph G and let $(T, \{V_u\}_{u\in V(T)})$ be a tree-decomposition of G such that T is a rooted tree of height at most h, for each $uv \in E(T)$ we have $|V_u \cap V_v| = 2$, and for each $u \in V(T)$ all weak and strong gadget extensions of $P[V_u]$ have dimension at most d. Then*

$$\dim(P) \leqslant \sum_{i=1}^{h+1} (16d)^i.$$

Proof. We prove the lemma by induction on h. In the base case $h = 0$, T consists of a single node w, and thus $P = Q_w(\varnothing, \varnothing)$, so $\dim(P) \leqslant d \leqslant 16d$.

We proceed to the inductive step. Assume that $h \geqslant 1$ and the lemma holds for tree-decompositions of height at most $h-1$. Let w denote the root of T, For each $v \in N_T(w)$, let T_v denote the component of $T - w$ containing

v, and let $P_v = P[\bigcup_{u \in V(T_v)} V_u]$. Observe that every edge of the cover graph of P_v which is not an edge of the cover graph of P must have both of its ends in $V_w \cap V_v$. Hence, the pair $(T_v, \{V_u\}_{u \in V(T_v)})$ is a tree-decomposition of the cover graph of P_v, and the gadget extensions of $P[V_u]$ with $u \in V(T_v)$ are the same in both tree-deompositions with a small exception: for $u = v$, the gadget extensions of $P[V_u]$ in P_v do not contain the elements a_w^* and b_w^*. Nevertheless, for each $u \in V(T_v)$, each gadget extension of $P[V_u]$ in P_v is a subposet of a gadget extension of $P[V_u]$ in P. Hence, by induction hypothesis, for each $v \in N_T(w)$ we have

$$\dim(P_v) \leqslant \sum_{i=1}^{h} (16d)^i.$$

For each $v \in N_T(w)$, let $Z_v = V_w \cap V_v$, and let z_v^1 and z_v^2 denote the elements of Z_v, in any order.

For each $x \in V(G) \backslash V_w$ there exists a unique node $v \in N_T(w)$ such that $x \in P_v$, and we denote that node by $v(x)$. We define two functions σ_{U} and σ_{D} assigning subsets of $\{1, 2\}$ to elements of P:

$$\sigma_{\mathrm{U}}(x) = \begin{cases} \varnothing & \text{if } x \in V_w, \\ \{i \in \{1,2\} : z^i_{v(x)} \in \mathrm{U}_P(x)\} & \text{if } x \notin V_w. \end{cases}$$

and

$$\sigma_{\mathrm{D}}(x) = \begin{cases} \varnothing & \text{if } x \in V_w, \\ \{i \in \{1,2\} : z^i_{v(x)} \in \mathrm{D}_P(x)\} & \text{if } x \notin V_w. \end{cases}$$

The sets $\sigma_{\mathrm{U}}^{-1}(S)$ with $S \subseteq \{1, 2\}$ partition the ground set of P, and likewise do the sets $\sigma_{\mathrm{D}}^{-1}(S)$ with $S \subseteq \{1, 2\}$. We have

$$\dim(P) = \dim_P(\mathrm{Inc}(P)) \leqslant \sum_{S_{\mathrm{U}} \subseteq \{1,2\}} \sum_{S_{\mathrm{D}} \subseteq \{1,2\}} \dim_P(\sigma_{\mathrm{U}}^{-1}(S_{\mathrm{U}}), \sigma_{\mathrm{D}}^{-1}(S_{\mathrm{D}}))$$

Let $(S_{\mathrm{U}}^{\max}, S_{\mathrm{D}}^{\max})$ be a pair of subsets of $\{1, 2\}$ which maximizes the value of $\dim_P(\sigma_{\mathrm{U}}^{-1}(S_{\mathrm{U}}^{\max}), \sigma_{\mathrm{D}}^{-1}(S_{\mathrm{D}}^{\max}))$, let $A = \sigma_{\mathrm{U}}^{-1}(S_{\mathrm{U}}^{\max})$ and $B = \sigma_{\mathrm{D}}^{-1}(S_{\mathrm{D}}^{\max})$. Since there are 16 distinct pairs $(S_{\mathrm{U}}, S_{\mathrm{D}})$ of subsets of $\{1, 2\}$, we have

$$\dim(P) \leqslant 16 \cdot \dim_P(A, B),$$

so it suffices to partition $\mathrm{Inc}_P(A, B)$ into a sufficiently small number of reversible sets.

For each $v \in N_T(w)$, let $X_v = \{z_v^i : i \in S_{\mathrm{D}}^{\max}\}$ and $Y_v = \{z_v^i : i \in S_{\mathrm{U}}^{\max}\}$. Let $\mathcal{X} = \{X_v\}_{v \in N_T(w)}$ and $\mathcal{Y} = \{Y_v\}_{v \in N_T(w)}$. Let $Q = Q_w(\mathcal{X}, \mathcal{Y})$ and $Q' = Q'_w(\mathcal{X}, \mathcal{Y})$. For $a \in A$ we denote by $a^{\downarrow}$ an element of Q and Q' defined as

$$a^{\downarrow} = \begin{cases} a & \text{if } a \in V_w, \\ a^*_{v(a)} & \text{if } a \notin V_w. \end{cases}$$

Similarly, for $b \in B$, we denote by $b^{\uparrow}$ a n element of Q and Q' defined as

$$b^{\uparrow} = \begin{cases} b & \text{if } b \in V_w, \\ b^*_{v(b)} & \text{if } b \notin V_w. \end{cases}$$

Let $a \in A$ and $b \in B$. By the definition of gadget extensions, we have

$$\mathrm{U}_P(a) \cap V_w = \mathrm{U}_Q(a^{\downarrow}) \cap V_w = \mathrm{U}_{Q'}(a^{\downarrow}) \cap V_w,$$

and

$$\mathrm{D}_P(b) \cap V_w = \mathrm{D}_Q(b^{\uparrow}) \cap V_w = \mathrm{D}_{Q'}(b^{\uparrow}) \cap V_w.$$

If there does not exist $v \in N_T(w)$ such that a and b are elements of $P_v - Z_v$, then any witnessing path from a to b in P or from $a^{\downarrow}$ to $b^{\uparrow}$ in Q or Q' contains a vertex from V_w, and hence the following are equivalent: (1) $a \leqslant b$ in P, (2) $a^{\downarrow} \leqslant b^{\uparrow}$ in Q, and (3) $a^{\downarrow} \leqslant b^{\uparrow}$ in Q'. On the other hand, if a and b are elements of $P_v - Z_v$ for some $v \in N_T(w)$, then $a^{\downarrow} < b^{\uparrow}$ in Q', and if additionally $a \parallel b$ in P, then $a^{\downarrow} \parallel b^{\uparrow}$ in Q.

We partition the set $\mathrm{Inc}_P(A, B)$ into sets I_1 and I_2 where

$$I_1 = \mathrm{Inc}(A, B) \cap \bigcup_{v \in N_T(w)} \mathrm{Inc}(P_v - Z_v) \quad \text{and} \quad I_2 = \mathrm{Inc}_P(A, B) \backslash I_1.$$

Let $D = \sum_{i=1}^{h} (16d)^i$ so that for each $v \in N_T(w)$ we have $\dim(P_v) \leqslant D$. We claim that $\dim_P(I_1) \leqslant dD$. We have $\dim(Q) \leqslant d$, so let us partition $\mathrm{Inc}(Q)$ into reversible sets $I_Q^1, \ldots, I_Q^d$.

For each $v \in N_T(w)$ we have $\dim(P_v - Z_v) \leqslant \dim(P_v) \leqslant D$, so let us partition $\mathrm{Inc}(P_v - Z_v)$ into D reversible sets $I_{P_v}^1, \ldots, I_{P_v}^D$. For each $j \in \{1, \ldots, D\}$, let $I_P^j = \bigcup_{v \in N_T(w)} I_{P_v}^j$. Hence, for any $j \in \{1, \ldots, D\}$ and $v \in N_T(w)$ the set $I_P^j \cap \mathrm{Inc}(P_v - Z_v)$ is reversible.

For each $(a,b) \in I_1$ we have $v(a) = v(b)$, and thus $a^{\downarrow} \parallel b^{\uparrow}$ in Q. To prove $\dim_P(I_1) \leqslant dD$ it suffices to show that for each $j_0 \in \{1,\ldots,d\}$ and each $j_1 \in \{1,\ldots,D\}$, the set

$$I_1^{j_0,j_1} := \{(a,b) \in I_1 : (a^{\downarrow}, b^{\uparrow}) \in I_Q^{j_0}, (a,b) \in I_P^{j_1}\}$$

is reversible. Suppose to the contrary that it is not. Fix a strict alternating cycle $((a_1,b_1),\ldots,(a_k,b_k))$ in $I_1^{j_0,j_1}$ so that $(a_i^{\downarrow}, b_i^{\uparrow}) \in I_Q^{j_0}$ and $(a_i,b_i) \in I_P^{j_1}$ for all $i \in \{1,\ldots,k\}$. For each $i \in \{1,\ldots,k\}$, $v(a_i)$ and $v(b_i)$ are the same node which we denote by v_i. The set $I_Q^{j_0}$ is reversible in Q, so the pairs $(a_i^{\downarrow}, b_i^{\uparrow})$ do not form an alternating cycle in Q. Hence there must exist $i \in \{1,\ldots,k\}$ such that $a_i^{\downarrow} \nleqslant b_{i+1}^{\uparrow}$ in Q (cyclically). As $a_i \leqslant b_{i+1}$ in P, it must be the case that $v_i = v_{i+1}$. Let us assume without loss of generality that $v_1 = v_2$.

The set $I_P^{j_1} \cap \operatorname{Inc}(P_{v_1} - Z_{v_1})$ is reversible, so, not all pairs in the cycle $((a_1,b_1),\ldots,(a_k,b_k))$ belong to $\operatorname{Inc}(P_{v_1} - Z_{v_1})$. Hence there must exist $i \in \{2,\ldots,k\}$ such that $v_1 = v_{i-1} \neq v_i$. A witnessing path from a_{i-1} to b_i has to intersect Z_{v_1} in an element z. Since $v_1 = v_2 = v_{i-1}$ and $\sigma_{\mathrm{U}}(a_1) = \sigma_{\mathrm{U}}(a_2) = \sigma_{\mathrm{U}}(a_{i-1})$, we have $a_1 \leqslant z \leqslant b_i$ and $a_2 \leqslant z \leqslant b_i$ in P, which contradicts strictness of the cycle. Hence $\dim_P(I_1) \leqslant dD$.

Next, we prove that $\dim(I_2) \leqslant d$. By our assumption, $\dim(Q') \leqslant d$, so let us partition $\operatorname{Inc}(Q')$ into reversible sets $I_{Q'}^1, \ldots, I_{Q'}^d$. We claim that for each $(a,b) \in I_2$ we have $a^{\downarrow} \parallel b^{\uparrow}$ in Q'. Since $a \nleqslant b$ in P and there does not exist $v \in N_T(w)$ such that $(a,b) \in \operatorname{Inc}(P_v - Z_v)$, we have $a^{\downarrow} \nleqslant b^{\uparrow}$ in Q', so suppose that we have $a^{\downarrow} > b^{\uparrow}$ in Q'. This implies that $a^{\downarrow}$ is not minimal and $b^{\uparrow}$ is not maximal in Q', so $\{a^{\downarrow}, b^{\uparrow}\} \subseteq V_w$, and therefore $a = a^{\downarrow} > b^{\uparrow} = b$ in $P[V_w]$ contradicting a and b being incomparable. Therefore, for each alternating cycle $((a_1,b_1),\ldots,(a_k,b_k))$ in I_2, the sequence $((a_1^{\downarrow}, b_1^{\uparrow}),\ldots,(a_k^{\downarrow}, b_k^{\uparrow}))$ is an alternating cycle in Q'. Hence for each $j \in \{1,\ldots,d\}$ the set

$$\{(a,b) \in I_2 : (a^{\downarrow}, b^{\uparrow}) \in I_{Q'}^j\}$$

is reversible. This proves $\dim_P(I_2) \leqslant d$.

Summarizing, we have

$$\begin{aligned}\dim(P) &\leqslant 16 \cdot \dim_P(A, B)\\ &\leqslant 16 \cdot (\dim_P(I_1) + \dim_P(I_2))\\ &\leqslant 16 \cdot (dD + d)\\ &= 16d \cdot \sum_{i=0}^{h} (16d)^i\\ &= \sum_{i=1}^{h+1} (16d)^i\end{aligned}$$

as required. □

For a labeled graph (G, L), let $G * L$ denote a graph obtained from G where for each $c \in L$ we add a copy c' and all possible edges between the vertices in $N_G(c) \cup \{c, c'\}$. Observe that if G is the 2-sum of a tree structure $(T, \{(G_u, L_u)\}_{u \in V(T)})$ and $(T, \{V_u\}_{u \in V(T)})$ is the tree-decomposition such that $V_u = V(G_u) \backslash L_u$ for each $u \in V(T)$, then for any poset P whose cover graph is G, each gadget extension of $P[V_u]$ is isomorphic to a subgraph of $G_u * L_u$; indeed, for each $v \in N_T(u)$ and the vertex $c \in L_u \cap L_v$, the vertices c and c' of $G * L$ can correspond to the elements a_v^* and b_v^* of a gadget extension of $P[V_u]$.

Lemma 3.11. *Let (G, L) be a labeled graph with $G \in \mathcal{P} \cup \mathcal{A}_m$. Then there exists a set $U \subseteq V(G * L)$ with $|U| \leqslant 2m$ such that every component of $(G * L) - U$ is a subgraph of a graph from $\mathcal{P}$.*

Proof. Suppose first that G is a graph from $\mathcal{P}$ with a reference cycle C. Let C^* be the Hamiltonian cycle in $G * L$ obtained from C by replacing each $c \in L$ with the vertices c and c' appearing next to each other. The resulting cycle C^* is a valid reference cycle for $G * L$ because any new chords are between the four consecutive vertices from $N_G(c) \cup \{c, c'\}$ for some $c \in L$, and these chords do not cross any other chords (because vertices in L are pairwise nonadjacent). Hence $G * L \in \mathcal{P}$, and the lemma is satisfied by $U = \varnothing$.

Now suppose that $G \in \mathcal{A}_m$, that is G is an augmentation of a graph G^0 on at most m vertices with strips $H_1, \ldots, H_k$. Every vertex in a strip has degree at least 2, so if a vertex $x \in L$ belongs to $V(H_i)$ for some $i \in \{1, \ldots, k\}$, then both neighbors of x in G belong to H_i. Hence every $x \in L$ has both neighbors in one of the subgraphs G^0, $H_1, \dots, H_k$. This implies

$$G * L = (G^0 * (L \cap V(G_0))) \cup \bigcup_{i=1}^{k} (H_i * (L \cap V(H_i))).$$

Let $U = V(G^0 * (L \cap V(G_0)))$. We have $|U| \leqslant 2|V(G^0)| \leqslant 2m$, and every component H' of $(G * L) - U$ is of the form

$$(H_i - (V(G^0) \cap V(H_i))) * (L \cap V(H_i) \backslash V(G_0)).$$

Let us show that any such component is a subgraph of a graph from $\mathcal{P}$. Let us represent H_i as $G_i - F$ like in the definition of a strip, so that $G_i \in \mathcal{P}$ and $F \subseteq E(G_i)$ is a set of edges with ends in corners of H_i. All corners of H_i belong to G^0, so every vertex from $L \cap V(H_i) \backslash V(G^0)$, has degree 2 not only in H_i, but also in G_i. Therefore, H' is a subgraph of $G_i * (L \cap V(H_i) \backslash V(G_0))$. Since $(G_i, L \cap V(H_i) \backslash V(G_0))$ is a labeled graph and $G_i \in \mathcal{P}$, we deduce that H' is a subgraph of a graph from $\mathcal{P}$, as claimed. $\square$

3.4 The proof

Let us prove Theorem 3.1.

By Theorem 3.4, there exists a positive integer m such that all $K_{2,n}$-minor-free graphs belong to $\mathcal{B}((\mathcal{P} \cup \mathcal{A}_m)^{(m)})$ Let us fix such m. We will show that every poset with a $K_{2,n}$-minor-free cover graph has dimension at most $(96 \cdot 4^m)^{m+2} + 2$.

Let P_0 be a poset with a $K_{2,n}$-minor-free cover graph G_0, let G be a block of G_0, and let $P = P_0[V(G)]$. Since G_0 does not have a $K_{2,n}$-minor, we have $G \in (\mathcal{P} \cup \mathcal{A}_m)^{(m)}$. Let $(T, \{(G_u, L_u)\}_{u \in V(T)})$ be a tree structure whose 2-sum is G such that the height of T is at most m and for each $u \in V(T)$ we have $G_u \in \mathcal{P} \cup \mathcal{A}_m$. For each $u \in V(T)$ let $V_u = V(G_u) \backslash L_u$, so that $(T, \{V_u\}_{u \in V(T)})$ forms a tree-decomposition of G. Fix any $u \in V(T)$. Every gadget extension of $P[V_u]$ has cover graph isomorphic to a subgraph of $G_u * L_u$. By Lemma 3.11, there exists a subset $U \subseteq V(G_u * L_u)$ with $|U| \leqslant 2m$ such that every component of $(G_u * L_u) - U$ is a subgraph of a graph from $\mathcal{P}$. By Lemmas 3.5 and 1.5, if in a poset every component of the cover graph is a subgraph of a graph from $\mathcal{P}$, then the dimension is at most 6. Hence, by Lemma 1.4 every gadget extension of $P[V_u]$ has dimension at most $6 \cdot 2^{2m}$. Since u was chosen arbitrarily, the Lemma 3.10 implies that $\dim(P) \leqslant \sum_{i=1}^{m+1}(16 \cdot 6 \cdot 2^{2m})^i \leqslant (96 \cdot 4^m)^{m+2}$, and since the block G of G_0 was also chosen arbitrarily, Theorem 1.6 implies that $\dim(P_0) \leqslant (96 \cdot 4^m)^{m+2} + 2$, so the proof is complete.

Chapter 4

Excluding a ladder

The Grid-Minor Theorem shows that the size of a largest $n \times n$ grid-minor is tied to treewidth: there exists a function $f(n)$ such that in any graph G, if n is the largest integer such that G has an $n \times n$ grid-minor, then we have $n \leqslant \mathrm{tw}(G) \leqslant f(n)$.

What is the structure of graphs excluding $k \times n$ grid-minors for a fixed value of k? In the case $k = 1$, a $1 \times n$ grid is simply a path on n vertices, and a graph has an $1 \times n$ grid-minor if and only if it contains a path on n vertices (as a subgraph).

The size of a longest path in a graph G is tied to its *treedepth* $\mathrm{td}(G)$, which is a graph parameter defined recursively as follows.

$$\mathrm{td}(G) = \begin{cases} 0 & \text{if } V(G) = \varnothing, \\ 1 + \min\{\mathrm{td}(G - \{x\}) : x \in V(G)\} & \text{if } G \text{ is connected,} \\ \max\{\mathrm{td}(C) : C \text{ is a component of } G\} & \text{otherwise.} \end{cases}$$

If n is the number of vertices in a longest path in a graph G, then we have $\lceil \log_2 n \rceil \leqslant \mathrm{td}(G) \leqslant n$.

$2 \times n$ grids are called ladders, and in this chapter we show that the size of a largest ladder-minor is tied to a variant of treedepth obtained by replacing components with blocks in the recursive definition of the parameter. As a consequence, we prove that posets without long ladder-minors in their cover graphs have bounded dimension.

Theorem 4.1 (Huynh, Joret, Micek, Seweryn, Wollan [12])**.** *For every positive integer n there exists an integer d such that every poset excluding a $2 \times n$ grid-minor has dimension at most d.*

4.1 Ladders and a variant of treedepth

For $n \geqslant 1$, we call the $2 \times n$ grid a *ladder* and denote it by L_n. Hence, the vertex set of L_n is $\{1,2\} \times \{1,\ldots,n\}$ and two vertices (i,j) and (i',j') are adjacent in L_n if $|i-i'|+|j-j'|=1$.

For a graph G, we recursively define a parameter $\mathrm{td}_2(G)$ as follows.

$$\mathrm{td}_2(G) = \begin{cases} 0 & \text{if } V(G) = \varnothing, \\ 1 + \min\{\mathrm{td}(G - \{x\}) : x \in V(G)\} & \text{if } G \text{ has exactly } 1 \text{ block,} \\ \max\{\mathrm{td}(B) : B \text{ is a block of } G\} & \text{otherwise.} \end{cases}$$

Clearly, $\mathrm{td}_2(H) \leqslant \mathrm{td}_2(G)$ whenever $H \subseteq G$, and every nonempty graph G has a block B such that $\mathrm{td}_2(B) = \mathrm{td}_2(G)$. We have $\mathrm{td}_2(G) = 0$ if and only if G is empty, $\mathrm{td}_2(G) \leqslant 1$ if and only if $E(G) = \varnothing$, and $\mathrm{td}_2(G) \leqslant 2$ if and only if G is a forest.

Theorem 4.2. *In any graph G, if n is the largest integer such that G has an L_n-minor, then*

$$\lfloor \log_2 n \rfloor + 2 \leqslant \mathrm{td}_2(G) \leqslant (n+1)(n^2+2).$$

In particular, Theorem 4.2 implies that a graph class $\mathcal{G}$ excludes an L_n-minor for some n if and only if the graphs $G \in \mathcal{G}$ have bounded value of $\mathrm{td}_2(G)$.

In the proof of Theorem 4.2, we use the following property of the parameter $\mathrm{td}_2(G)$.

Lemma 4.3. *For every graph G and a k-element subset $X \subseteq V(G)$, we have $\mathrm{td}_2(G) \leqslant \mathrm{td}_2(G - X) + k$.*

Proof. We prove the lemma by induction on k. The lemma is trivial for $k = 0$, so assume that $k \geqslant 1$ and the lemma holds for $(k-1)$-element subsets of vertices. Let $X \subseteq V(G)$ satisfy $|X| = k$, and let $x_0 \in X$. Let B be a block of G such that $\mathrm{td}_2(B) = \mathrm{td}_2(G)$. If $x_0 \notin V(B)$, then we have $\mathrm{td}_2(G) = \mathrm{td}_2(B) \leqslant \mathrm{td}_2(G - \{x_0\})$, and if $x_0 \in V(B)$, then

$$\begin{aligned} \mathrm{td}_2(G) = \mathrm{td}_2(B) &= 1 + \min\{\mathrm{td}(B - \{x\}) : x \in V(B)\} \\ &\leqslant 1 + \mathrm{td}_2(B - \{x_0\}) \\ &\leqslant 1 + \mathrm{td}_2(G - \{x_0\}). \end{aligned}$$

Hence we always have $\mathrm{td}_2(G) \leqslant 1 + \mathrm{td}_2(G - \{x_0\})$. By the induction hypothesis we have

$$\mathrm{td}_2(G - \{x_0\}) \leqslant \mathrm{td}_2((G - \{x_0\}) - (X \backslash \{x_0\})) + (k - 1) = \mathrm{td}_2(G - X) + k - 1.$$

Hence $\mathrm{td}_2(G) \leqslant \mathrm{td}_2(G - X) + k$, which completes the inductive proof. $\square$

The proof of Theorem 4.2 relies on two classical results: Menger's Theorem and Erdős-Szekeres Theorem.

Theorem 4.4 (Menger's Theorem [22]). *Let G be a graph, let A and B be subsets of $V(G)$. Then the minimum size of a vertex subset separating A and B is equal to the maximum number of pairwise disjoint A–B paths in G.*

Theorem 4.5 (Erdős-Szekeres Theorem [8]). *Every sequence of $(n-1)^2 + 1$ distinct integers contains an increasing or decreasing subsequence of length n.*

We call a sequence $(P_1, P_2; Q_1, \ldots, Q_n)$ an *n-ladder model* in a graph G if

(1) P_1 and P_2 are disjoint paths in G, each with an end in $V(Q_1)$,

(2) $Q_1, \ldots, Q_n$ are pairwise disjoint $V(P_1)$–$V(P_2)$ paths in G, and

(3) each of P_1 and P_2 intersects the paths $Q_1, \ldots, Q_n$ in that order.

Note that the paths P_1 and P_2 are not required to have an end in $V(Q_n)$. Clearly, a graph G contains an n-ladder model if and only if L_n is a topological minor of G. Since each vertex in L_n has degree at most 3, this is equivalent to L_n being a minor of G. An n-ladder model $(P_1, P_2; Q_1, \ldots, Q_n)$ is *rooted at* a pair of vertices (z_1, z_2) if each P_i is a $V(Q_1)$–z_i path.

Lemma 4.6. *Let n and t be positive integers, let $s = (n-1)^2 + 2$, let G be a 2-connected graph, and let z_1 and z_2 be distinct vertices of G. If $\mathrm{td}_2(G) > ts$, then at least one of the following holds:*

(1) G has an L_n-minor, or

(2) G has a t-ladder model rooted at (z_1, z_2).

Proof. We prove the lemma by induction on t. Suppose first that $t = 1$. Since G is 2-connected, it is connected, so there exists a z_1–z_2 path Q_1 in G, and the desired 1-ladder model rooted at (z_1, z_2) in G can be defined as $(P_1, P_2; Q_1)$ where each P_i is the trivial path $G[\{z_i\}]$.

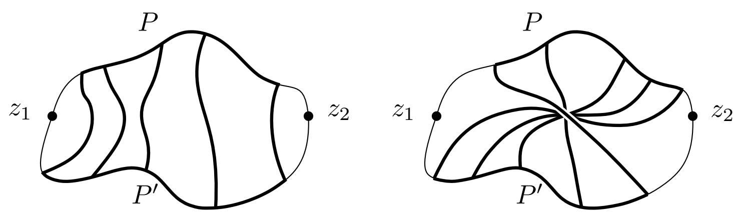


Figure 4.1: Disjoint $V(P)$–$V(P')$ paths forming an n-ladder model.

Now suppose that $t \geqslant 2$ and the lemma holds for $t-1$. Since G is 2-connected, there exist internally disjoint z_1–z_2 paths P and P'. Let us fix any such P and P'. By Menger's Theorem, either there exist $s+1$ pairwise disjoint $V(P)$–$V(P')$ paths, or there exists a set of at most s vertices which separates $V(P)$ and $V(P')$. We consider these two cases separately.

Suppose first that there exist pairwise disjoint $V(P)$–$V(P')$ paths Q_1, $\ldots$, Q_{s+1}. For $i \in \{1,\ldots,s+1\}$, let x_i and x'_i denote the ends of Q_i on P and P', respectively. Without loss of generality, we assume that the vertices x_1, $\ldots$, x_{s+1} appear on P in that order. Let $\pi\colon \{1,\ldots,s+1\} \to \{1,\ldots,s+1\}$ be a permutation such that the vertices $x'_{\pi(1)}$, $\ldots$, $x'_{\pi(s+1)}$ appear on P' in that order. Since the paths P and P' are internally disjoint, the paths $Q_2, \ldots, Q_s$ are nontrivial. We have $s-1=(n-1)^2+1$, so by Erdős-Szekeres Theorem there exist indices $2 \leqslant i_1 < \cdots < i_n \leqslant s$ such that either $\pi(i_1) < \cdots < \pi(i_n)$, or $\pi(i_1) > \cdots > \pi(i_n)$. In either case $(x_{i_1}Px_{i_n}, x'_{i_1}P'x'_{i_n}; Q_{i_1}, \ldots, Q_{i_n})$ is an n-ladder model in G (see Figure 4.1). Thus G has an L_n-minor.

Now suppose that there exists a set $X \subseteq V(G)$ with $|X| \leqslant s$ which separates $V(P)$ and $V(P')$. We have $\mathrm{td}_2(G) > ts$, so by Lemma 4.3 we have

$$\mathrm{td}_2(G-X) \geqslant \mathrm{td}_2(G) - s > ts - s = (t-1)s.$$

Let B be a block of $G-X$ with $\mathrm{td}_2(B) = \mathrm{td}_2(G-X) > (t-1)s$. In particular, we have $\mathrm{td}(B) > 2$, so $|V(B)| > 2$, which means that B is 2-connected. As X separates $V(P)$ and $V(P')$, the block B intersects at most one of the sets $V(P)$ and $V(P')$. Without loss of generality we assume that B is disjoint from $V(P)$. Since G is 2-connected, there exist disjoint $V(P)$–$V(B)$ paths R_1 and R_2. For $i \in \{1,2\}$, let x_i and z'_i denote the ends of R_i lying in $V(P)$ and $V(B)$, respectively. Without loss of generality we assume that the vertices z_1, x_1, x_2, z_2 lie on P in that order (with a possibility that $z_1 = x_1$ and/or $x_2 = z_2$). We have $\mathrm{td}_2(B) > (t-1)s$, so we can apply the induction hypothesis to B and the vertices z'_1 and z'_2. If B has an L_n-minor,

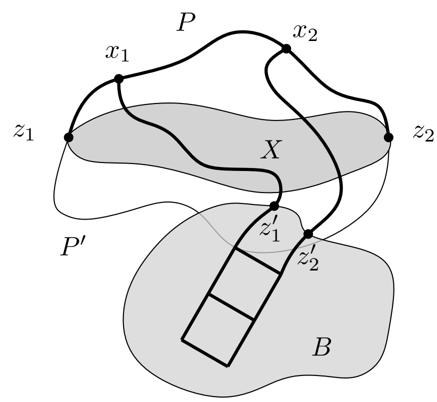


Figure 4.2: Extending a $(t-1)$-ladder model in B to a t-ladder model rooted at (z_1, z_2).

then so does G. Let us hence assume that there is a $(t-1)$-ladder model $(P_1, P_2; Q_1, \ldots, Q_{t-1})$ rooted at (z'_1, z'_2) in B. We can extend it to an t-ladder model $(P'_1, P'_2; Q_1, \ldots, Q_t)$ rooted at (z_1, z_2) where $P'_i = P_i \cup R_i \cup x_i P z_i$ and $Q_t = x_1 P x_2$, see Figure 4.2. This completes the inductive proof. $\square$

Proof of Theorem 4.2. Lemma 4.6 applied with $t = n$ shows that a graph G has an L_n-minor whenever $\mathrm{td}_2(G) > n((n-1)^2 + 2)$. Hence, if n is the largest integer such that G has an L_n-minor, then $\mathrm{td}_2(G) \leqslant (n+1)(n^2+2)$. To show that $\mathrm{td}_2(G) \geqslant \lfloor \log_2 n \rfloor + 2$, it suffices to prove that $\mathrm{td}_2(G) \geqslant d + 2$ whenever G has an L_{2^d}-minor. We prove this by induction on d. In the base case $d = 0$, G has an L_1-minor, and thus $E(G) \neq \emptyset$, so $\mathrm{td}_2(G) \geqslant 2$. For the inductive step, suppose that $d \geqslant 1$ and G has an L_{2^d}-minor. Since L_{2^d} is 2-connected, G has a block B with an L_{2^d}-minor. Hence B contains two disjoint 2^{d-1}-ladder models. By definition of $\mathrm{td}_2(G)$, the block B has a vertex x such that $\mathrm{td}(B) = 1 + \mathrm{td}(B - \{x\})$. At least one of the two 2^{d-1}-ladder models survives in $B - \{x\}$, so by induction hypothesis we have $\mathrm{td}_2(B - \{x\}) \geqslant (d-1) + 2 = d + 1$. Hence

$$\mathrm{td}_2(G) \geqslant \mathrm{td}_2(B) = 1 + \mathrm{td}_2(B - \{x\}) \geqslant 1 + ((d-1) + 2) = d + 2.$$

The proof is complete. $\square$

Lemma 4.7. *Let P be a poset with a cover graph G and let $m = \mathrm{td}_2(G)$. Then $\dim(P) \leqslant 2^{m+1} - 2$.*

Proof. We prove the lemma by induction on m. In the base case $m = 1$, there are no edges in the cover graph of P, and thus $\dim(P) \leqslant 2 = 2^{m+1} - 2$. For the inductive step, assume that $m \geqslant 2$. Let B be a block of G which maximizes the dimension $\dim(P[V(B)])$. By Theorem 1.6, we have $\dim(P) \leqslant \dim(P[V(B)]) + 2$. We also have $\mathrm{td}_2(B) \leqslant \mathrm{td}_2(G) = m$, so there exists a vertex $x \in V(B)$ such that $\mathrm{td}_2(B - x) \leqslant m - 1$. Hence for each $H \subseteq B - x$ we have $\mathrm{td}_2(H) \leqslant m - 1$, so by induction hypothesis we can apply Lemma 1.4 with $X = \{x\}$ to deduce that $\dim(P[V(B)]) \leqslant 2 \cdot (2^m - 2) = 2^{m+1} - 4$. Hence

$$\dim(P) \leqslant \dim(P[V(B)]) + 2 \leqslant 2^{m+1} - 4 + 2 = 2^{m+1} - 2.$$

□

Proof of Theorem 4.1. If P is a poset with an L_n-minor-free cover graph G, then by Theorem 4.2 we have $\mathrm{td}_2(G) \leqslant (n+1)(n^2+2)$, and by Lemma 4.7 we have $\dim(P) \leqslant 2^{(n+1)(n^2+2)+1} - 2$, so $d = 2^{(n+1)(n^2+2)+1} - 2$ satisfies the theorem. □

Let us mention another application of Theorem 4.2. It is a well-known fact that any two longest paths in a connected graph G intersect. This is equivalent to saying that if a connected graph G contains two disjoint paths each on n vertices, then it contains a path on $n + 1$ vertices. Ladder minors have a similar property.

Theorem 4.8 (Huynh, Joret, Micek, Seweryn, Wollan [12]). *For every n, there exists an integer k such that every 3-connected graph G which contains k pairwise disjoint copies of L_n as a minor, has an L_{n+1}-minor.*

The proof of Theorem 4.8 is not included in this thesis because of its technical nature.

4.2 Centered colorings

An alternative way to define treedepth is via centered colorings. A *vertex-coloring* of a graph G is a function $\phi\colon V(G) \to \mathbb{N}$, and the values of ϕ are called *colors*. Let G be a graph with a fixed vertex-coloring ϕ. In a subgraph $H \subseteq G$, a vertex $x \in V(G)$ is called a *center* if the color of x is unique in H, that is $\phi(x) \neq \phi(x')$ for every $x' \in V(H)\backslash\{x\}$. We call ϕ a *centered coloring* if every connected subgraph of G has a center. It turns out that for any graph

G, the minimum number of colors used by a centered coloring of G is equal to $\operatorname{td}(G)$.

We can analogously define a variant of centered coloring related to our graph parameter $\operatorname{td}_2(G)$. Let us call a vertex-coloring ϕ of G a 2-*connected centered coloring* if $\phi(x) \neq \phi(y)$ for each $xy \in E(G)$ and every 2-connected subgraph of G has a center.

Lemma 4.9. *The minimum number of colors used by a* 2*-connected centered coloring of a graph G is exactly* $\operatorname{td}_2(G)$*.*

Proof. We prove the lemma by induction on the number of vertices in G. The base case $V(G) = \varnothing$ is trivial.

For the inductive step, assume that $|V(G)| \geqslant 1$. Let m be the smallest number of colors in a 2-connected centered coloring of G, and let us show that $\operatorname{td}_2(G) = m$.

Suppose first that G does not have a cutvertex, so it is a single block. Fix a 2-connected centered coloring of G which uses m colors, and let x be a center of G. The coloring uses $m-1$ colors on $V(G - \{x\})$, so by induction hypothesis $\operatorname{td}_2(G - \{x\}) \leqslant m - 1$. Hence $\operatorname{td}_2(G) \leqslant \operatorname{td}_2(G - \{x\}) + 1 \leqslant m$. Now let $x \in V(G)$ be a vertex such that $\operatorname{td}_2(G) = \operatorname{td}_2(G - \{x\}) + 1$. By induction hypothesis, $G - \{x\}$ has a 2-connected centered coloring using $\operatorname{td}_2(G) - 1$ colors. Extend such a coloring by assigning a brand new color to x. The resulting coloring is a 2-connected centered coloring of G which uses $\operatorname{td}_2(G)$ colors, so $m \leqslant \operatorname{td}_2(G)$.

Now let $B_1, \ldots, B_k$ be the blocks of G and suppose that $k \geqslant 2$. Each of the blocks admits a 2-connected centered coloring using m colors, so by induction hypothesis, we have $\operatorname{td}_2(B_i) \leqslant m$ for each $i \in \{1, \ldots, k\}$, and hence $\operatorname{td}_2(G) \leqslant m$. By induction hypothesis, each block B_i admits a 2-connected centered coloring using at most $\operatorname{td}_2(G)$ colors. After renaming the colors in the colorings of the blocks, we may assume that they agree on the cutvertices of G. Combining the colorings we obtain a 2-connected centered coloring of G which uses $\operatorname{td}_2(G)$ colors, so $m \leqslant \operatorname{td}_2(G)$. $\square$

Centered colorings are related to linear colorings. A *linear coloring* of a graph G is a vertex-coloring of G such that every path in G has a center. We denote by $\chi_{\text{lin}}(G)$ the minimum number of colors used by a linear coloring of G. Every centered coloring is linear, and therefore we always have $\chi_{\text{lin}}(G) \leqslant \operatorname{td}(G)$. If P is a path on 2^m vertices, then $\chi_{\text{lin}}(P) > m$, and therefore the length of a longest path in a graph G is less than $2^{\chi_{\text{lin}}(G)}$. Hence the parameters $\chi_{\mathrm{lin}}(G)$ and $\mathrm{td}(G)$ are tied:

$$\chi_{\mathrm{lin}}(G) \leqslant \mathrm{td}(G) < 2^{\chi_{\mathrm{lin}}(G)}.$$

Analogously to linear colorings we can define *cycle centered coloring* as a vertex-coloring in which every cycle has a center. Note that in such a coloring some pairs of adjacent vertices may have the same color. Using Theorem 4.2, we show that the minimum number used by a cycle centered coloring of a graph G is tied to $\mathrm{td}_2(G)$.

Lemma 4.10. *Let G be a graph, let ϕ be a cycle centered coloring of G using at most m colors, and let $(P_1, P_2; Q_1, \ldots, Q_n)$ be an n-ladder model in G. If ϕ uses exactly the same set of colors on the paths $Q_1, \ldots, Q_n$, then $n < 2^m$.*

Proof. We prove the lemma by induction on m. The lemma holds in the base case $m = 0$: if ϕ uses 0 colors, then G must be an empty graph and thus $n = 0$. For the inductive step, let us assume that $m \geqslant 1$, and towards a contradiction, suppose that $n \geqslant 2^m$. Without loss of generality we assume that each of the paths P_1 and P_2 has an end in $V(Q_n)$. Consider the cycle $C = P_1 \cup P_2 \cup Q_1 \cup Q_n$, and let x be a center of C. The vertex x must be a center of the union $H := P_1 \cup P_2 \cup Q_1 \cup \cdots \cup Q_n$: the color of x is unique in C and if there existed $x' \in V(Q_i)$ with $\phi(x') = \phi(x)$, then by our assumption on ϕ we would find vertices of color $\phi(x)$ on both Q_1 and Q_n, contradicting x being a center of C. Since $n \geqslant 2^m$, our n-ladder model contains two disjoint 2^{m-1}-ladder models of the form $(P_1', P_2'; Q_1', \ldots, Q_{2^{m-1}}')$ where $P_i' \subseteq P_i$ for $i \in \{1, 2\}$, and $\{Q_1', \ldots, Q_{2^{m-1}}'\} \subseteq \{Q_1, \ldots, Q_n\}$. One of these models does not contain x, fix such $(P_1', P_2'; Q_1', \ldots, Q_{2^{m-1}}')$. By induction hypothesis, ϕ uses more than $m-1$ colors on that model, which together with the color $\phi(x)$ give more than m colors, contradiction. Hence indeed $n < 2^m$. $\square$

Theorem 4.11. *Let $\mathcal{G}$ be class of graphs. The following are equivalent:*

(1) *there exists an integer n such that no graph in $\mathcal{G}$ has an L_n-minor.*

(2) *there exists an integer m such that $\mathrm{td}_2(G) \leqslant m$ for every $G \in \mathcal{G}$,*

(3) *there exists an integer m such that every graph in $\mathcal{G}$ has a 2-connected centered coloring using at most m colors;*

(4) *there exists an integer m such that every graph in $\mathcal{G}$ has a cycle centered coloring using at most m colors.*

Proof. By Theorem 4.2 the items (1) and (2) are equivalent, and by Lemma 4.9 the items (2) and (3) are equivalent. Since every 2-connected centered coloring is a cycle centered coloring, the item (3) implies the item (4) To complete the proof we show that (4) implies (1). Namely, we show that if a graph G admits a cycle centered coloring using at most m colors, then G does not have an L_{4^m}-minor.

Suppose to the contrary that the graph G admits a cycle centered coloring $\phi\colon V(G) \to \{1, \dots, m\}$ and L_{4^m} is a minor of G. Let $(P_1, P_2; Q_1, \dots, Q_{4^m})$ be a 4^m-ladder model in G. The coloring ϕ uses a nonempty subset of $\{1, \dots, m\}$ on each $V(Q_i)$. Since there are $2^m - 1$ nonempty subsets of colors, by the pigeonhole principle there exists indices $1 \leqslant i_1 < \cdots < i_{2^m} \leqslant 4^m$ such that ϕ colors the paths $Q_{i_1}, \dots, Q_{i_{2^m}}$ with the same set of colors. Let P_1' and P_2' be $V(Q_{i_1})$–$V(Q_{i_{2^m}})$ paths contained in P_1 and P_2, respectively. This way we obtain a 2^m-ladder model $(P_1', P_2'; Q_{i_1}, \dots, Q_{i_{2^m}})$ which contradicts Lemma 4.10. This completes the proof. □

Let $\chi_{\mathrm{cyc}}(G)$ denote the minimum number of colors used by a cycle centered coloring of a graph G. We conclude this section with a short discussion about the asymptotic of the function tying $\mathrm{td}_2(G)$ with $\chi_{\mathrm{cyc}}(G)$.

As we have already mentioned, for any graph G we have $\mathrm{td}(G) \leqslant 2^{\chi_{\mathrm{lin}}(G)}$. What is the best bound on $\mathrm{td}(G)$ in terms of $\chi_{\mathrm{lin}}(G)$? Kun et al. [21] constructed graphs $R_1, R_2, \dots$ such that $\lim_{n\to\infty} \frac{\chi_{\mathrm{lin}}(R_n)}{\mathrm{td}(R_n)} = 2$, and they conjectured that $\mathrm{td}(G) \leqslant 2\,\chi_{\mathrm{lin}}(G)$ for every graph G. They also gave a polynomial bound on $\mathrm{td}(G)$ in terms of $\chi_{\mathrm{lin}}(G)$, and the best known bound is by Bose et al. [2], who showed that $\mathrm{td}(G) \in (\chi_{\mathrm{lin}}(G))^{10+o(1)}$. These results suggests that $\mathrm{td}_2(G)$ may be linearly bounded in terms of $\chi_{\mathrm{cyc}}(G)$.

Conjecture 4.12. *There exists a constant c such that for every graph G we have*

$$\mathrm{td}_2(G) \leqslant c \cdot \chi_{\mathrm{cyc}}(G).$$

Chapter 5

k-Outerplanarity

Felsner, Trotter and Wiechert [9] showed that posets with outerplanar cover graphs have dimension at most 4. A well-studied and useful generalization of outerplanar graphs are k-outerplanar graphs. A planar drawing of a graph is *k-outerplanar* if after k-fold removal of the vertices on the boundary of the outer face there are no vertices left, and a *k-outerplanar* graph is a graph which has a k-outerplanar drawing. For each $k \geqslant 1$, k-outerplanar graphs form a minor-closed class of graphs. In this chapter, we show that posets with k-outerplanar cover graphs have dimension $\mathcal{O}(k^3)$.

Theorem 5.1. *There exists a function $f(k) \in \mathcal{O}(k^3)$ such that every poset with a k-outerplanar cover graph has dimension at most $f(k)$.*

As a consequence of this result, we improve the bound on the dimension of posets with planar cover graphs in terms of their height.

Theorem 5.2. *There exists a function $g(h) \in \mathcal{O}(h^3)$ such that every poset of height h with a planar cover graph has dimension at most $g(h)$.*

Previously, the best known bound was $\mathcal{O}(h^6)$.

5.1 Min-max reduction and unfolding

In this section we introduce two standard techniques from dimension theory: min-max reduction and unfolding.

The *min-max dimension* of a poset P is $\dim_P(\mathrm{Min}(P), \mathrm{Max}(P))$. The following well-known lemma shows that in order to bound the dimension of a poset, it suffices to bound the min-max dimension of a poset with similar properties as P.

Lemma 5.3 (Min-max reduction)**.** *For each poset P there exists a poset P' such that*

(1) *the cover graph of P' can be obtained from the cover graph of P by adding zero or more degree-1 vertices,*

(2) *the height of P' is equal to the height of P, and*

(3) $\dim(P) \leqslant \dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P'))$.

Proof. Let P' be a superposet of P obtained by adding the following new elements to P: for every non-minimal element $x \in P$ introduce a new minimal element x^- covered only by x, and for every non-maximal element $x \in P$ introduce a new maximal element x^+ covering only x. Furthermore, for each $x \in \mathrm{Min}(P)$, let x^- denote the element x itself, and similarly, for each $x \in \mathrm{Max}(P)$ let x^+ denote the element x. Observe that for any elements a and b of P, if $a \leqslant b$ in P, then $a^- \leqslant b^+$ in P', and if $a \parallel b$ in P, then $a^- \parallel b^+$ in P'. The cover graph of P' is obtained from the cover graph of P by adding degree-1 vertices, and P' has the same height as P. It remains to show that $\dim(P) \leqslant d$, where $d = \dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P'))$.

Let $\{I'_1, \ldots, I'_d\}$ be a partition of $\mathrm{Inc}_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P'))$ into reversible sets. For each $j \in \{1, \ldots, d\}$, let $I_j = \{(a, b) \in \mathrm{Inc}(P) : (a^-, b^+) \in I'_j\}$. For every $(a, b) \in \mathrm{Inc}(P)$ we have $(a^-, b^+) \in \mathrm{Inc}_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P'))$, so $\{I_1, \ldots, I_d\}$ is a partition of $\mathrm{Inc}(P)$. Observe that each I_j is reversible in P; otherwise, some I_j would contain an alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ and the set I'_j would contain the alternating cycle $((a_1^-, b_1^+), \ldots, (a_k^-, b_k^+))$, contradicting its reversibility. Hence, $\dim(P) \leqslant d$. □

Let us observe that adding a degree-1 vertex to a graph preserves its k-outerplanarity. For suppose that G is a graph with a k-outerplanar drawing, and for each $i \in \{1, \ldots, k\}$, let V_i denote the set of vertices lying on the

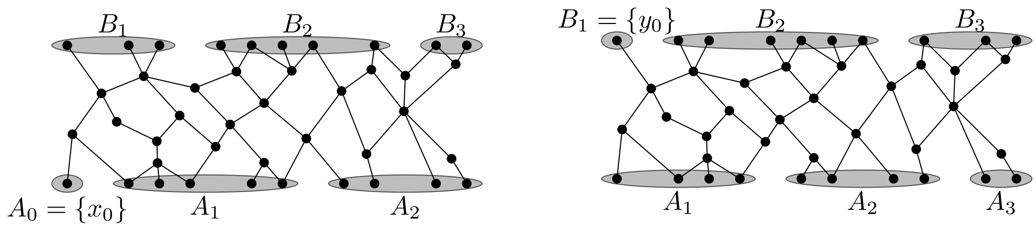


Figure 5.1: Unfolding a poset from a minimal element x_0 and from a maximal element y_0.

boundary of the outer face after $(i-1)$-fold removal of the vertices on the boundary of the outer face. Hence $V(G) = V_1 \cup \cdots \cup V_k$, and if G' is the graph obtained from G by adding a degree-1 vertex x attached to a vertex $y \in V_i$, then we can extend the drawing of G to a drawing of G' such that x is on the outer face of $G[V_i \cup \cdots \cup V_k]$. This way, in the iterative process of removing the vertices from the boundary of the outer face, the vertex x will be removed together with y in the i-th iteration, so the drawing of G' is k-outerplanar.

Let P be a connected poset with at least two elements (so that $\mathrm{Min}(P) \cap \mathrm{Max}(P) = \varnothing$). Let $x_0 \in \mathrm{Min}(P) \cup \mathrm{Max}(P)$. Define an infinite alternating sequence of sets $(A_0, B_1, A_1, B_2, \ldots)$ as follows. If $x_0 \in \mathrm{Min}(P)$, then let $A_0 = \{x_0\}$ and $B_1 = \mathrm{U}_P(x_0) \cap \mathrm{Max}(P)$, and if $x_0 \in \mathrm{Max}(P)$, then let $A_0 = \varnothing$ and $B_1 = \{x_0\}$. For every $i \geqslant 1$, define inductively

$$\begin{aligned} A_i &= (\mathrm{D}_P(B_i) \cap \mathrm{Min}(P)) \backslash A_{i-1}, \quad \text{and} \\ B_{i+1} &= (\mathrm{U}_P(A_i) \cap \mathrm{Max}(P)) \backslash B_i. \end{aligned}$$

(See Figure 5.1.) Such a sequence $(A_0, B_1, A_1, B_2, \ldots)$ is the *unfolding* of P from x_0. Since P is connected, the sets A_0, A_1, … partition $\mathrm{Min}(P)$, and the sets B_1, B_2, … partition $\mathrm{Max}(P)$. Note that the set A_0 may be empty, and since P is finite, starting from some point all sets in the unfolding are empty. Moreover, an element of A_{i_1} can be comparable with an element of B_{i_2} only if $i_2 \in \{i_1, i_1 + 1\}$, and for each $i \geqslant 1$, every element of A_i is comparable with an element of B_i and every element of B_i is comparable with an element of A_{i-1} (unless $x_0 \in \mathrm{Max}(P)$ and $i = 1$).

The following lemma is well-known.

Lemma 5.4. *Let $(A_0, B_1, A_1, B_2, \ldots)$ be an unfolding of a poset P. Then there* *exists an index $i \geqslant 1$ such that*

$$\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) \leqslant 2 \cdot \max\{\dim_P(A_i, B_i), \dim_P(A_i, B_{i+1})\}.$$

Proof. Let us partition $\mathrm{Inc}_P(\mathrm{Min}(P), \mathrm{Max}(P))$ into two sets $I_<$ and $I_>$, so that for each pair $(a, b) \in \mathrm{Inc}_P(\mathrm{Min}(P), \mathrm{Max}(P))$ with $a \in A_{i_1}$ and $b \in B_{i_2}$ we have $(a, b) \in I_<$ if A_{i_1} appears in the unfolding earlier than B_{i_2} (that is $i_1 < i_2$) and $(a, b) \in I_>$ if A_{i_1} appears in the unfolding later than B_{i_2} (that is $i_1 \geqslant i_2$). Let d denote the largest among all of the values $\dim_P(A_i, B_i)$ and $\dim_P(A_i, B_{i+1})$ with $i \geqslant 1$. To complete the proof it suffices to show that $\dim_P(I_<) \leqslant d$ and $\dim_P(I_>) \leqslant d$. The proofs of these bounds are dual, so we only show the latter.

For each $i \geqslant 1$, let $\{I_i^1, \ldots, I_i^d\}$ be a partition of $\mathrm{Inc}_P(A_i, B_i)$ into d reversible sets, and for every $j \in \{1, \ldots, d\}$, define $I^j = \bigcup_{i \geqslant 1} I_i^j$. Let $I_0 = \bigcup_{i_1 > i_2} \mathrm{Inc}_P(A_{i_1}, B_{i_2})$. Note that the sets I^1, …, I^d, I_0 partition $I_>$. We claim that for each $j \in \{1, \ldots, d\}$, the set $I^j \cup I_0$ is reversible. Towards a contradiction, suppose that there is an alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ in $I^j \cup I_0$. For each $i \in \{1, \ldots, k\}$, if i_1 is the index such that $a_i \in A_{i_1}$, then the index i_2 such that $b_i \in B_{i_2}$ satisfies $i_1 \geqslant i_2$ (because $(a_i, b_i) \in I_>$), and the index i_2 such that $b_{i+1} \in B_{i_2}$ satisfies $i_2 \in \{i_1, i_1 + 1\}$, so, in particular, $i_2 \geqslant i_1$. Since these inequalities hold cyclically for all i, there must exist an index i_1 such that for all $i \in \{1, \ldots, k\}$ we have $(a_i, b_i) \in I^j \cap \mathrm{Inc}(A_{i_1}, B_{i_1}) = I_{i_1}^j$, contradicting reversibility of $I_{i_1}^j$. Hence $I^j \cup I_0$ is indeed reversible, and in particular I^j is reversible. Therefore, $\{I^1 \cup I_0, I^2, \ldots I^d\}$ is a partition into d reversible sets witnessing that $\dim_P(I_>) \leqslant d$. $\square$

Lemma 5.4 can be reformulated as follows.

Lemma 5.5. *Let P be a connected poset with at least two elements, and let $x_0 \in \mathrm{Min}(P) \cup \mathrm{Max}(P)$. Then there exist $P' \in \{P, P^d\}$ and an index $i \geqslant 1$ such that in the unfolding $(A_0, B_1, A_1, B_2, \ldots)$ of P' from x_0 we have $x_0 \notin \mathrm{U}_{P'}(A_i)$ and*

$$\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) \leqslant 2 \cdot \dim_{P'}(A_i, B_i).$$

Proof. Without loss of generality assume that x_0 is a minimal element in P. Let $(A'_0, B'_1, A'_1, B'_2, \ldots)$ be the unfolding of P from x_0. Then the unfolding of P^d from x_0 is $(A''_0, B''_1, A''_1, B''_2, \ldots)$ where $A''_0 = \varnothing$ and for $i \geqslant 1$ we have $B''_i = A'_{i-1}$ and $A''_i = B'_i$. By Lemma 5.4 there exist $i \geqslant 1$ and $j \in \{i, i+1\}$

such that $\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) \leqslant 2 \cdot \dim_P(A'_i, B'_j)$. If $j = i$, then $x_0 \notin \mathrm{U}_P(A_i)$, so P and i satisfy the lemma, and if $j = i + 1$, then

$$\begin{aligned}\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) &\leqslant 2 \cdot \dim_P(A'_i, B'_{i+1}) \\ &= 2 \cdot \dim_{P^d}(B'_{i+1}, A'_i) = 2 \cdot \dim_{P^d}(A''_{i+1}, B''_{i+1})\end{aligned}$$

and $x_0 \notin \mathrm{D}_P(B'_{i+1}) = \mathrm{U}_{P^d}(A''_{i+1})$, so P^d and $i + 1$ satisfy the lemma. $\square$

Lemma 5.5 allows us to reduce a poset to another one whose min-max dimension is at most 2 times smaller, and which has stronger structural properties than the original poset.

Lemma 5.6. *Let P be a connected poset with at least two elements, let $x_0 \in \mathrm{Min}(P) \cup \mathrm{Max}(P)$, and let G be the cover graph of P. Then for some $P' \in \{P, P^d\}$ there exist a convex subposet Q of P' and a component C of $G - Q$ such that $x_0 \in V(C)$, $\mathrm{Max}(Q) \subseteq \mathrm{U}_{P'}(V(C))$, $\mathrm{Min}(Q) \cap \mathrm{D}_{P'}(V(C)) = \varnothing$, and*

$$\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) \leqslant 2 \cdot \dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q)).$$

The following easy lemma will be used in the proof of Lemma 5.6 (and also later on, in the proof of Theorem 5.1).

Lemma 5.7. *For every poset P with two subsets of elements A and B we have*

$$\dim_P(A, B) = \dim_P(A, B \cap \mathrm{U}_P(A))$$

Proof. Since $B \cap \mathrm{U}_P(A) \subseteq B$, we have $\dim_P(A, B \cap \mathrm{U}_P(A)) \leqslant \dim_P(A, B)$. Let $d = \dim_P(A, B \cap \mathrm{U}_P(A))$. It remains to argue that $\dim_P(A, B) \leqslant d$. Let $\{I_1, \ldots, I_d\}$ be a partition of $\mathrm{Inc}_P(A, B \cap \mathrm{U}_P(A))$ into reversible sets. If $((a_1, b_1), \ldots, (a_k, b_k))$ is an alternating cycle in $\mathrm{Inc}_P(A, B)$, then for each $i \in \{1, \ldots, k\}$ we have $a_{i-1} \leqslant b_i$ in P, and thus $b_i \in \mathrm{U}_P(A)$. Hence no alternating cycle in $\mathrm{Inc}_P(A, B)$ contains a pair from $\mathrm{Inc}_P(A, B \setminus \mathrm{U}_P(A))$, so the set $I_1 \cup \mathrm{Inc}_P(A, B \setminus \mathrm{U}_P(A))$ is reversible. Therefore the partition $\{I_1 \cup \mathrm{Inc}_P(A, B \setminus \mathrm{U}_P(A)), I_2, \ldots, I_d\}$ witnesses that $\dim_P(A, B) \leqslant d$. $\square$

Proof of Lemma 5.6. Apply Lemma 5.5 to P and x_0 to obtain $P' \in \{P, P^d\}$ and an index $i \geqslant 1$ so that for the unfolding $(A_0, B_1, A_1, B_2, \ldots)$ of P' from x_0 we have $x_0 \notin \mathrm{U}_{P'}(A_i)$ and

$$\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) = \dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P')) \leqslant 2 \cdot \dim_{P'}(A_i, B_i).$$

Let $Q = P'[\mathrm{U}_{P'}(A_i) \cap \mathrm{D}_{P'}(B_i)]$. We have $\mathrm{Min}(Q) = A_i$ since every element of A_i is comparable with an element of B_i in Q, and $\mathrm{Max}(Q) = B_i \cap \mathrm{U}_Q(A_i)$, so by Lemma 5.7 we have

$$\dim_{P'}(A_i, B_i) = \dim_{P'}(A_i, B_i \cap \mathrm{U}_{P''}(A_i)) = \dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q)).$$

Hence, $\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) \leqslant 2 \cdot \dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q))$.

Since $x_0 \notin \mathrm{U}_{P'}(A_i)$, it is not the case that x_0 is maximal and $i = 1$. In particular, $x_0 \notin Q$ and $\mathrm{Max}(Q) \subseteq \mathrm{U}_{P'}(A_{i-1})$. Let C be the component of $G - Q$ containing x_0. By definition of unfolding, for each $a \in A_{i-1}$ there is a (not necessarily witnessing) x_0–a path in $G - Q$, and every x_0–$\mathrm{U}_{P'}(A_i)$ path in G contains an element of Q. Hence, $\mathrm{Max}(Q) \subseteq \mathrm{U}_{P'}(A_{i-1}) \subseteq \mathrm{U}_{P'}(V(C))$ and $\mathrm{Min}(Q) \cap \mathrm{D}_{P'}(V(C)) = A_i \cap \mathrm{D}_{P'}(V(C)) = \varnothing$.

□

When P is a poset with a planar cover graph, Lemma 5.6 gives us a poset Q in which every maximal element is comparable with an element on the boundary of the outer face.

Lemma 5.8. *For every height-h poset P with a fixed planar drawing of its cover graph G, there exists a poset Q of height at most h such that its cover graph H is a subgraph of G, every maximal element of Q is comparable with an element on the boundary of the outer face in the inherited drawing of H, and*

$$\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) \leqslant 2 \cdot \dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q)).$$

Proof. If $\dim_P(\mathrm{Min}(P), \mathrm{Max}(P)) \leqslant 2$, then the lemma is satisfied with the empty poset as Q. Hence we assume that $\dim(\mathrm{Min}(P), \mathrm{Max}(P)) \geqslant 3$. By Lemma 1.5 we may assume that P is connected.

No two minimal elements of P are adjacent in G, so there must exists a non-minimal element on the boundary of the outer face of G. Let P' be a superposet of P obtained by adding a minimal element x_0 covered only by one non-minimal element of P on the boundary of the outer face of G, let G' denote the cover graph of P', and extend the drawing of G to a planar drawing of G' by adding x_0 and the incident edge on the outer face.

Let $P'' \in \{P', (P')^d\}$, $Q \subseteq P''$ and $C \subseteq G - Q$ be obtained by applying Lemma 5.5 to P' and x_0. We have

$$\begin{aligned}\dim_P(\mathrm{Min}(P), \mathrm{Min}(P)) &\leqslant \dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P')) \\ &= \dim_{P''}(\mathrm{Min}(P''), \mathrm{Max}(P'')) \\ &\leqslant 2 \cdot \dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q)),\end{aligned}$$

and the height of Q is at most h. The ground set of Q is disjoint from $V(C)$, so in particular, $x_0 \notin Q$, so the cover graph H of Q is a subgraph of G. It remains to argue that every $b \in \mathrm{Max}(Q)$ is comparable with an element on the boundary of the outer face of H.

Since $\mathrm{Max}(Q) \subseteq \mathrm{U}_{P''}(V(C))$, for every $b \in \mathrm{Max}(Q)$ there exists a witnessing path W from an element $x \in V(C)$ to b in P''. C is a connected subgraph of G' which contains the vertex x_0 belonging the the outer face of H, so the path W has to intersect the boundary of the outer face of H. This proves that every $b \in \mathrm{Max}(Q)$ is comparable in Q with an element on the boundary of the outer face of H. $\square$

5.2 The roadmap

In this section we formulate three lemmas and show how they imply Theorems 5.1 and 5.2. Lemma 5.10 is a result by Kozik, Micek and Trotter [20], and the proofs of Lemmas 5.9 and 5.11 are presented in the Sections 5.3 and 5.4, respectively.

Let P be a poset with a planar drawing of its cover graph, and let $I \subseteq \mathrm{Inc}(P)$. If x_0 and y_0 are two vertices on the boundary of the outer face such that $x_0 < y_0$ in P and for each $(a, b) \in I$ we have $a \leqslant y_0$ and $b \geqslant x_0$ in P, we say that I is *doubly exposed by* (x_0, y_0) in the drawing. The pair (x_0, y_0) is called a *min-max pair* if $x_0 \in \mathrm{Min}(P)$ and $y_0 \in \mathrm{Max}(P)$. Note that if (x_0, y_0) is a min-max pair, then $P - \{x_0, y_0\}$ is a convex subposet of P, and thus, the cover graph of $P - \{x_0, y_0\}$ is a subgraph of the cover graph of P. The following is the first lemma used in the proof of the main theorem.

Lemma 5.9. *For every poset P with a k-outerplanar cover graph, there exist a poset R, a drawing of the cover graph of R and a subset $I \subseteq \mathrm{Inc}(R)$ doubly exposed by a min-max pair (x_0, y_0) such that the inherited drawing of the cover graph of $R - \{x_0, y_0\}$ is k-outerplanar, and*

$$\dim(P) \leqslant 4k \cdot \dim_R(I).$$

A *standard example* of size n is a poset consisting of n minimal elements $a_1, \ldots, a_n$ and n maximal elements $b_1, \ldots, b_n$ such that $a_i < b_j$ if and only if $i \neq j$. For $n \geqslant 3$, a standard example of size n is a smallest and canonical poset of dimension n. Given a poset P, a subset $I \subseteq \mathrm{Inc}(P)$ such that for any distinct $(a_1, b_1), (a_2, b_2) \in I$ in we have $a_1 < b_2$ and $a_2 < b_1$ in P is also called a *standard example*. For any subset $I \subset \mathrm{Inc}(P)$, we denote by $\rho_P(I)$ the size of a largest standard example contained in I. The second lemma is a result by Kozik, Micek and Trotter [20].

Lemma 5.10 ([20]). *Let P be a poset with a planar drawing of its cover graph, and let $I \subseteq \mathrm{Inc}(P)$ be a doubly exposed set in the drawing. Then*

$$\dim_P(I) \leqslant \rho_P(I)^2.$$

We note that very recently, Micek, Smith Blake and Trotter [23] announced that they improved the bound in Lemma 5.10 from quadratic to linear. This improvement, together with our proof gives an $\mathcal{O}(k^2)$ bound for the dimension of posets with k-outerplanar cover graphs.

The third lemma is as follows.

Lemma 5.11. *Let P be a poset with a planar drawing of its cover graph, let $I \subseteq \mathrm{Inc}(P)$ be a set doubly exposed by a min-max pair (x_0, y_0) such that the inherited drawing of the cover graph of $P - \{x_0, y_0\}$ is k-outerplanar. Then*

$$\rho_P(I) < 440(k+1).$$

The multiplicative factor 440 in Lemma 5.11 is quite large and suboptimal, but we did not try to optimize it as the proof is already long and technical.

In the proof of Lemma 5.11, we show that if in a poset with a planar drawing of its cover graph there is a doubly exposed standard example of size $440(k+1)$, then it contains a subposet isomorphic to Kelly_{4k+5}, where Kelly_n denotes the *Kelly poset* defined as follows. The ground set of Kelly_n is the family

$$\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\} \cup \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\} \cup \{\boldsymbol{c}_2, \ldots, \boldsymbol{c}_{n-2}\} \cup \{\boldsymbol{d}_2, \ldots, \boldsymbol{d}_{n-2}\}$$

of subsets of $\{1, \ldots, n\}$, where $\boldsymbol{a}_i = \{i\}$, $\boldsymbol{b}_i = \{1, \ldots, n\} \backslash \{i\}$, $\boldsymbol{c}_i = \{1, \ldots, i\}$, $\boldsymbol{d}_i = \{i+1, \ldots, n\}$, and we have $\boldsymbol{x} \leqslant \boldsymbol{y}$ in Kelly_n when $\boldsymbol{x} \subseteq \boldsymbol{y}$ (See Figure 5.2). Kelly posets were discovered by Kelly [19] as a construction of planar posets with unbounded dimension (since Kelly_n contains a standard example of size n, its dimension is at least n). We show that the occurrence of a subposet isomorphic to Kelly_{4k+5} prevents the drawing of the cover graph from being k-outerplanar.

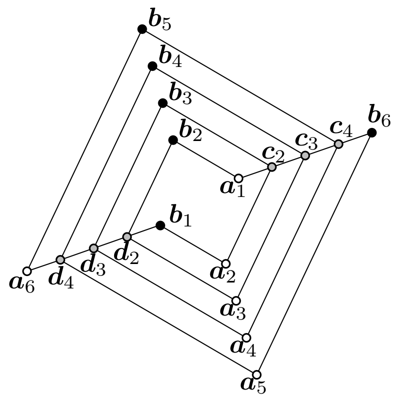


Figure 5.2: The poset Kelly_6.

Proof of Theorem 5.1. Let P be a poset with a k-outerplanar cover graph. By Lemma 5.9 there exist a poset R with a planar drawing of its cover graph and a subset $I \subseteq \mathrm{Inc}(P)$ which is doubly exposed by a min-max pair (x_0, y_0) such that $\dim(P) \leqslant 4k \cdot \dim_R(I)$ and the inherited drawing of the cover graph of $R - \{x_0, y_0\}$ is k-outerplanar. Hence, by Lemmas 5.10 and 5.11,

$$\dim(P) \leqslant 4k \cdot \dim_R(I) \leqslant 4k \cdot \rho_R(I)^2 < 4k \cdot (440(k+1))^2.$$

□

Proof of Theorem 5.2. Let P be a height-h poset with a planar cover graph. Adding degree-1 vertices to a graph preserves its planarity, so by Lemma 5.3 there exists a height-h poset P' with a planar cover graph G such that $\dim(P) \leqslant \dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P'))$. Apply Lemma 5.8 to such P' to obtain a poset Q of height at most h with a planar drawing of its cover graph H such that every $b \in \mathrm{Max}(Q)$ is comparable with an element on the boundary of the outer face and

$$\begin{aligned}\dim(P) &\leqslant \dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P')) \\ &\leqslant 2 \cdot \dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q)) \leqslant 2 \cdot \dim(Q).\end{aligned}$$

We claim that the drawing of H is $(2h-1)$-outerplanar. For every $x \in Q$ there exist $b \in \mathrm{Max}(Q)$ and an element y from the boundary of the outer face such that $x \leqslant b$ and $y \leqslant b$ in Q. Since the height of Q is at most h, the union of witnessing paths from x to b and from y to b contains an x–y path on at most $2h-1$ vertices. Hence, after removing the vertices from the boundary of the outer face in H at most $2h-1$ times, x will be removed. This proves that H is $(2h-1)$-outerplanar.

We just showed that for every height-h poset P with a planar cover graph there exists a poset Q with a $(2h-1)$-outerplanar cover graph such that $\dim(P) \leqslant 2 \cdot \dim(Q)$. Hence, if $f(k) \in \mathcal{O}(k^3)$ is a function satisfying Theorem 5.1, then the function $g(h) := 2 \cdot f(2h-1)$ satisfies Theorem 5.2. $\square$

5.3 Reduction to doubly exposed posets

In this section we prove Lemma 5.9.

Let P be a poset with a k-outerplanar cover graph. If $\dim(P) \leqslant 4k$, then the lemma is satisfied by any poset R with a k-outerplanar cover graph and any set I which is doubly exposed by a min-max pair (even $I = \varnothing$). Therefore we may assume that $\dim(P) > 4k$. Adding degree-1 vertices preserves k-outerplanarity, so by Lemma 5.3 there exists a poset P' with a k-outerplanar cover graph such that

$$4k < \dim(P) \leqslant \dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P')).$$

Let G be the cover graph of P' and let us fix a k-outerplanar drawing of G. When a planar drawing of a graph G' is clear from the context (for instance, when $G' \subseteq G$), we denote by $\partial G'$ the set of vertices of G' which lie on the boundary of the outer face in the drawing of G'.

By Lemma 5.8, there exists a poset Q such that the cover graph H of Q is a subgraph of G, every $b \in \mathrm{Max}(Q)$ is comparable in Q with an element in ∂H and

$$\dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P')) \leqslant 2 \cdot \dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q)).$$

Fix such Q and H.

Let $H_1 = H$, and define recursively $H_i = H_{i-1} - \partial H_{i-1}$ for $i \in \{2, \ldots, k\}$. Since the drawing of G is k-outerplanar, the inherited drawing of H is k-outerplanar as well, so the sets $\partial H_1, \ldots, \partial H_k$ partition the set $V(H)$. Define a function $\alpha\colon \mathrm{Min}(Q) \to \{1, \ldots, k\}$ such that for each $a \in \mathrm{Min}(Q)$, $\alpha(a)$ is the smallest index i such that a is comparable with an element from ∂H_i in Q. For each $i \in \{1, \ldots, k\}$, let $A_i = \alpha^{-1}(i)$, let $Q'_i = Q[\mathrm{U}_Q(A_i)]$, and let H'_i denote its cover graph $H[\mathrm{U}_Q(A_i)]$ of Q'_i. Observe that $H'_i \subseteq H_i$. Every $a \in \mathrm{Min}(Q'_i)$ is comparable with an element of $\partial H'_i$: since $\mathrm{Min}(Q'_i) = A_i$, there exists $x \in \partial H_i$ comparable with a, and since $H'_i \subseteq H_i$, such an element x belongs to $\partial H'_i$.

Let Q' denote one of the posets $Q'_1, \ldots, Q'_k$ which has the largest min-max dimension. Let H' denote the cover graph of Q'. We have

$$\begin{aligned}
\dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q)) &\leqslant \sum_{i=1}^{k} \dim_Q(A_i, \mathrm{Max}(Q)) \\
&= \sum_{i=1}^{k} \dim_Q(A_i, \mathrm{U}_Q(A_i) \cap \mathrm{Max}(Q)) \quad \text{by Lemma 5.7} \\
&= \sum_{i=1}^{k} \dim_{Q'_i}(\mathrm{Min}(Q'_i), \mathrm{Max}(Q'_i)) \\
&\leqslant k \cdot \dim_{Q'}(\mathrm{Min}(Q'), \mathrm{Max}(Q')),
\end{aligned}$$

and every minimal element of Q' is comparable with an element from $\partial H'$. Furthermore, since Q' is a convex subposet of Q, also every maximal element of Q' is comparable with an element from $\partial H'$.

We already know that

$$\begin{aligned}
4k < \dim(P) &\leqslant \dim_{P'}(\mathrm{Min}(P'), \mathrm{Max}(P')) \\
&\leqslant 2 \cdot \dim_Q(\mathrm{Min}(Q), \mathrm{Max}(Q)) \\
&\leqslant 2k \cdot \dim_{Q'}(\mathrm{Min}(Q'), \mathrm{Max}(Q')),
\end{aligned}$$

so $\dim_{Q'}(\mathrm{Min}(Q'), \mathrm{Max}(Q')) > 2$. By Lemma 1.5, Q' has a component Q'' of the same min-max dimension as Q'. Fix such a component Q'', so that

$$2 < \dim_{Q'}(\mathrm{Min}(Q'), \mathrm{Max}(Q')) = \dim_{Q''}(\mathrm{Min}(Q''), \mathrm{Max}(Q'')).$$

and let H'' denote the cover graph of Q''.

It remains to construct a poset R with a subset $I \subseteq \mathrm{Inc}(R)$ doubly exposed by a min-max pair (x_0, y_0) such that $\dim_{Q''}(\mathrm{Min}(Q''), \mathrm{Max}(Q'')) \leqslant 2 \cdot \dim_R(I)$ and the cover graph of $R - \{x_0, y_0\}$ is k-outerplanar. To achieve this we need to unfold the poset.

Since $\dim_{Q''}(\mathrm{Min}(Q''), \mathrm{Max}(Q'')) > 2$, Q'' is not a 1-element poset, so H'' must contain two adjacent vertices on the boundary of the outer face. At most one of those vertices can be a minimal element of Q'', so $\partial H''$ contains a non-minimal element of Q''. Let Q''_+ be a superposet of Q'' obtained by adding a minimal element x_0 covered by a non-minimal element of Q'' from

$\partial H''$, and let H''_+ denote the cover graph of Q''_+. Extend the drawing of H'' to a planar drawing of H''_+ with x_0 on the boundary of the outer face. Let $Q'''_+ \in \{Q''_+, (Q''_+)^d\}$, $R_0 \subseteq Q'''_+$ and $C \subseteq H''_+ - R_0$ be obtained by applying Lemma 5.6 to Q''_+ and x_0, so that

$$\begin{aligned} 2 < \dim_{Q''}(\mathrm{Min}(Q''), \mathrm{Max}(Q'')) &\leqslant \dim_{Q''_+}(\mathrm{Min}(Q''_+), \mathrm{Max}(Q''_+)) \\ &\leqslant 2 \cdot \dim_{R_0}(\mathrm{Min}(R_0), \mathrm{Max}(R_0)). \end{aligned}$$

In particular, $\dim_{R_0}(\mathrm{Min}(R_0), \mathrm{Max}(R_0)) > 1$, so R_0 is not empty. Let J_0 denote the cover graph of R_0, and note that J_0 is a subgraph of H'' and hence the induced drawing of J_0 is k-outerplanar. We shall construct the poset R as a superposet of R_0 obtained by adding a minimal element x_0 and a maximal element y_0 so that $\mathrm{Inc}_{R_0}(\mathrm{Min}(R_0), \mathrm{Max}(R_0))$ is doubly exposed by (x_0, y_0).

The graph C is a component of $H''_+ - R_0$, so in H''_+, the vertices of C are adjacent only to each other and to elements of R_0. Since $\mathrm{D}_{Q'''_+}(V(C)) \cap \mathrm{Min}(R_0) = \varnothing$, no element of R_0 is covered by an element of $V(C)$ in Q'''_+. In particular, $V(C)$ is convex in Q'''_+.

We distinguish two subsets D_1 and D_2 of ∂J_0. Let D_1 denote the set of those vertices $y \in \partial J_0$ which cover some element of $V(C)$ in Q'''_+, and let $D_2 = \partial(H''_+[V(C) \cup V(J_0)]) \backslash V(C)$. Obtain the poset R from R_0 by adding a minimal element x_0 covered by the elements in the set $\mathrm{Min}(R_0[D_1])$ and a maximal element y_0 covering the elements in the set $\mathrm{Max}(R_0[D_2])$, see Figure 5.3. This way we have $D_1 \subseteq \mathrm{U}_R(x_0)$. Since $\mathrm{Max}(R_0) \subseteq \mathrm{U}_{Q'''_+}(V(C))$, we have $\mathrm{Max}(R_0) \subseteq \mathrm{U}_{Q'''_+}(D_1)$, and therefore $\mathrm{Max}(R_0) \subseteq \mathrm{U}_R(x_0)$. Moreover, for every $a \in \mathrm{Min}(R_0)$, a witnessing path in Q'''_+ from a to an element of $\partial H''$ is disjoint from C, and thus intersects $\partial(H''[V(C) \cup V(J_0)])$ in an element of D_2. Since $D_2 \subseteq \mathrm{D}_R(y_0)$, this implies $\mathrm{Min}(R_0) \subseteq \mathrm{D}_R(y_0)$. Finally, observe that some element of D_2 is adjacent to a vertex of C in H'', so $D_1 \cap D_2 \neq \varnothing$. Since $D_1 \subseteq \mathrm{U}_R(x_0)$ and $D_2 \subseteq \mathrm{D}_R(y_0)$, this implies that $x_0 < y_0$ in R.

It remains to show that there exists a planar drawing of the cover graph of R with x_0 and y_0 on the boundary of the outer face. The cover graph of $R - \{y_0\}$ is a minor of H''_+ obtained by contracting all edges in $V(C)$ to x_0 and deleting some vertices and edges. Since C contains the vertex x_0 which lies in $\partial H''_+$, the drawing of J_0 can be extended to a planar drawing of the cover graph of $R - \{y_0\}$ with x_0 in the same point as in the drawing of H''_+, such that x_0 and all elements of D_2 are still on the boundary of the outer face. Since y_0 is adjacent in the cover graph of R only to elements in D_2, we

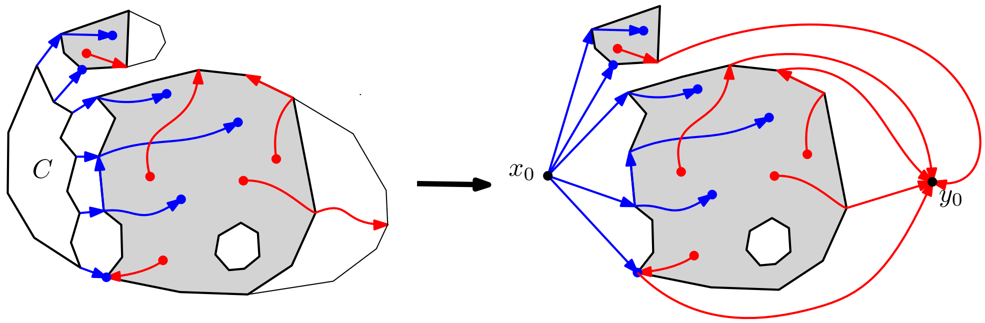


Figure 5.3: Obtaining R from Q'''_+. The gray part represents R_0, the blue vertices are the elements of $\mathrm{Max}(R_0)$, the red vertices are the elements of $\mathrm{Min}(R_0)$, and an arrow from x to y represents a witnessing path from x to y.

can extend the drawing of the cover graph of $R - \{y_0\}$ as described above to a planar drawing of R with x_0 and y_0 on the boundary of the outer face.

5.4 From a standard example to a Kelly subposet

In this section we prove Lemma 5.11. The setting of this lemma is the same as the one considered in [20], and we use some terminology and notation from there. However, our terminology and notation are not completely consistent with the final version of [20] as our proof is based on an early version of that manuscript.

Throughout this section we assume that P is a poset with a fixed planar drawing of its cover graph G, and $x_0 \in \mathrm{Min}(P)$ and $y_0 \in \mathrm{Max}(P)$ are two elements of P with $x_0 < y_0$ in P which lie on the boundary of the outer face in the drawing of G. Let $A = \mathrm{D}_P(y_0)$ and $B = \mathrm{U}_P(x_0)$, so that every set doubly exposed by (x_0, y_0) is a subset of $\mathrm{Inc}_P(A, B)$. Thus, we need to show that if $\mathrm{Inc}_P(A, B)$ contains a standard example of size $440(k + 1)$ then the inherited drawing of the cover graph of $P - \{x_0, y_0\}$ is not k-outerplanar.

If H is a nonempty subgraph of a witnessing path in P, we denote by $\min(H)$ the only minimal element of $P[V(H)]$, and by $\max(H)$ the only maximal element of $P[V(H)]$.

Whenever $x \leqslant y$ in P, there exists at least one witnessing path from x to y in P. We find it convenient to fix a "canonical" witnessing path $W(x, y)$ from x and y. We require these paths to have the property that the intersection $W(x_1, y_1) \cap W(x_2, y_2)$ of any two of them is either empty or a path of the form $W(x, y)$. One way to construct such paths is as follows. For any $x, y \in P$ with $x \leqslant y$ in P, let $W(x, y)$ be the witnessing path $z_0 \cdots z_p$ with $z_0 = x$ and $z_p = y$ for which the sequence $(z_0, \ldots, z_p)$ is earliest in the lexicographical order with respect to any fixed linear order on the ground set of P. This way for any $i, j \in \{0, \ldots, p\}$ with $i \leqslant j$, the z_i–z_j subpath of $W(x, y)$ is $W(z_i, z_j)$. Hence, if the intersection $W(x_1, y_1) \cap W(x_2, y_2)$ is nonempty, then $W(x_1, y_1) \cap W(x_2, y_2) = W(x, y)$ where $x = \min(W(x_1, y_1) \cap W(x_2, y_2))$ and $y = \max(W(x_1, y_1) \cap W(x_2, y_2))$.

By our choice of the witnessing paths $W(x, y)$, for any $a_1, a_2 \in A$, the intersection $W(a_1, y_0) \cap W(a_2, y_0)$ is a path with an end in y_0. Hence we can define a rooted tree

$$S = \bigcup_{a \in A} W(a, y_0)$$

with y_0 as the root. Similarly we define a rooted tree

$$T = \bigcup_{b \in B} W(x_0, b)$$

with x_0 as the root. Observe that for any vertices x and y, if x is a descendant of y in S or an ancestor of y in T, then $x \leqslant y$ in P. We refer to S as the *red tree* and to T as the *blue tree*.

In the drawing of G, add in the outer face an imaginary edge $e_{-\infty}$ attached to x_0 and an imaginary edge $e_{+\infty}$ attached to y_0. We use the imaginary edges to define partial orderings of the vertex sets of the trees S and T. For $U \in \{T, S\}$, we define a strict partial order $\prec_U$ on $V(U)$ as follows. Let x and y be two vertices of U, let z be their lowest common ancestor and let e be an edge between z and the parent of z in U (or the imaginary edge incident with z if z is the root of U). We write $x \prec_U y$ if $z \notin \{x, y\}$ and the edge e and the paths zUx and zUy leave the vertex in a clockwise manner in our drawing. See Figure 5.4. Clearly, $\prec_U$ is a strict partial order. Observe that if $x, y \in V(U)$ are incomparable in P, then none of them is an ancestor of the other in U, and therefore either $x \prec_U y$ or $y \prec_U x$.

For a cycle C in G, the *region bounded by* C is the bounded face of C together with the points on the closed curve representing C. Clearly, every connected subgraph of G which contains a vertex in the region bounded by C and a vertex outside the region bounded by C has a nonempty intersection with C. For a subset X of elements of P we say that a vertex x is

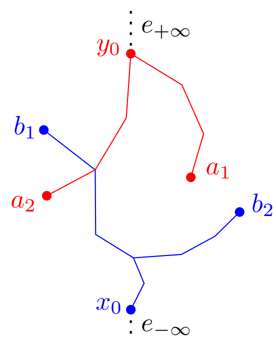


Figure 5.4: $a_1 \prec_S a_2$ and $b_1 \prec_T b_2$.

enclosed by X if there exists a cycle C in G such that $V(C) \subseteq X$ and x lies in the region bounded by C.

Lemma 5.12. *Let* $\{(a_1, b_1), \ldots, (a_m, b_m)\} \subseteq \mathrm{Inc}_P(A, B)$ *be a standard example and let* $i, j \in \{1, \ldots, m\}$ *be distinct. Then* a_i *is not enclosed by* $\mathrm{U}_P(a_j)$ *and* b_i *is not enclosed by* $\mathrm{D}_P(b_j)$.

Proof. Towards a contradiction, suppose that a_i is enclosed by $\mathrm{U}_P(a_j)$. Let C be a cycle in G such that $V(C) \subseteq \mathrm{U}_P(a_j)$ and a_i lies in the region bounded by C. The graph $H = W(x_0, b_j) \cup W(a_i, b_j)$ is a connected subgraph of G. Since x_0 lies on the boundary of the outer face of G, either $x_0 \in V(C)$ or x_0 lies outside the region bounded by C, Hence H intersects C in a vertex z. Since $V(C) \subseteq \mathrm{U}_P(a_j)$ we have $a_j \leqslant z$ in P, and since $V(H) \subseteq \mathrm{D}_P(b_j)$, we have $z \leqslant b_j$ in P. This implies $a_j \leqslant z \leqslant b_j$ in P, which is a contradiction. Hence a_i is not enclosed by $\mathrm{U}_P(a_j)$. The proof that b_i is not enclosed by $\mathrm{D}_P(b_j)$ is dual. $\square$

We generalize the notation xUy for the x–y subpath of a tree U to multiple trees. If trees $U_1, \ldots, U_p$ are subgraphs of G and $z_0, \ldots, z_p$ are vertices of G with $\{z_{i-1}, z_i\} \subseteq V(U_i)$ for each $i \in \{1, \ldots, p\}$, then by $z_0U_1z_1U_2 \ldots U_pz_p$ we denote the union $\bigcup_{i=1}^{p} z_{i-1}U_iz_i$. We only use this notation to denote a path or a cycle.

Lemma 5.13. *Let* $\{(a_1, b_1), \ldots, (a_m, b_m)\} \subseteq \mathrm{Inc}_P(A, B)$ *be a standard example, let* $i, j, k \in \{1, \ldots, m\}$*, and let* W *be a witnessing path in* P.

(1) *If* $a_i \prec_S a_j \prec_S a_k$ *and* W *intersects both* a_iSy_0 *and* a_kSy_0*, then* W *intersects* a_jSy_0.

(2) *If $b_i \prec_T b_j \prec_T b_k$ and W intersects x_0Tb_i and x_0Tb_k, then W intersects x_0Tb_j.*

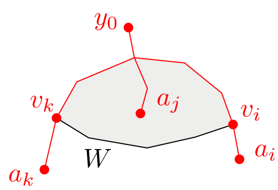


Figure 5.5: The vertex a_j lies in the region bounded by the cycle $v_iWv_kSv_i$.

Proof. For the proof of (1), we may assume that W has its ends on a_iSy_0 and a_kSy_0 and no inner vertex of W lies on any of these two paths. Let v_i and v_k denote the ends of W on the paths a_iSy_0 and a_kSy_0 respectively. If W is disjoint from a_jSy_0, then none of the vertices v_i and v_k is an ancestor of a_j in S, so we have $v_i \prec_S a_j \prec_S v_k$ and a_j lies in the region bounded by the cycle $C = v_iWv_kSv_i$. See Figure 5.5. However, if $v_i < v_k$ in P, then $V(C) \subseteq \mathrm{U}_P(v_i) \subseteq \mathrm{U}_P(a_i)$, and if $v_k < v_i$ in P, then $V(C) \subseteq \mathrm{U}_P(v_k) \subseteq \mathrm{U}_P(a_k)$. By Lemma 5.12 none of these can hold, so W must intersect a_jSy_0. This proves (1), and the proof of (2) is dual. $\square$

For every pair of elements $a \in A$ and $b \in B$ with $a \leqslant b$ in P we define two vertices $v(a,b)$ and $u(a,b)$ as follows. If the paths aSy_0 and x_0Tb intersect, then let $v(a,b)$ and $u(a,b)$ be one and the same arbitrary vertex on $aSy_0 \cap x_0Tb$, and if the paths aSy_0 and x_0Tb are disjoint, then let $v(a,b) = \max(W(a,b) \cap aSy_0)$ and $u(a,b) = \min(W(a,b) \cap x_0Tb)$. This way we have $a \leqslant v(a,b) \leqslant u(a,b) \leqslant b$ in P. The *separating path* $N(a,b)$ associated with the comparability $a \leqslant b$ is an x_0–y_0 path defined as $N(a,b) = x_0TuWvSy_0$, where $u = u(a,b)$, $v = v(a,b)$ and $W = W(a,b)$. (See Figure 5.6.) The witnessing paths $x_0Tu(a,b)$, $W(v(a,b), u(a,b))$ and $v(a,b)Sy_0$ are referred to as the *blue part*, the *black part* and the *red part* of $N(a,b)$, respectively.

Every x_0–y_0 path in G splits the graph into two parts: "left" and "right". To formalize this, let N be an x_0–y_0 path in G and let $z \in V(G)\backslash V(N)$. Choose a z–$V(N)$ path M and let w denote the end of M lying on N. Let $wNe_{-\infty}$ denote the path obtained from wNx_0 by adding the edge $e_{-\infty}$ attached to x_0, and let $wNe_{+\infty}$ denote the path obtained from wNy_0 by adding the edge $e_{+\infty}$ attached to y_0. Since the drawing of G is planar and the vertices x_0 and y_0 lie on the boundary of the outer face, either for every choice

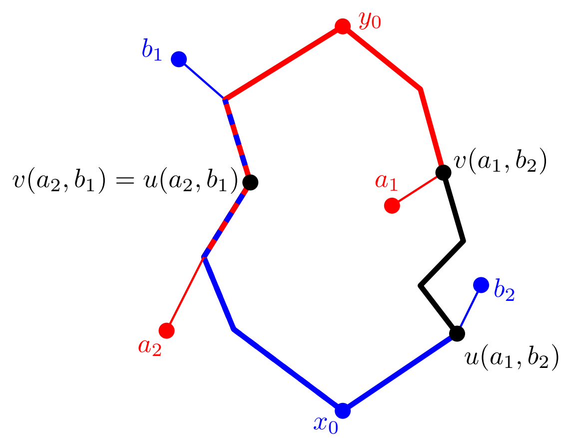


Figure 5.6: $a_2 \leqslant b_1$ and $a_1 \leqslant b_2$ in P. The x_0–y_0 paths $N(a_2, b_1)$ and $N(a_1, b_2)$ are bolded.

of M the paths $wNe_{-\infty}$, $wNe_{+\infty}$ and M leave the vertex w in a clockwise manner, or for every choice of M the paths $wNe_{-\infty}$, $wNe_{+\infty}$ and M leave the vertex w in a counter-clockwise manner. In the former case we say that z is *right of* N and in the latter case, we say that z is *left of* N. For instance, in Figure 5.6, the vertices a_2 and b_1 are left of $N(a_2, b_1)$, and the vertices a_1 and b_2 are right of $N(a_2, b_1)$.

The following is a simple but very useful consequence of the definition of being left/right to an x_0–y_0 path.

Lemma 5.14. *Let* $\{(a_1, b_1), \ldots, (a_m, b_m)\} \subseteq \mathrm{Inc}_P(A, B)$ *be a standard example and let* $i, j \in \{1, \ldots, m\}$ *be distinct.*

(1) *For every* $v \in A$*, if* vSy_0 *is disjoint from the black and the blue part of* $N(a_i, b_j)$*, then* $v(a_i, b_j) \prec_S v$ *if and only if* v *is left of* $N(a_i, b_j)$*.*

(2) *For every* $u \in B$*, if* x_0Tu *is disjoint from the red and the black part of* $N(a_i, b_j)$*, then* $u \prec_T u(a_i, b_j)$ *if and only if* u *is left of* $N(a_i, b_j)$*.*

Proof. The items (1) and (2) are dual, so we only prove (1). Let $w = \min(vSy_0 \cap N(a_i, b_j))$, so that vSw intersects $N(a_i, b_j)$ only in w. Since vSy_0 is disjoint from the black and the blue part of $N(a_i, b_j)$, the vertex w lies on the red part of $N(a_i, b_j)$ and is distinct from $v(a_i, b_j)$. Hence, the equivalence of $v(a_i, b_j) \prec_S v$ and v being left of $N(a_i, b_j)$ is a tautology. □

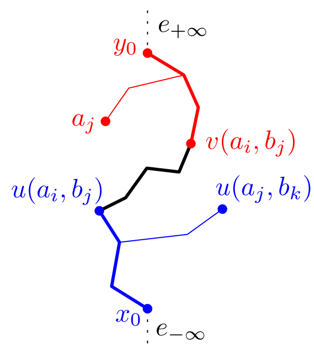


Figure 5.7: The vertex a_j is left of $N(a_i, b_j)$ and the vertex $u(a_j, b_k)$ is not.

Lemma 5.15. *Let $\{(a_1, b_1), \dots, (a_m, b_m)\} \subseteq \mathrm{Inc}_P(A, B)$ be a standard example, and let $i, j, k \in \{1, \dots, m\}$ be such that $a_i \prec_S a_j$ and $b_j \prec_T b_k$. Then $v(a_i, b_j) \leqslant u(a_j, b_k)$ in P.*

Proof. The vertex $v(a_i, b_j)$ is not an ancestor of a_j in S as that would imply $a_j \leqslant v(a_i, b_j) \leqslant b_j$ in P. Since $v(a_i, b_j)$ is an ancestor of a_i in S and $a_i \prec_S a_j$ in P, this implies $v(a_i, b_j) \prec_S a_j$. The path $a_j S y_0$ is disjoint from the black and the blue part of $N(a_i, b_j)$ as otherwise we would have $a_j \leqslant u(a_i, b_j) \leqslant b_j$ in P. Hence, by Lemma 5.14, a_j is left of $N(a_i, b_j)$.

Since $b_j \prec_T b_k$, it is impossible for $u(a_j, b_k) \prec_T u(a_i, b_j)$ to hold. Hence, if $u(a_j, b_k)$ is left of $N(a_i, b_j)$, then by Lemma 5.14, the path $x_0 T u(a_j, b_k)$ intersects the blue or the black part of $N(a_i, b_j)$, and therefore we have $v(a_i, b_j) \leqslant u(a_j, b_k)$ in P, so the lemma is satisfied. Hence we assume that $u(a_j, b_k)$ is not left of $N(a_i, b_j)$ (See Figure 5.7). As a_j is left of $N(a_i, b_j)$, the witnessing path $W(a_j, u(a_j, b_k))$ intersects $N(a_i, b_j)$ in a vertex z. The vertex z does not lie on the black or the blue part of $N(a_i, b_j)$ as that would imply $a_j \leqslant z \leqslant u(a_i, b_j) \leqslant b_j$ in P. Hence z lies on the red part of $N(a_i, b_j)$, and therefore we have $v(a_i, b_j) \leqslant z \leqslant u(a_j, b_k)$ in P. $\square$

In a standard example $\{(a_1, b_1), \dots, (a_m, b_m)\} \subseteq \mathrm{Inc}_P(A, B)$, for distinct $i, j \in \{1, \dots, m\}$, we have $a_i \parallel a_j$ and $b_i \parallel b_j$ in P, and therefore we have either $a_i \prec_S a_j$ or $a_j \prec_S a_i$, and we have either $b_i \prec_T b_j$ or $b_j \prec_T b_i$.

Lemma 5.16. *Let $\{(a_1, b_1), \dots, (a_m, b_m)\} \subseteq \mathrm{Inc}_P(A, B)$ be a standard example and let $i, j \in \{1, \dots, m\}$ be distinct. Then $a_i \prec_S a_j$ if and only if $b_i \prec_T b_j$.*

Proof. Suppose to the contrary that there exist $i, j \in \{1, \dots, m\}$ such that

$a_i \prec_S a_j$ and $b_j \prec_T b_i$. By Lemma 5.15 we have $a_i \leqslant v(a_i, b_j) \leqslant u(a_j, b_i) \leqslant b_i$ in P, which is a contradiction. $\square$

For two pairs $(a, b), (a', b') \in \mathrm{Inc}_P(A, B)$, let us write $(a, b) \prec (a', b')$ if $a \prec_S a$ and $b \prec_T b'$. Lemma 5.16 implies that the pairs of every standard example in $\mathrm{Inc}_P(A, B)$ are linearly ordered by $\prec$.

5.4.1 Finding a tree-disjoint standard example

For a standard example I in $\mathrm{Inc}_P(A, B)$ we define trees

$$S(I) = \bigcup_{(a,b)\in I} aSy_0 \quad \text{and} \quad T(I) = \bigcup_{(a,b)\in I} x_0Tb.$$

We say that I is *tree-disjoint* if the trees $S(I)$ and $T(I)$ are disjoint. In this section we prove the following.

Lemma 5.17. *Let $m \geqslant 1$. If $\mathrm{Inc}_P(A, B)$ contains a standard example of size m, then $\mathrm{Inc}_P(A, B)$ contains a tree-disjoint standard example of size $\lceil m/11 \rceil$.*

Given two pairs $(a, b), (a', b') \in \mathrm{Inc}_P(A, B)$ belonging to one standard example, we write $(a, b) \to (a', b')$ when the paths aSy_0 and x_0Tb' have a nonempty intersection. Note that the relation $\to$ is independent of the order $\prec$, and for a pair with $(a, b) \to (a', b')$ we can have either $(a, b) \prec (a', b')$ or $(a', b') \prec (a, b)$. If $\{(a_1, b_1), \ldots, (a_p, b_p)\} \subseteq \mathrm{Inc}_P(A, B)$ is a standard example with $(a_i, b_i) \to (a_{i+1}, b_{i+1})$ for each $i \in \{1, \ldots, p-1\}$, then we call the sequence $(a_1, b_1) \to \cdots \to (a_p, b_p)$ a *directed path*. A directed path $(a_1, b_1) \to \cdots \to (a_p, b_p)$ is *increasing* if $(a_1, b_1) \prec \cdots \prec (a_p, b_p)$, and *decreasing* if $(a_p, b_p) \prec \cdots \prec (a_1, b_1)$. Figure 5.8 shows an increasing directed path consisting of 6 pairs.

Lemma 5.18. *Every increasing or decreasing directed path in $\mathrm{Inc}_P(A, B)$ consists of at most 6 pairs.*

Proof. Because of symmetry, it suffices to show that every increasing path has at most 6 pairs. Suppose to the contrary that there exists a directed path $(a_1, b_1) \to \cdots \to (a_7, b_7)$ with $(a_1, b_1) \prec \cdots \prec (a_7, b_7)$. For every $i \in \{1, \ldots, 6\}$, the paths a_iSy_0 and x_0Tb_{i+1} intersect, so the black part of $N(a_i, b_{i+1})$ consists of one vertex which we denote by c_i. By Lemma 5.15, for each $i \in \{1, \ldots, 5\}$ we have $c_i = v(a_i, b_{i+1}) \leqslant u(a_{i+1}, b_{i+2}) = c_{i+1}$ in P, so

$$x_0 \leqslant c_1 \leqslant \cdots \leqslant c_6 \leqslant y_0$$

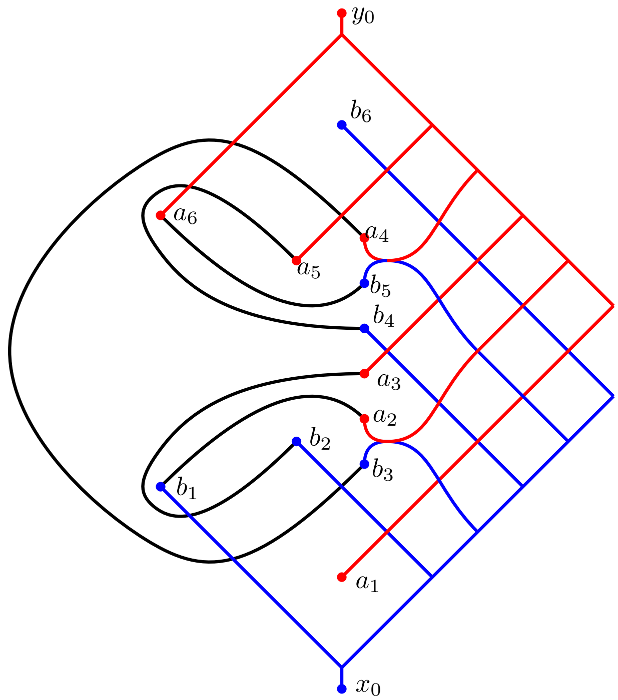


Figure 5.8: An increasing path $(a_1, b_1) \to \cdots \to (a_6, b_6)$. The paths $W(a_{i+1}, b_i)$ are drawn in black. The union of these paths and the red and blue trees contains a witnessing path from a_i to b_j for each pair of distinct i and j.

holds in P. Let $W_0 = x_0Tc_1W(c_1, c_2)c_2 \cdots c_5W(c_5, c_6)c_6Sy_0$.

For each $i \in \{1, \ldots, 7\}$, let $s_i = \min(a_iSy_0 \cap W_0)$, and let $t_i = \max(x_0Tb_i \cap W_0)$. Thus, s_i is the only vertex of a_iSs_i which lies on W_0, and t_i is the only vertex of t_iTb_i which lies on W_0. Since each c_i lies on both x_0Tb_{i+1} and a_iSy_0, we have $s_i \leqslant c_i \leqslant t_{i+1}$ in P for each $i \in \{1, \ldots, 6\}$. Moreover, we have $t_i < s_i$ in P for each $i \in \{1, \ldots, 7\}$ as otherwise we would have $a_i \leqslant s_i \leqslant t_i \leqslant b_i$ in P. Hence we have

$$x_0 \leqslant t_1 < s_1 \leqslant t_2 < s_2 \leqslant \cdots \leqslant t_7 < s_7 \leqslant y_0 \quad \text{in } P.$$

Claim 5.18.1. *For any $i, j \in \{1, \ldots, 7\}$ with $i > j$, the witnessing path $W(a_i, b_j)$ is disjoint from W_0.*

Proof. Suppose to the contrary that $W(a_i, b_j)$ intersects W_0 in a vertex w. Since $j < i$, the vertices s_j and t_i of W_0 satisfy $x_0 \leqslant s_j \leqslant t_i \leqslant y_0$ in P, so we have $w \leqslant t_i$ or $s_j \leqslant w$ in P. In the former case we have $a_i \leqslant w \leqslant t_i \leqslant b_i$ in P, and in the latter we have $a_j \leqslant s_j \leqslant w \leqslant b_j$ in P. As both cases lead to a contradiction, the claim follows. $\Diamond$

Claim 5.18.2. *The vertices $a_2, \dots, a_7$ and $b_1, \dots, b_6$ are left of W_0.*

Proof. Let us first show that b_1 is left of W_0. In the tree T, c_1 is an ancestor of b_2, and the vertex t_1 is an ancestor of c_1 since $x_0W_0c_1 = x_0Tc_1$. Since $t_1 < c_1$ in P and $b_1 \prec_T b_2$, this means that $b_1 \prec_T c_1$, and therefore b_1 is left of W_0.

By Claim 5.18.1, for each $i \in \{2, \dots, 7\}$, the witnessing path $W(a_i, b_1)$ is disjoint from W_0. Since b_1 is left of W_0, this implies that the vertices a_2, $\dots$, a_7 are left of W_0. Again by Claim 5.18.1, for each $j \in \{1, \dots, 6\}$ the witnessing path $W(a_7, b_j)$ is disjoint from W_0. Since a_7 is left of W_0, this implies that the vertices $b_1, \dots, b_6$ are left of W_0. $\Diamond$

Claim 5.18.3. *The paths a_1Ss_1, $\dots$, a_7Ss_7 are pairwise disjoint, and the paths $t_1Tb_1, \dots, t_7Tb_7$ are pairwise disjoint.*

Proof. For any $i \in \{1, \dots, 7\}$ and $v \in V(a_iSs_i)$, we have $\min(vSy_0 \cap W_0) = s_i$. Since the vertices s_1, $\dots$, s_7 are pairwise distinct, this implies that the paths $a_1Ss_1, \dots, a_7Ss_7$ are pairwise disjoint. The paths $t_1Tb_1, \dots, t_7Tb_7$ are pairwise disjoint by a dual argument. $\Diamond$

Claim 5.18.4. *For any $i, j \in \{2, \dots, 6\}$ with $j \neq i + 1$, the paths a_iSs_i and t_jTb_j are disjoint.*

Proof. The witnessing path a_iSy_0 intersects x_0Tb_{i+1} and is disjoint from x_0Tb_i since $a_i \parallel b_i$ in P. Hence, by Lemma 5.13, the path a_iSy_0 is disjoint from the paths x_0Tb_1, $\dots$, x_0Tb_i, and therefore a_iSs_i is disjoint from t_jTb_j if $j \leqslant i$. Now suppose that $j \geqslant i + 2$. We have $s_i < t_j$ in P, so for any $v \in V(a_iSs_i)$ and $u \in V(t_jTb_j)$ we have $v < u$ in P. Therefore the paths a_iSs_i and t_jTb_j are disjoint. $\Diamond$

By Claims 5.18.2 and 5.18.1, for any $i, j \in \{1, \dots, 7\}$ with $i > j$, all vertices of $W(a_i, b_j)$ are left of W_0. In particular, all vertices of the paths $a_iSv(a_i, b_j)$ and $u(a_i, b_j)Tb_j$ are left of W_0. Therefore, the vertices s_i and t_j lie on $N(a_i, b_j)$ and all inner vertices of the path $t_jN(a_i, b_j)s_i$ are left of W_0.

Claim 5.18.5. *Either the black part of $N(a_4, b_2)$ intersects a_6Ss_6 or the black part of $N(a_6, b_4)$ intersects t_2Tt_6.* (*See Figure 5.9.*)

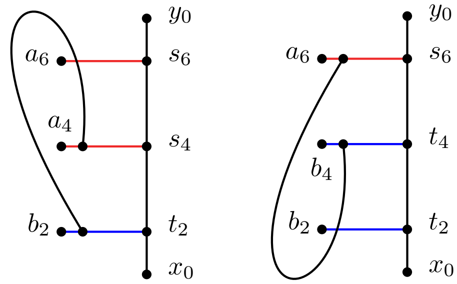


Figure 5.9: Two possible outcomes of Claim 5.18.5

Proof. Since $t_2 < t_4 < s_4 < s_6$ in P, the paths $t_2N(a_4, b_2)s_4$ and $t_4N(a_6, b_4)s_6$ must intersect in a vertex z. By Claims 5.18.3 and 5.18.4, the paths t_2Tb_2, t_4Tb_4, a_4Ss_4 and a_6Ss_6 are pairwise disjoint. Hence, z must lie on the black part of $N(a_4, b_2)$ or $N(a_6, b_4)$. If z lies on the black part of $N(a_4, b_2)$, then z must lie on the red part of $N(a_6, b_4)$ as otherwise we would have $a_4 \leqslant z \leqslant u(a_6, b_4) \leqslant b_4$ in P. If z lies on the black part of $N(a_6, b_4)$, then z must lie on the blue part of $N(a_4, b_2)$ as otherwise we would have $a_4 \leqslant v(a_4, b_2) \leqslant z \leqslant b_4$ in P. $\Diamond$

The two alternatives in the statement of Claim 5.18.5 are dual, so without loss of generality we assume that the black part of $N(a_4, b_2)$ intersects a_6Ss_6. Let $W = W(a_4, b_2)$, let $v = v(a_4, b_2)$ and let v' denote any vertex of the intersection of a_6Ss_6 with the black part of $N(a_4, b_2)$. We claim that the witnessing paths $vWv'Ss_6$ and $vSs_4W_0s_6$ are internally disjoint. By Claim 5.18.1 the paths vWv' and $s_4W_0s_6$ are disjoint, and by Claim 5.18.3, the paths $v'Ss_6$ and vSs_4 are disjoint. Since $v = v(a_4, b_2)$, the paths vWv' and vSs_4 are internally disjoint, and by definition of s_6 the paths $v'Ss_6$ and $s_4W_0s_6$ are internally disjoint. Hence the witnessing paths $vWv'Ss_6$ and $vSs_4W_0s_6$ are internally disjoint and their union is a cycle which we denote by C.

We have $V(C) \subseteq \mathrm{D}_P(s_6) \subseteq \mathrm{D}_P(t_7) \subseteq \mathrm{D}_P(b_7)$, so by Lemma 5.12, b_6 does not lie in the region bounded by C. The intersection of C with W_0 is $s_4W_0s_6$ and t_6 is an inner vertex of $s_4W_0s_6$. Hence the path t_6Tb_6 has to intersect the cycle C in a vertex z distinct from t_6. Since t_6 is the only vertex of t_6Tb_6 on W_0, the vertex z does not lie on W_0, and by Claim 5.18.4, z does not lie on vSs_4 or $v'Ss_6$. Thus, z lies on vWv', which implies that W is a witnessing path intersecting both x_0Tb_2 and x_0Tb_6. (See Figure 5.10.) By Lemma 5.13, W intersects x_0Tb_4. Since $W = W(a_4, b_2)$, this implies $a_4 \leqslant b_4$ in P, which

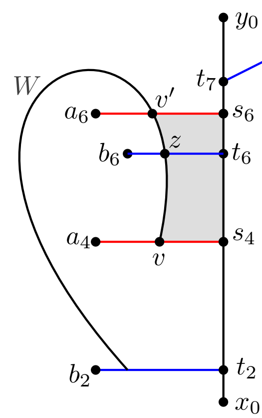


Figure 5.10: Illustration of Lemma 5.18. The gray area is the region bounded by C.

is a contradiction. Hence there is no increasing directed path on more than 6 pairs. ☐

Proof of Lemma 5.17. Let $I \subseteq \mathrm{Inc}_P(A, B)$ be a standard example of size m. For each $(a, b) \in I$, let $p(a, b) \in \{1, \ldots, 6\}$ denote the maximum number of pairs in an increasing directed path which starts with (a, b) and has all pairs from the set I, and let $q(a, b) \in \{1, \ldots, 6\}$ denote the maximum number of pairs in a decreasing directed path with starts with (a, b) and has all pairs from the set I.

We claim that for every $(a, b) \in I$ we have $p(a, b) = 1$ or $q(a, b) = 1$. Suppose to the contrary that $p(a, b) \geqslant 2$ and $q(a, b) \geqslant 2$. Therefore there exist pairs $(a', b'), (a'', b'') \in I$ with $(a, b) \to (a', b')$ and $(a, b) \to (a'', b'')$ such that $(a', b') \prec (a, b) \prec (a'', b'')$, and thus $b' \prec_T b \prec_T b''$. The witnessing path aSy_0 intersects x_0Tb' and x_0Tb'', so by Lemma 5.13 it intersects x_0Tb. This implies $a \leqslant b$ in P, which is a contradiction, so indeed $p(a, b) = 1$ or $q(a, b) = 1$.

There are 11 different pairs (p, q) with $p, q \in \{1, \ldots, 6\}$ such that $p = 1$ or $q = 1$. Hence, by the pigeonhole principle, there exist a pair (p, q) and a subset $I' \subseteq I$ with $|I'| = \lceil m/11 \rceil$ such that $p(a, b) = p$ and $q(a, b) = q$ for each $(a, b) \in I'$. We claim that the standard example I' is tree-disjoint. Suppose to the contrary that the trees $S(I')$ and $T(I')$ intersect. Hence there exist pairs $(a, b), (a', b') \in I'$ such that aSy_0 intersects x_0Tb', that is $(a, b) \to (a', b')$. The pairs (a, b) and (a', b') must be distinct, so either $(a, b) \prec (a', b')$ or

$(a', b') \prec (a, b)$. If $(a, b) \prec (a', b')$, then any increasing directed path starting with (a', b') can be extended by prepending (a, b), so $p(a, b) > p(a', b')$, and if $(a', b') \prec (a, b)$, then any decreasing directed path starting with (a', b') can be extended by prepending (a, b), so $q(a, b) > q(a', b')$. Hence, either $p(a, b) \neq p(a', b')$, or $q(a, b) \neq q(a', b')$, which is a contradiction. This completes the proof. $\square$

5.4.2 Finding a path-separated standard example

For $m \geqslant 1$, we say that a standard example $\{(a_1, b_1), \ldots, (a_{m+2}, b_{m+2})\}$ in $\mathrm{Inc}_P(A, B)$ with $(a_1, b_1) \prec \cdots \prec (a_{m+2}, b_{m+2})$ is *path-separated* if it is tree-disjoint and either

(1) there exist $a^* \in \{a_{m+1}, a_{m+2}\}$ and $b^* \in \{b_{m+1}, b_{m+2}\}$ with $a^* \leqslant b^*$ in P such that $a_1, \ldots, a_m$ are right of $N(a^*, b^*)$, and $b_1, \ldots, b_m$ are left of $N(a^*, b^*)$, or

(2) there exist $a^* \in \{a_1, a_2\}$ and $b^* \in \{b_1, b_2\}$ with $a^* \leqslant b^*$ in P such that $a_3, \ldots, a_{m+2}$ are left of $N(a^*, b^*)$, and $b_3, \ldots, b_{m+2}$ are right of $N(a^*, b^*)$.

In this subsection we prove the following.

Lemma 5.19. *Let $m \geqslant 1$. If $\mathrm{Inc}_P(A, B)$ contains a tree-disjoint standard example of size $2m + 1$, then it contains a path-separated standard example of size $m + 2$.*

We prove it with a sequence of lemmas. Let us first observe that in the case of tree-disjoint standard examples, the statement of Lemma 5.14 simplifies a bit.

Lemma 5.20. *Let $I = \{(a_1, b_1), \ldots, (a_m, b_m)\} \subseteq \mathrm{Inc}_P(A, B)$ be a tree-disjoint standard example and let $i, j \in \{1, \ldots, m\}$ be distinct.*

(1) For every $v \in V(S(I))$, if vSy_0 is disjoint from the black part of $N(a_i, b_j)$, then $v(a_i, b_j) \prec_S v$ if and only if v is left of $N(a_i, b_j)$.

(2) For every $u \in V(T(I))$, if x_0Tu is disjoint from the black part of $N(a_i, b_j)$, then $u \prec_T u(a_i, b_j)$ if and only if u is left of $N(a_i, b_j)$.

Proof. Since the standard example is tree-disjoint, for any $v \in V(S(I))$, the path vSy_0 does not intersect the blue part of $N(a_i, b_j)$, and for any $u \in V(T(I))$, the path x_0Tu does not intersect the red part of $N(a_i, b_j)$, so the lemma is an immediate consequence of Lemma 5.14. $\square$

Lemma 5.21. *Let $\{(a_1, b_1), \ldots, (a_m, b_m)\}$ be a standard example in* $\mathrm{Inc}_P(A, B)$ *with*

$$(a_1, b_1) \prec \cdots \prec (a_m, b_m).$$

and let $i, j \in \{1, \ldots, m\}$ satisfy $i < j$. Then the vertices $a_j, \ldots, a_m$ and $b_1, \ldots, b_i$ are left of $N(a_i, b_j)$.

Proof. Let $k \in \{j, \ldots, m\}$. The path $W(a_i, b_j)$ intersects $a_i S y_0$ and is disjoint from $a_j S y_0$ since $a_j \parallel b_j$ in P. We have $i < j \leqslant k$, so by Lemma 5.13 the path $a_k S y_0$ is disjoint from $W(a_i, b_j)$. Since $a_j \parallel b_j$ in P, the vertex $v(a_i, b_j)$ is not an ancestor of a_j. As $a_i \prec_S a_j$, this implies $v(a_i, b_j) \prec_S a_j$, and thus $v(a_i, b_j) \prec_S a_k$. By Lemma 5.20, a_k is left of $N(a_i, b_j)$. Dual arguments show that the vertices $b_1, \ldots, b_i$ are left of $N(a_i, b_j)$. $\square$

Observe that if $\{(a_1, b_1), \ldots, (a_m, b_m)\}$ is a tree-disjoint standard example in $\mathrm{Inc}_P(A, B)$ and $i, j \in \{1, \ldots, m\}$ are distinct, then $N(a_i, b_j)$ does not contain any a_k with $k \neq i$: since the standard example is tree-disjoint, a_k does not lie on the blue part, and if a_k lied on the red or the black part of $N(a_i, b_j)$, we would have $a_i \leqslant a_k$ in P, which is impossible. Hence, every a_k with $k \neq i$ is either left or right of $N(a_i, b_j)$. By a symmetric argument, every b_k with $k \neq j$ is either left of or right of $N(a_i, b_j)$.

Lemma 5.22. *Let $\{(a_1, b_1), \ldots, (a_m, b_m)\}$ be a tree-disjoint standard example in* $\mathrm{Inc}_P(A, B)$*, and let $i, j, k \in \{1, \ldots, m\}$ be such that $(a_i, b_i) \prec (a_j, b_j) \prec (a_k, b_k)$. Then a_i is right of $N(a_j, b_k)$ or b_k is right of $N(a_i, b_j)$.*

Proof. Suppose that a_i is left of $N(a_j, b_k)$, and let us show that b_k is right of $N(a_i, b_j)$. We have $a_i \prec_S a_j$ and $v(a_j, b_k)$ is an ancestor of a_j in S, so we do not have $v(a_j, b_k) \prec_S a_i$. By Lemma 5.20, the path $a_i S y_0$ must intersect the black part of $N(a_j, b_k)$. By our choice of the canonical witnessing paths, the intersection $a_i S y_0 \cap W(v(a_j, b_k), u(a_j, b_k))$ is a witnessing path of the form $W(w_1, w_2)$ (with a possibility that $v(a_j, b_k) = w_1 = w_2$). Observe that all vertices of $a_i S w_1$ except w_1 are left of $N(a_j, b_k)$, and all vertices of $w_2 S y_0$ except w_2 either lie on the red part of $N(a_j, b_k)$ or are right of $N(a_j, b_k)$. Since the standard example is separated, we have $w_2 < u(a_j, b_k)$ in P, and we have $v(a_i, b_j) < w_1$ in P as otherwise we would have $a_j \leqslant v(a_j, b_k) \leqslant w_1 \leqslant v(a_i, b_j) \leqslant b_j$ in P. See Figure 5.11.

Observe that the path $W(w_2, b_k)$ intersects $N(a_i, b_j)$ only in w_2; indeed, by our choice of w_2, $W(w_2, b_k)$ intersects the red part of $N(a_i, b_j)$ only in w_2, and if $W(w_2, b_k)$ intersected the black or the blue part of $N(a_i, b_j)$, we would

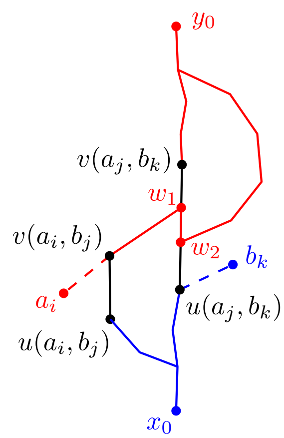


Figure 5.11: If a_i is left of $N(a_j, b_k)$, then b_k must be right of $N(a_i, b_j)$.

have $a_j \leqslant w_2 \leqslant u(a_i, b_j) \leqslant b_j$ in P. The paths $w_2 N(a_i, b_j) x_0$, $w_2 N(a_i, b_j) y_0$ and $W(w_2, b_k)$ leave the vertex w_2 in a clockwise manner, so b_k is right of $N(a_i, b_j)$. $\square$

Lemma 5.23. *Let $\{(a_1, b_1), \ldots, (a_m, b_m)\}$ be a tree-disjoint standard example in* $\mathrm{Inc}_P(A, B)$ *with*

$$(a_1, b_1) \prec \cdots \prec (a_m, b_m),$$

and let $i, j, k \in \{1, \ldots, m\}$ satisfy $i < j < k$.

(1) *If a_i is right of $N(a_j, b_k)$, then all a_1, …, a_i are right of $N(a_j, b_k)$.*

(2) *If b_k is right of $N(a_i, b_j)$, then all b_j, …, b_m are right of $N(a_i, b_j)$.*

Proof. Because of symmetry, we only prove (1). Let $N = N(a_j, b_k)$, and suppose towards a contradiction that for some $\ell \in \{1, \ldots, i-1\}$, the vertex a_ℓ is left of N. Since $a_\ell \prec_S a_j$, we do not have $v(a_i, b_j) \prec_S a_\ell$, so by Lemma 5.20, the path $a_\ell S y_0$ intersects the black part of $N(a_j, b_k)$. Let $z = \min(a_\ell S y_0 \cap W(v(a_j, b_k), u(a_j, b_k)))$, and consider the cycle $C = zSv(a_j, b_k)Nz$. We have $a_\ell \prec_S a_i \prec_S a_j$ and a_i is right of N, so the vertex a_i clearly lies in the region bounded by C. Since $a_j \leqslant v(a_j, b_k) \leqslant z$ in P, we have $V(C) \subseteq \mathrm{U}_P(a_j)$, so a_k is enclosed by $\mathrm{U}_P(a_j)$, contrary to Lemma 5.12. Hence a_ℓ is right of N. $\square$

Proof of Lemma 5.19. Let $\{(a_1, b_1), \ldots, (a_{2m+1}, b_{2m+1})\}$ be a tree-disjoint standard example in $\mathrm{Inc}_P(A, B)$, and assume without loss of generality that

$(a_1, b_1) \prec \ldots \prec (a_{2m+1}, b_{2m+1})$. By Lemma 5.22, a_m is right of $N(a_{m+1}, b_{m+2})$ or b_{m+2} is right of $N(a_m, b_{m+1})$.

Suppose that a_m is right of $N(a_{m+1}, b_{m+2})$. By Lemma 5.23, the vertices a_1, …, a_{m-1} are also right of $N(a_{m+1}, b_{m+2})$. By Lemma 5.21, the vertices b_1, …, b_m are left of $N(a_{m+1}, b_{m+2})$. Hence $\{(a_1, b_1), \ldots, (a_{m+2}, b_{m+2})\}$ is a path-separated standard example of size $m + 2$ in $\mathrm{Inc}_P(A, B)$. By symmetric arguments, if b_{m+2} is right of $N(a_m, b_{m+1})$, then the vertices b_{m+2}, …, b_{2m+1} are right of $N(a_m, b_{m+1})$, and the vertices a_{m+2}, …, a_{2m+1} are left of $N(a_m, b_{m+1})$, and therefore $\{(a_m, b_m), \ldots, (a_{2m+1}, b_{2m+1})\}$ is a path-separated standard example of size $m + 2$ in $\mathrm{Inc}_P(A, B)$. □

5.4.3 Finding a Kelly subposet

We prove one more lemma before the proof of Lemma 5.4.

Lemma 5.24. *Let $\{(a_1, b_1), \ldots, (a_m, b_m)\}$ be a tree-disjoint standard example with*

$$(a_1, b_1) \prec \cdots \prec (a_m, b_m),$$

let $i, j, k \in \{1, \ldots, m-2\}$ satisfy $i < j < k$ and suppose that b_m is right of the paths $N(a_i, b_{i+1})$ and $N(a_k, b_{k+1})$. Then b_m is right of $N(a_j, b_{j+1})$.

Proof. For each $\ell \in \{i, j, k\}$, let $v_\ell = v(a_\ell, b_{\ell+1})$, $u_{\ell+1} = u(a_\ell, b_{\ell+1})$, $W_{\ell,\ell+1} = W(a_\ell, b_{\ell+1})$ and $N_{\ell,\ell+1} = N(a_\ell, b_{\ell+1})$.

Suppose that the claim is not true, that is b_m is right of $N_{i,i+1}$ and $N_{k,k+1}$ and left of $N_{j,j+1}$.

Consider the union $H = N_{i,i+1} \cup N_{j,j+1} \cup N_{k,k+1}$ and its drawing inherited from the drawing of G. Since b_m is right of $N_{i,i+1}$ (and $N_{k,k+1}$) and left of $N_{j,j+1}$, it is easy to see that b_m does not lie on the outer face of H. Hence, the boundary of the face of H containing b_m is a cycle, which we denote by C. Observe that no vertex in the region bounded by C is left of $N_{i,i+1}$ or $N_{k,k+1}$, or right of $N_{j,j+1}$. We complete the proof by showing that $V(C) \subseteq \mathrm{D}_P(b_{k+1})$ and hence b_m is enclosed by $\mathrm{D}_P(b_{k+1})$, contradicting Lemma 5.12.

Let us redraw the cycle C as a circle so that the clockwise cyclic ordering of the vertices is the same as in the drawing of G and the edges are represented as arcs of equal length. We orient the edges of C so that each edge xy with $x < y$ in P is oriented from x to y. Now, it suffices to show that for every vertex y without an outgoing edge we have $y \leqslant b_{k+1}$ in P.

Assign to each edge of C one of the colors: red, black or blue, so that every edge of a given color belongs to the part of the same color in at least one of the paths $N_{i,i+1}$, $N_{j,j+1}$ and $N_{k,k+1}$. See Figure 5.12.

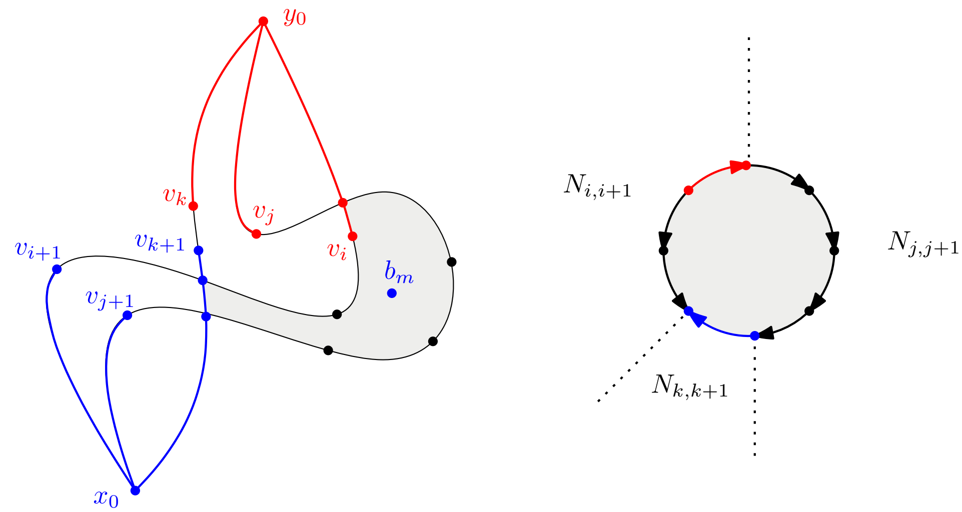


Figure 5.12: A potential configuration in Lemma 5.24 and the corresponding orientation of the edges on the circle.

We claim that every red edge of C belongs to the red part of $N_{i,i+1}$. Suppose that it is not the case. Hence, there exists a vertex v on C which lies on the red part of $N_{j,j+1}$ or $N_{k,k+1}$ but does not lie on the red part of $N_{i,i+1}$. The witnessing path $W_{i,i+1}$ intersects a_iSy_0 and is disjoint from $a_{i+1}Sy_0$ since $a_{i+1} \parallel b_{i+1}$ in P. Hence, by Lemma 5.13, the path $W_{i,i+1}$ is disjoint from the paths v_jSy_0 and v_kSy_0. In particular, v is not a descendant of v_i in S, so $v_i \prec_S v$. Since v lies on C, it is not left of $N_{i,i+1}$, so, by Lemma 5.20, the path vSy_0 intersects the black part of $N_{i,i+1}$. Hence, the witnessing path $W_{i,i+1}$ intersects one of the paths a_jSy_0 or a_kSy_0, which, as we already argued, is not possible. Hence indeed every red edge in C belongs to the red part of $N_{i,i+1}$. Since the region bounded by C is on the right side of $N_{i,i+1}$, this implies that all red edges in C are oriented clockwise.

Let $y \in V(C)$, let x_1 and x_2 be the neighbors of y in C, and suppose that the edges x_1y and x_2y are oriented towards y. Since all red edges are oriented clockwise, the edges x_1y and x_2y are not both red. Since the standard example is tree-disjoint, it is impossible that one of the edges x_1y and x_2y is blue and the other one is red. It is also impossible that both edges x_1y

and x_2y are blue: a blue edge from x_i to y means that x_i is the parent of y in T, and we have $x_1 \neq x_2$. Hence at least one of the edges x_1y and x_2y is black, so in particular, y lies on the black part of $N_{i,i+1}$, $N_{j,j+1}$ or $N_{k,k+1}$.

If y lies on the black part of $N_{k,k+1}$, then we have $y \leqslant u_{k+1} \leqslant b_{k+1}$ in P. Let us hence assume that y lies on the black part of $N(a_\ell, b_{\ell+1})$ for some $\ell \in \{i, j\}$. Since y is a vertex of C, it is not left of $N_{k,k+1}$, and since $i < j < k$, we have $\ell + 1 \leqslant k$, so by Lemma 5.21, $b_{\ell+1}$ is left of $N_{k,k+1}$. Hence, the path $yW_{\ell,\ell+1}b_{\ell+1}$ must intersect $N_{k,k+1}$ in a vertex z. The vertex z does not belong to the red part of $N_{k,k+1}$ because then $W_{\ell,\ell+1}$ would be a witnessing path intersecting $a_\ell S y_0$ and $a_k S y_0$, so by Lemma 5.13, the path $W_{\ell,\ell+1}$ would intersect $a_{\ell+1} S y_0$ and we would have $a_{\ell+1} \leqslant b_{\ell+1}$ in P. Hence z belongs to the blue or the black part of $N_{k,k+1}$, which implies $y \leqslant z \leqslant u_{k+1} \leqslant b_{k+1}$ in P. Therefore $V(C) \subseteq \mathrm{D}_P(b_{k+1})$, so b_m is enclosed by $\mathrm{D}_P(b_{k+1})$, which by Lemma 5.12 is a contradiction. This concludes the proof. □

Proof of Lemma 5.4. Suppose that $\mathrm{Inc}_P(A, B)$ contains a standard example of size $440(k+1)$. We need to show that the inherited drawing of the cover graph of $P - \{x_0, y_0\}$ is not k-outerplanar. By Lemma 5.17, there exists a tree-disjoint standard example of size $40(k+1)$ in $\mathrm{Inc}_P(A, B)$, and by Lemma 5.19 there exists a path-separated standard example of size $m+2$ in $\mathrm{Inc}_P(A, B)$ where $m = 20k + 19$. Let us fix any such a standard example $I = \{(a_1, b_1), \ldots, (a_{m+2}, b_{m+2})\}$ with

$$(a_1, b_1) \prec \ldots \prec (a_{m+2}, b_{m+2}).$$

Let a^* and b^* be vertices witnessing that I is path-separated, let $N^* = N(a^*, b^*)$, $v^* = v(a^*, b^*)$ and $u^* = u(a^*, b^*)$. Because of symmetry, we may assume without loss of generality that $a^* \in \{a_{m+1}, a_{m+2}\}$, $b^* \in \{b_{m+1}, b_{m+2}\}$, the vertices a_1, ..., a_m are right of N^* and the vertices $b_1, \ldots, b_m$ are left of N^*.

For each $i \in \{1, \ldots, m-1\}$, a_i is right of N^* and b_{i+1} is left of N^*. Hence the path $W(a_i, b_{i+1})$ must intersect N^*. Observe that $W(a_i, b_{i+1})$ does not intersect the red part of N^* as otherwise, by Lemma 5.13, the path $W(a_i, b_{i+1})$ would intersect $a_{i+1} S y_0$, implying $a_{i+1} \leqslant b_{i+1}$ in P. Hence the path $W(a_i, b_{i+1})$ must intersect the black part $W(v^*, u^*)$ or the blue part $x_0 T u^*$ of N^*, and v^* is not right of $N(a_i, b_{i+1})$.

We claim that for each $i \in \{2, \ldots, m-2\}$, the path $W(a_i, b_{i+1})$ is disjoint from $u^* T b^*$. Suppose to the contrary that for some $i \in \{2, \ldots, m-2\}$ the path $W(a_i, b_{i+1})$ intersects $u^* T b^*$. This implies that $u^* \leqslant b_{i+1}$ in P. Since

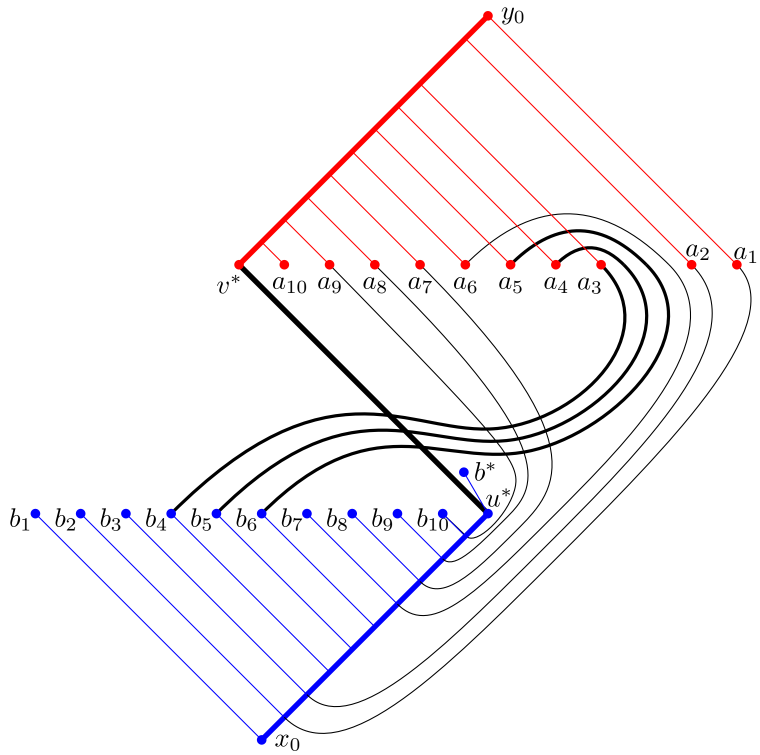


Figure 5.13: In this example, the vertex b^* is right of the paths $N(a_3, b_4)$, $N(a_4, b_5)$ and $N(a_5, b_6)$, so $\Xi^+ = \{3, 4, 5\}$.

the path $W(a_{i+1}, b_{i+2})$ intersects the black or the blue part of N^*, we obtain $a_{i+1} \leqslant u^* \leqslant b_{i+1}$ in P, which is a contradiction. Hence the path u^*Tb^* is disjoint from $W(a_i, b_{i+1})$ for each $i \in \{2, \ldots, m-2\}$. Since $b_{i+1} \prec_T b^*$, this implies that if b^* is left of $N(a_i, b_{i+1})$, then u^* is left of $N(a_i, b_{i+1})$ too, and if b^* is right of $N(a_i, b_{i+1})$, then u^* is either right of $N(a_i, b_{i+1})$ or on the blue part of $N(a_i, b_{i+1})$. In particular, u^* is left of $N(a_i, b_{i+1})$ if and only if b^* is left of $N(a_i, b_{i+1})$.

Let Ξ^+ denote the set of all indices $i \in \{2, \ldots, m-2\}$ such that b^* is right of $N(a_i, b_{i+1})$ (and thus u^* is not left of $N(a_i, b_{i+1})$). The set Ξ^+ consists of consecutive indices; this follows from Lemma 5.24 applied to the standard example $\{(a_1, b_1) \ldots, (a_m, b_m), (a, b^*)\}$ where $a \in \{a_{m+1}, a_{m+2}\}$ is the element which belongs to one pair with b^* in our original standard example. See Figure 5.13.

For each $i \in \{2, \ldots, m-2\}$, if $i \in \Xi^+$, then the vertex u^* is not left of

$N(a_i, b_{i+1})$, and therefore $W(a_i, b_{i+1})$ intersects $W(v^*, u^*)$, and if $i \notin \Xi^+$, then the vertex u^* is left of $N(a_i, b_{i+1})$, and therefore $W(a_i, b_{i+1})$ intersects x_0Tu^*.

Let Ξ^- denote the set of all $i \in \{2, \ldots, m-2\}$ such that a^* is left of $N(a_{i+1}, b_i)$. By dual arguments, Ξ^- consists of consecutive indices, and for each $i \in \{2, \ldots, m-2\}$, if $i \in \Xi^-$, then the vertex v^* is not right of $N(a_{i+1}, b_i)$, and therefore $W(a_{i+1}, b_i)$ intersects $W(v^*, u^*)$, and if $i \notin \Xi^-$, then the vertex v^* is right of $N(a_{i+1}, b_i)$, and therefore $W(a_{i+1}, b_i)$ intersects v^*Sy_0.

Note that the sets Ξ^+ and Ξ^- are disjoint: If there existed i belonging to Ξ^+ and Ξ^-, then both witnessing paths $W(a_i, b_{i+1})$ and $W(a_{i+1}, b_i)$ would intersect $W(v^*, u^*)$ which would imply $a_i \leqslant b_i$ or $a_{i+1} \leqslant b_{i+1}$ in P.

Each of the sets Ξ^+ and Ξ^- is an interval of consecutive indices, and the endpoints of these intervals split the set $\{2, \ldots, m-2\}$ into at most five intervals. Since $\lceil (m-3)/5 \rceil = \lceil (20k+16)/5 \rceil = 4k+4$, there exist $4k+4$ consecutive indices in $\{2, \ldots, m-2\}$ such that either none of them belongs to Ξ^+ or all of them belong to Ξ^+, and either none of them belongs to Ξ^- or all of them belong to Ξ^-. Choose such $4k+4$ consecutive indices, and let i_1 denote the least of them, so that the set of these indices is $\Xi = \{i_1, \ldots, i_1 + 4k + 3\}$. Since $\Xi^+ \cap \Xi^- = \varnothing$, the set Ξ contains indices from at most one of the sets Ξ^+ and Ξ^-.

Let W^+ denote the witnessing path $W(v^*, u^*)$ if $\Xi \subseteq \Xi^+$, or the witnessing path x_0Tu^* if $\Xi \cap \Xi^+ = \varnothing$. Symmetrically, let W^- denote the witnessing path $W(v^*, u^*)$ if $\Xi \subseteq \Xi^+$, or the witnessing path v^*Sy_0 if $\Xi \cap \Xi^+ = \varnothing$. This way, for each $i \in \Xi$, the path $W(a_i, b_{i+1})$ intersects W^+, and the path $W(a_{i+1}, b_i)$ intersects W^-.

For each $i \in \{1, \ldots, 4k+5\}$, let $a'_i = a_{i_1+i-1}$, $b'_i = b_{i_1+i-1}$, and for each $i \in \{1, \ldots, 4k+4\}$, let $W'_{i,i+1} = W(a'_i, b'_{i+1})$ and $W'_{i+1,i} = W(a'_{i+1}, b'_i)$. The path $W'_{i,i+1}$ intersects W^+ and the path $W'_{i+1,i}$ intersects W^-, so we can define

$$
\begin{aligned}
c'_i &= \min(V(W'_{i,i+1}) \cap W^+),\\
c''_i &= \max(V(W'_{i,i+1}) \cap W^+),\\
d'_i &= \min(V(W'_{i+1,i}) \cap W^-),\\
d''_i &= \max(V(W'_{i+1,i}) \cap W^-).
\end{aligned}
$$

For each $i \in \{1, \ldots, 4k+3\}$, we do not have $c'_{i+1} \leqslant c''_i$ or $d'_i \leqslant d''_{i+1}$ in P as that would imply $a'_{i+1} \leqslant c'_{i+1} \leqslant c''_i \leqslant b'_{i+1}$ or $a'_{i+1} \leqslant d'_i \leqslant d''_{i+1} \leqslant b'_{i+1}$ in P. Hence, we have $c''_i < c'_{i+1}$ and $d''_{i+1} < d'_i$ in P, which means that

$$c'_1 \leqslant c''_1 < c'_2 \leqslant c''_2 < \cdots < c'_{4k+4} \leqslant c''_{4k+4}$$

and

$$d'_{4k+4} \leqslant d''_{4k+4} < d'_{4k+3} \leqslant d''_{4k+3} < \cdots < d'_1 \leqslant d''_1$$

hold in P. We note that the set

$$\{a'_1, \ldots, a'_{4k+5}\} \cup \{b'_1, \ldots, b'_{4k+5}\} \cup \{c'_2, \ldots, c'_{4k+3}\} \cup \{d'_2, \ldots, d'_{4k+3}\}$$

induces a copy of Kelly_{4k+5}.

We complete the proof by finding $k+1$ pairwise disjoint cycles in the cover graph of $P - \{x_0, y_0\}$ such that one of them lies in the region bounded by each of the remaining ones. Such cycles prevent the drawing of the cover graph of $P - \{x_0, y_0\}$ from being k-outerplanar since after k-fold removal of vertices on the boundary of the outer face we remove vertices from at most k of these cycles.

Suppose first that Ξ is disjoint from the sets Ξ^+ and Ξ^-, so $W^+ = x_0 T u^*$ and $W^- = v^* S y_0$. For each $i \in \{2, \ldots, 2k+2\}$, let $v_i = \max(W'_{i,i+1} \cap W'_{i,i-1})$ and $u_i = \min(W'_{i-1,i} \cap W'_{i+1,i})$, and define paths M_i^R, M_i^L and a cycle C_i as

$$\begin{aligned} M_i^R &= c'_i W'_{i,i+1} v_i W'_{i,i-1} d'_{i-1}, \\ M_i^L &= d''_i W'_{i+1,i} u_i W'_{i-1,i} c''_{i-1}, \\ C_i &= c'_i M_i^R d'_{i-1} S d''_i M_i^L c''_{i-1} T c'_i. \end{aligned}$$

(See Figure 5.14.) We have $V(M_i^R) \subseteq \mathrm{U}_P(v_i) \subseteq \mathrm{U}_P(a'_i)$ and $V(M_i^L) \subseteq \mathrm{D}_P(u_i) \subseteq \mathrm{D}_P(b'_i)$, so the paths M_i^R and M_i^L are disjoint and therefore C_i is indeed a cycle. Clearly, no vertex of M_i^R is left of N^* and no vertex of M_i^L is right of N^*.

Suppose that for some $i \in \{3, \ldots, 2k+2\}$, the paths M_{i-1}^R and M_i^R intersect in a vertex z. In particular, we have $a'_{i-1} \leqslant z$ and $a'_i \leqslant z$ in P. Since $a'_{i-1} \parallel b'_{i-1}$ and $d'_{i-1} \leqslant b'_{i-1}$ in P, the vertex z does not lie on the subpath $v_i W'_{i,i-1} d'_{i-1}$ of M_i^R. Since $a'_i \parallel b'_i$ and $c'_{i-1} \leqslant b'_i$ in P, the vertex z does not lie on the subpath $c'_{i-1} W'_{i-1,i} v_{i-1}$ of M_{i-1}^R. Hence,

$$M_{i-1}^R \cap M_i^R = v_{i-1} W'_{i-1,i-2} d'_{i-2} \cap c'_i W'_{i,i+1} v_i = W(v_{i-1}, d'_{i-2}) \cap W(v_i, c'_i),$$

so in particular $M_{i-1}^R \cap M_i^R$ is a path. Therefore, for each $i \in \{3, \ldots, 2k+1\}$, the path M_i^R is "sandwiched" between M_{i-1}^R and M_{i+1}^R: no vertex of the path M_i^R is right of $x_0 T c'_{i-1} M_{i-1}^R d'_{i-2} S y_0$ or left of $x_0 T c'_{i+1} M_{i+1}^R d'_i S y_0$. It is hence easy to see that the paths M_{i-1}^R and M_{i+1}^R are disjoint. By symmetric arguments, the paths M_{i-1}^L and M_{i+1}^L are disjoint. As a consequence, the

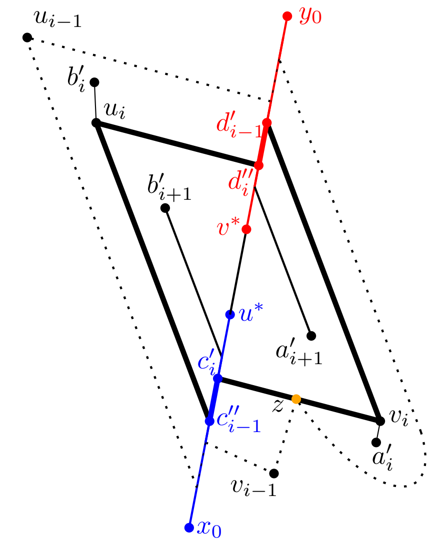


Figure 5.14: The cycle C_i is bolded. The cycle C_{i-1} (dotted) may intersect C_i in a vertex z (orange) lying on the intersection of $v_i M_i^R c'_i$ and $v_{i-1} M_{i-1}^R d''_i$.

cycle C_{i-1} and C_{i+1} are disjoint. Therefore, the cycles of the form C_{2j} with $j \in \{1, \dots, k+1\}$ are pairwise disjoint, and for each $j \in \{2, \dots, k+1\}$, the cycle C_{2j} lies in the region bounded by C_{2j-2}.

Let us show that for each $j \in \{1, \dots, k+1\}$, the cycle C_{2j} does not contain x_0. Since $a'_{2j-1} \leqslant c'_{2j-1}$ and $x_0 \leqslant b'_{2j-1}$ in P, we have $c_{2j-1} \neq x_0$. As $C_{2j} \cap x_0 T u^* = c''_{2j-1} T c'_{2j}$, this implies that C_{2j} does not contain x_0. A symmetric argument shows that C_{2j} does not contain y_0. Hence the cycles $C_2, \dots, C_{2k+2}$ prevent the drawing of the cover graph of $P - \{x_0, y_0\}$ from being k-outerplanar.

It remains to consider the cases when $\Xi \subseteq \Xi^+$ or $\Xi \subseteq \Xi^-$. Because of duality, we assume without loss of generality that $\Xi \subseteq \Xi^+$, and therefore $W^+ = W(v^*, u^*)$, and $W^- = v^* S y_0$.

Recall that $d'_{4k+4}, \dots, d'_1$ are distinct vertices which appear in that order on the witnessing path $W^- = v^* S y_0$. The paths $a'_i W'_{i,i-1} d'_{i-1}$ with $i \in \{2, \dots, 4k+5\}$ have no vertices left of N^*. We claim that these paths are pairwise disjoint. Suppose to the contrary that there exist indices $i, j \in \{2, \dots, 4k+5\}$ with $i < j$ such that the paths $a'_i W'_{i,i-1} d'_{i-1}$ and $a'_j W'_{j,j-1} d'_{j-1}$ intersect, and choose a pair of such indices with the smallest difference $j - i$. It is impossible that $j - i = 1$ as that would imply $a'_i \leqslant d'_{j-1} \leqslant b'_{j-1} = b'_i$ in P, so $j - i \geqslant 2$. By minimality of the difference $j - i$, the path

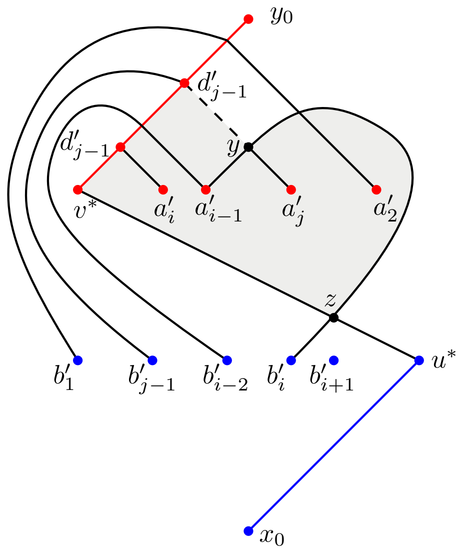


Figure 5.15: The witnessing path $W'_{i,i+1}$ must intersect the cycle C (bounding the shaded region) on the path $yW'_{j,j-1}d'_{j-1}$.

$a'_{i+1}W'_{i+1,i}d'_i$ is disjoint from $a'_iW'_{i,i-1}d'_{i-1}$ and $a'_jW'_{j,j-1}d'_{j-1}$. Now, for the vertex $w = \max(a'_iW'_{i,i-1}d'_{i-1} \cap a'_jW'_{j,j-1}d'_{j-1})$, the cycle $wW'_{j,j-1}d'_{j-1}Sd'_{i-1}W'_{i,i-1}w$ witnesses that a'_{i+1} is enclosed by $\mathrm{U}_P(a'_j)$. This contradicts Lemma 5.12, so the paths $a'_iW'_{i,i-1}d'_{i-1}$ must be pairwise disjoint.

Next, we show that for any $i, j \in \{2, \ldots, 4k+4\}$ with $j \leqslant i$, the path $W'_{i,i+1}$ intersects the path $W'_{j,j-1}$. We prove this by induction on i. The base case $i = 2$ holds true: the paths $W'_{2,3}$ and $W'_{2,1}$ intersect in the vertex a'_2. Let $i \in \{3, \ldots, 4k+4\}$. The paths $W'_{i,i+1}$ and $W'_{i,i-1}$ intersect in a'_i, so it suffices to show for $j \in \{2, \ldots, i-1\}$ that if $W'_{i-1,i}$ intersects $W'_{j,j-1}$, then $W'_{i,i+1}$ intersects $W'_{j,j-1}$ as well. Let $y = \max(W'_{i-1,i} \cap W'_{j,j-1})$, and let $z = \min(yW'_{i-1,i}b'_i \cap W(v^*, u^*))$. Consider the cycle $C = yW'_{i-1,i}zN^*d'_{j-1}W'_{j,j-1}y$, see Figure 5.15.

We claim that the vertex a'_i lies in the region bounded by C. The path $a'_iW'_{i,i-1}d'_{i-1}$ intersects v^*Sy_0 only in the vertex d'_{i-1} and is disjoint from the path $a'_jW'_{j,j-1}d'_{j-1}$. Furthermore, $a'_iW'_{i,i-1}d'_{i-1}$ is disjoint from $yW'_{i-1,i}zN^*v^*$ since $V(yW'_{i-1,i}zN^*v^*) \subseteq \mathrm{D}_P(z) \subseteq \mathrm{D}_P(b'_i)$. Hence $a'_iW'_{i,i-1}d'_{i-1}$ intersects C only in d'_{i-1} which is an inner vertex of $v^*Sd'_{j-1}$. As no vertex of $a'_iW'_{i,i-1}d'_{i-1}$ or C is left of N^*, the vertex a'_i must lie in the region bounded by C.

The vertex b'_{i+1} is left of N^*, so it does not lie in the region bounded by

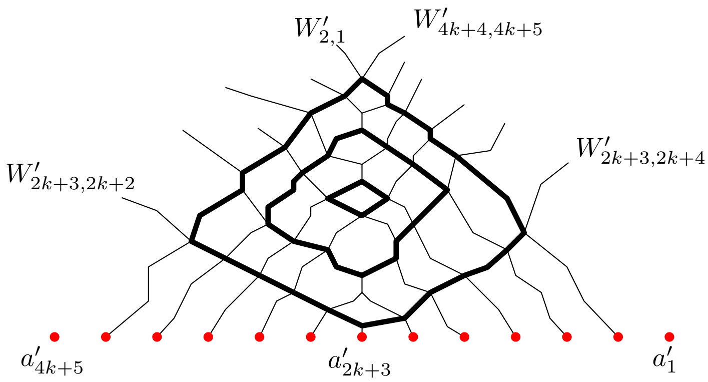


Figure 5.16: The cycles $C_1, \ldots C_{k+1}$ are bolded.

C. The path $W'_{i,i+1}$ must therefore intersect C. However, the path $W'_{i,i+1}$ is disjoint from $yW'_{i-1,i}zN^*v^*$ since $V(yW'_{i-1,i}zN^*v^*) \subseteq \mathrm{D}_P(z) \subseteq \mathrm{D}_P(b'_i)$. It is also disjoint from v^*Sy_0, so $W'_{i,i+1}$ must intersect the cycle C on the path $yW'_{j,j-1}d'_{j-1}$. This completes the inductive proof.

For any $j \in \{2, \ldots, 2k+3\}$ and $i \in \{2k+3, \ldots, 4k+4\}$, we have $j \leqslant i$, so the paths $W'_{i,i+1}$ and $W'_{j,j-1}$ intersect. For every $i \in \{2k+3, \ldots, 4k+3\}$, we have $a'_{i+1} \parallel b'_{i+1}$ in P, so there do not exist $x \in V(W'_{i,i+1})$ and $y \in V(W'_{i+1,i+2})$ such that $y \leqslant x$ in P. Hence, each witnessing path $W'_{j,j-1}$ with $j \in \{2, \ldots, 2k+3\}$ must intersect the paths $W'_{2k+3,2k+4}, \ldots, W'_{4k+4,4k+5}$ in that order. By a symmetric argument, each path $W'_{i,i+1}$ with $i \in \{2k+3, \ldots, 4k+4\}$ intersects the paths $W'_{2k+3,2k+2}, \ldots, W'_{2,1}$ in that order. Hence the paths $W'_{2,1}, \ldots, W'_{2k+3,2k+2}$ and $W'_{2k+3,2k+4}, \ldots, W'_{4k+4,4k+5}$ form a $(2k+2) \times (2k+2)$ grid. It is hence easy to see that there exist $k+1$ "nested" cycles $C_1, \ldots, C_{k+1}$ such that for each $\alpha \in \{1, \ldots, k+1\}$ we have

$$C_\alpha \subseteq \bigcup_{j \in \{1+\alpha, 2k+4-\alpha\}} W'_{j,j-1} \cup \bigcup_{i \in \{2k+2+\alpha, 4k+5-\alpha\}} W'_{i,i+1},$$

and for each $\alpha \in \{1, \ldots, k\}$, the cycle $C_{\alpha+1}$ has all vertices in the region bounded by C_α. (See Figure 5.16.)

Observe that none of the cycles $C_1, \ldots, C_{k+1}$ contains x_0 or y_0: every vertex z of any of these cycles lies on a witnessing path of the form W'_{i_1,i_2}, so $a'_{i_1} \leqslant z \leqslant b'_{i_2}$ in P. Since $x_0 \leqslant b'_{i_1}$ and $a'_{i_2} \leqslant y_0$ in P, we have $z \notin \{x_0, y_0\}$. Therefore, the cycles $C_1, \ldots, C_{k+1}$ witness that the drawing of the cover graph of $P - \{x_0, y_0\}$ is not k-outerplanar. The proof of Lemma 5.11 is complete. $\square$

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