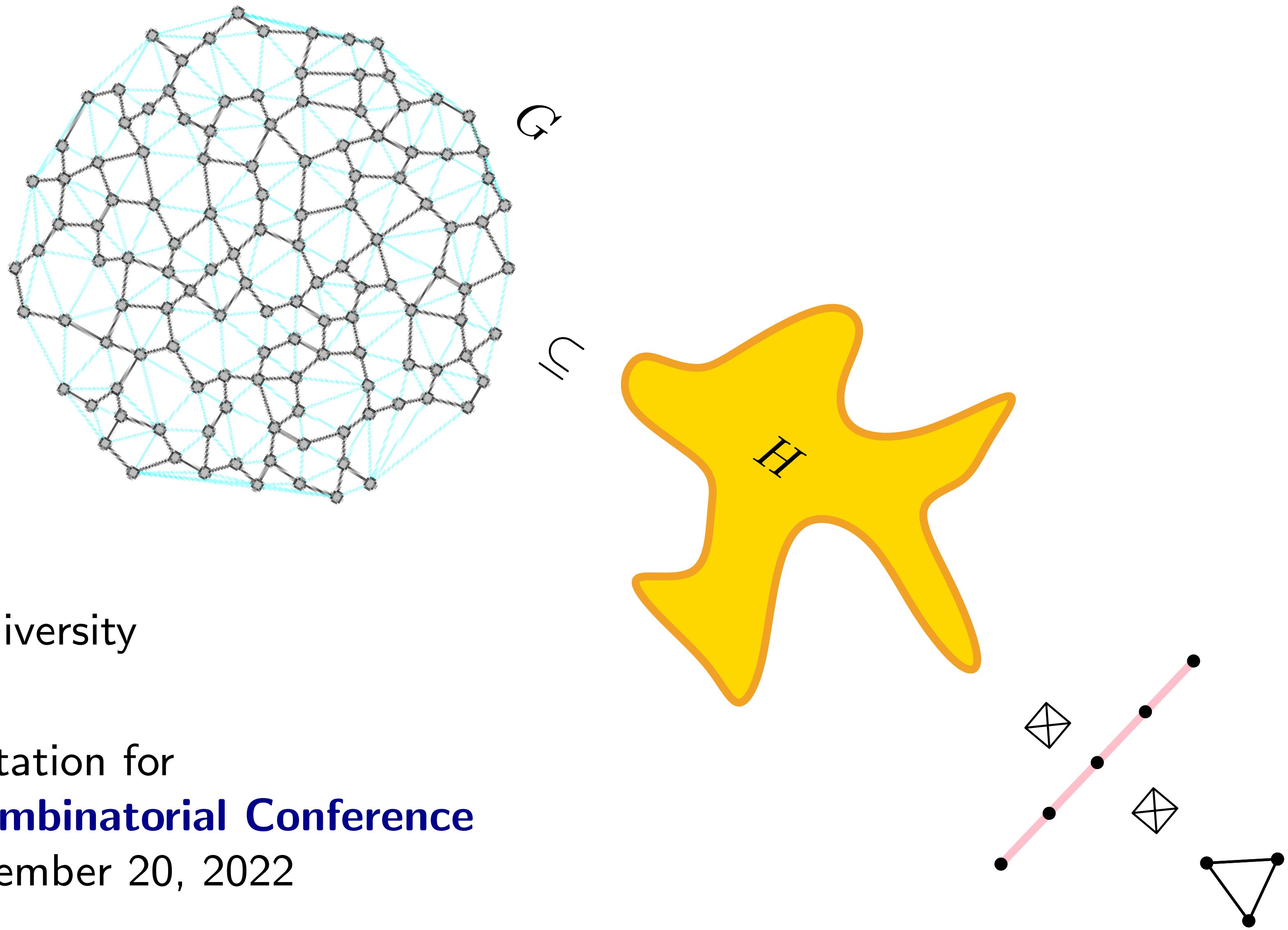
Product structure of planar graphs



Piotr Micek
Jagiellonian University

tutorial presentation for
9th Polish Combinatorial Conference
Będlewo, September 20, 2022

plan of tutorial

Part I statements
background
proof
variants / generalizations

Part II quick applications

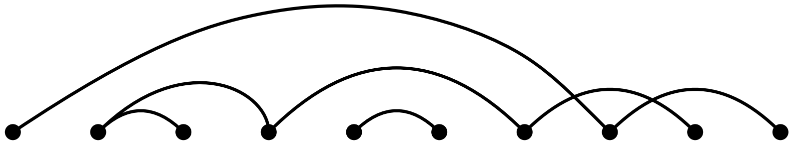
Part III application: adjacency labelling scheme
open problems / further research

the strategy is simple

- ▷ take a problem that is solved for **bounded treewidth** graphs but open for **planar** graphs
- ▷ use the product structure to lift the solution

queue number

$(G, <)$ graph with a linear order on vertices



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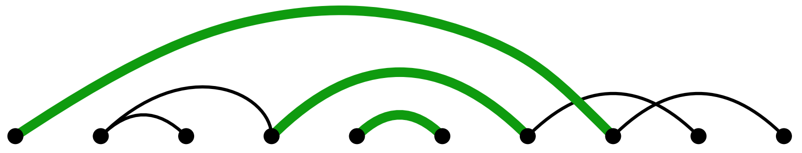
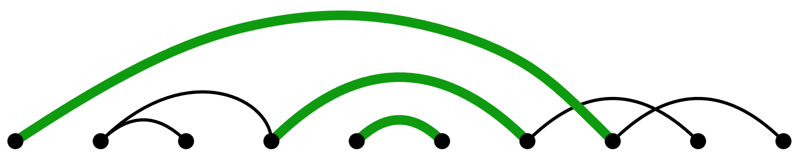


rainbow in $(G,<)$ of size 3

queue number

$(G, <)$ graph with a linear order on vertices



rainbow in $(G, <)$ of size 3

$\mathrm{qn}(G)$ queue number of G

smallest integer k such that there is an ordering $<$ of $V(G)$ with each rainbow of size $\leqslant k$

(Heath, Leighton, Rosenberg 1991)

Is there a constant C s.t. G planar $\Longrightarrow \mathrm{qn}(G) \leqslant C$?

history and related work

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G planar, n vertices $\implies \mathrm{qn}(G) \in O(\log^2(n))$

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- (Bekos et al. 2019)
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G planar, maximum degree $\Delta \implies \mathrm{qn}(G) \in O(\Delta^6)$

using **Product Structure Theorem**:

G is planar $\Rightarrow \mathrm{qn}(G) \leqslant 49$

proof

Lemma $\operatorname{qn}(H \boxtimes P) \leqslant 3 \cdot \operatorname{qn}(H) + 1$ for every path P

proof

Lemma $\mathrm{qn}(H \boxtimes P) \leqslant 3 \cdot \mathrm{qn}(H) + 1$ for every path P

Corollary For every planar graph G

$G \subseteq H \boxtimes P$ with $\mathrm{tw}(H) \leqslant 8$ and P a path

$$\mathrm{qn}(G) \leqslant \mathrm{qn}(H \boxtimes P) \leqslant 3\,\mathrm{qn}(H) + 1 \leqslant 3 \cdot (2^8 - 1) = 766$$

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Corollary For every planar graph G

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$$\mathrm{qn}(G) \underset{\text{monotonicity}}{\leqslant} \mathrm{qn}(H \boxtimes P) \underset{\text{lemma}}{\leqslant} 3\,\mathrm{qn}(H) + 1 \underset{\mathrm{tw}(H) \leqslant 8}{\leqslant} 3 \cdot (2^8 - 1) = 766$$

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Lemma $\operatorname{qn}(H \boxtimes P) \leqslant 3 \cdot \operatorname{qn}(H) + 1$ for every path P

Corollary For every planar graph G

$G \subseteq H \boxtimes P$ with $\operatorname{tw}(H) \leqslant 8$ and P a path

monotonicity lemma $\operatorname{tw}(H) \leqslant 8$

$$\operatorname{qn}(G) \leqslant \operatorname{qn}(H \boxtimes P) \leqslant 3\operatorname{qn}(H) + 1 \leqslant 3 \cdot (2^8 - 1) = 766$$

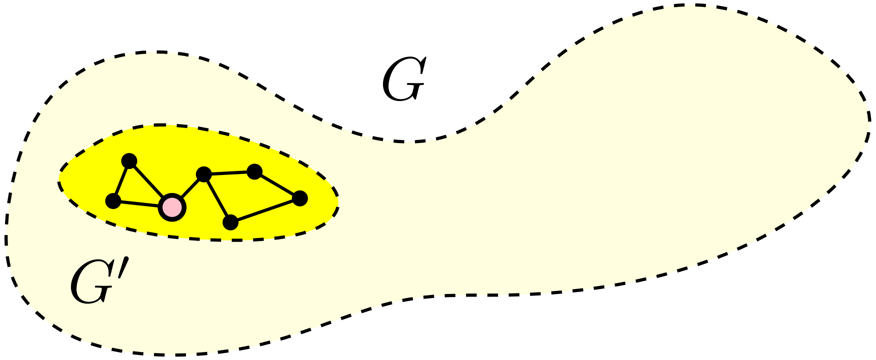
Proof of the lemma at a blackboard

centered colorings

color vertices of G in such a way that

$\forall$ (non-empty) connected subgraph G' of G

there is a color used exactly once in G'



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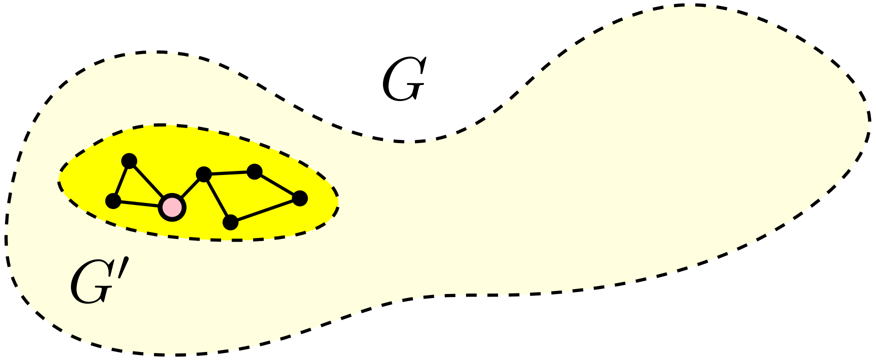

$\mathrm{cen}(G)$

min number of colors

in such a coloring of G

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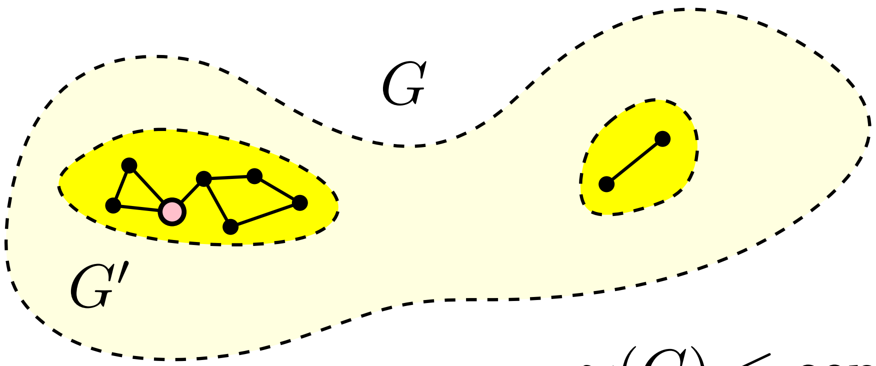

$\chi(G) \leqslant \mathrm{cen}(G)$

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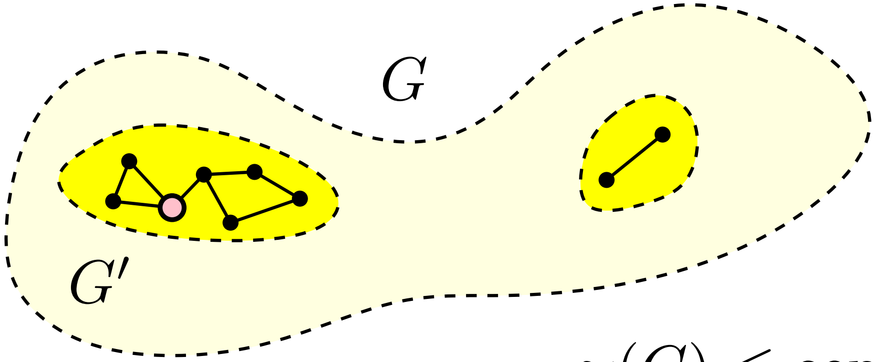


$\mathrm{cen}(G)$

min number of colors

in such a coloring of G

$$\chi(G) \leqslant \mathrm{cen}(G)$$

actually this peculiar parameter coincides with ...

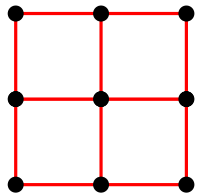
$$\mathrm{cen}(G) = \mathrm{td}(G)$$

the treedepth of G

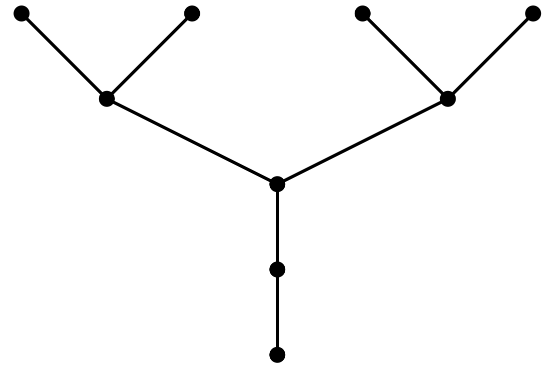
centered coloring

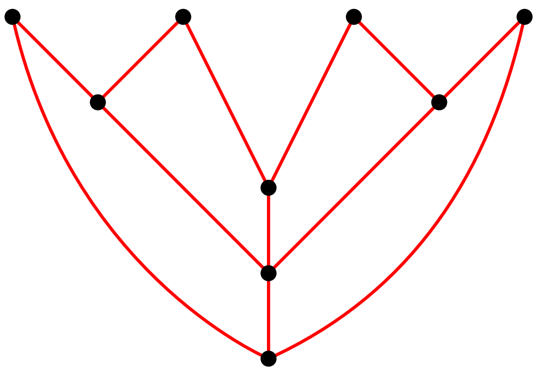
The treedepth of G is the minimum height of
a rooted forest F such that $G \subseteq \mathrm{clos}(F)$.

G

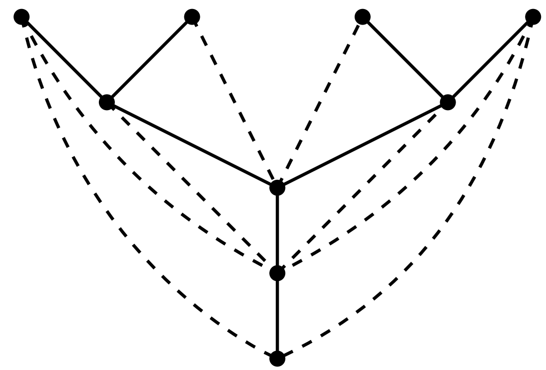


F





$\subseteq$



$\mathrm{clos}(F)$

so $\mathrm{td}(G) \leqslant 5$

centered coloring

The treedepth of G is the minimum height of
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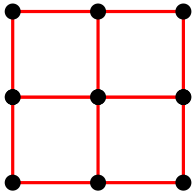
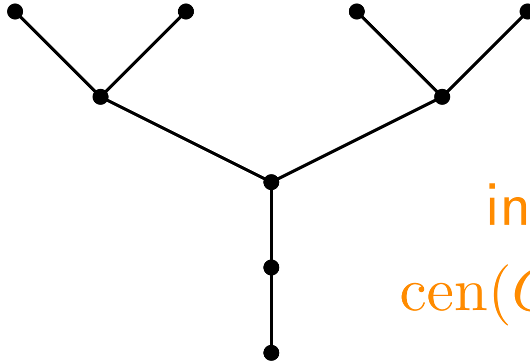
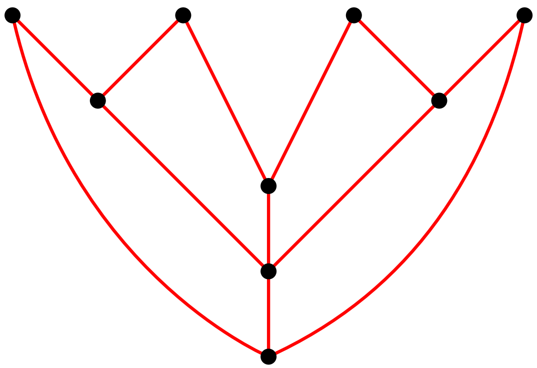
G



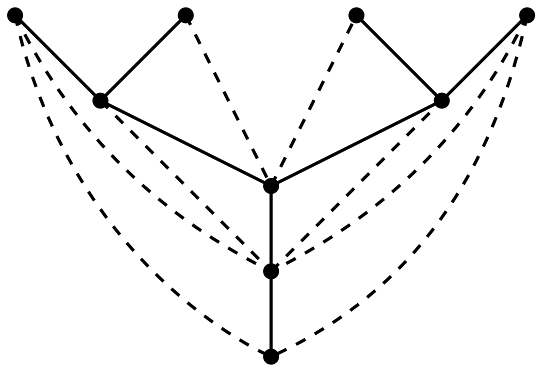
F



indeed
$\operatorname{cen}(G) = \operatorname{td}(G)$



$\subseteq$



$\operatorname{clos}(F)$

so $\operatorname{td}(G) \leqslant 5$

between $\chi(G)$ and $\mathrm{td}(G)$

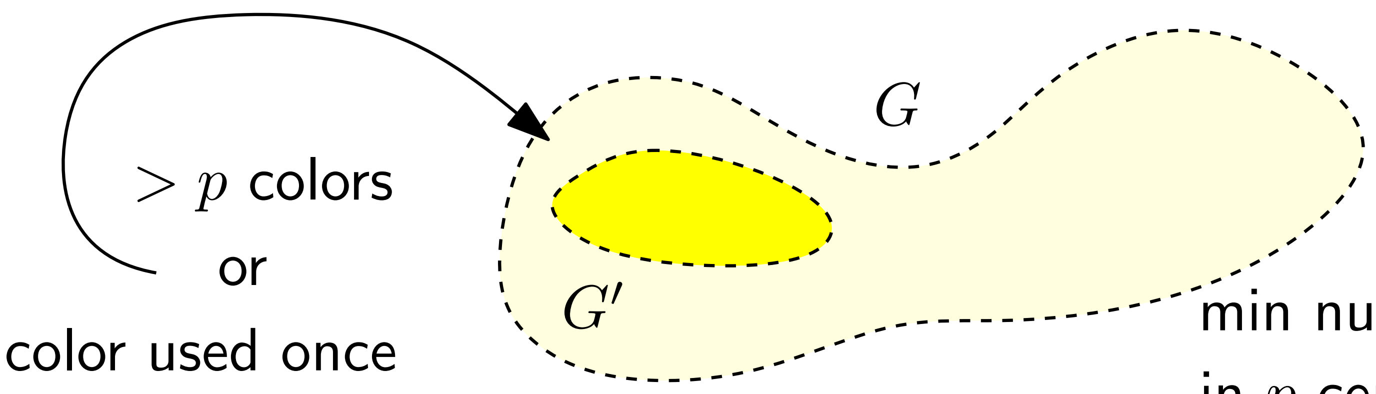
here is an idea how to interpolate between the two:

let $p \in \{1, 2, \ldots, \infty\}$

A vertex coloring ϕ of G is p-centered if

$\forall$ (non-empty) connected subgraph G' of G

either ϕ uses more than p colors on G', or
there is a color that appears exactly once on G'



$\chi_p(G)$
min number of colors
in p-centered coloring of G

p-centered colorings

$$\chi(G) = \chi_1(G) \leqslant \chi_2(G) \leqslant \cdots \leqslant \chi_\infty(G) = \mathrm{td}(G)$$

1-centered coloring $\equiv$ proper coloring

2-centered coloring $\equiv$ star coloring

∞-centered coloring $\equiv$ centered coloring

this family of parameters captures

important concepts in sparsity

p-centered colorings

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A class of graphs $\mathcal{C}$ has bounded expansion iff

$$\exists_{f} \ \forall_{p \geqslant 1} \ \forall_{G \in \mathcal{C}} \qquad \chi_p(G) \leqslant f(p)$$

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this includes planar graphs, bounded degree graphs, and more

bounds for planar graphs

when G is planar

Is $\chi_p(G)$ bounded by a polynomial in p? (Dvořák 2016)

$\chi_p(G) = \mathcal{O}(p^{19})$ (Mi. Pilipczuk, Siebertz 2019)

$\chi_p(G) = \mathcal{O}(p^3 \log p)$ (Dębski, Felsner, PM, Schröder 2020)

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we will need results for graphs of bounded tw

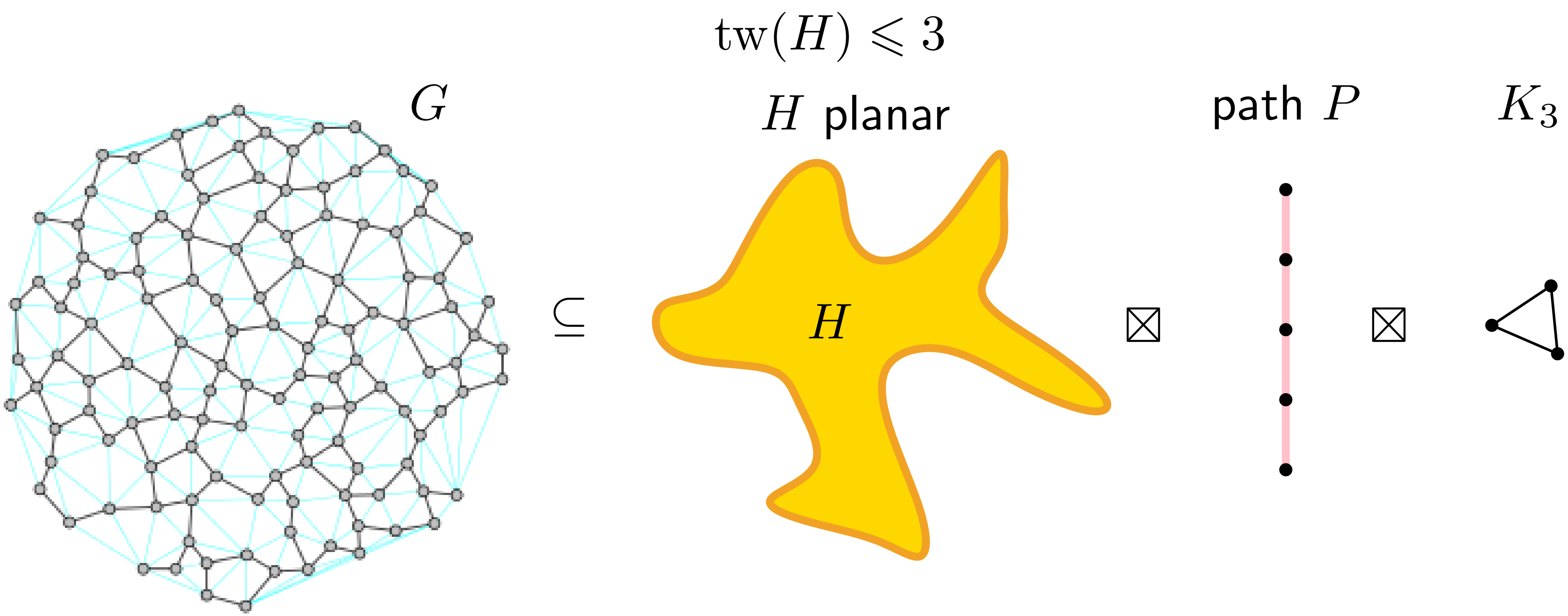
$\chi_p(G) \leqslant \binom{p+k}{k} = \mathcal{O}(p^k)$ when $\mathrm{tw}(G) \leqslant k$ (PS 2019)

$\chi_p(G) = \mathcal{O}(p^{k-1} \log p)$ when $\mathrm{stw}(G) \leqslant k$ $(k \geqslant 2)$ (DFMS 2020)

product structure theorem

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2020)

For every **planar graph** G, we have



proof strategy

Lemma $\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \chi_p(H_2)$

proof strategy

Lemma $\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \chi_p(H_2)$

Wrong!

proof strategy

Lemma $$\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \chi_p(H_2)$$

Wrong!

Lemma $$\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \overline{\chi_p}(H_2)$$

Correct

proof strategy

Lemma $\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \chi_p(H_2)$ **Wrong!**

Lemma $\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \overline{\chi_p}(H_2)$ **Correct**

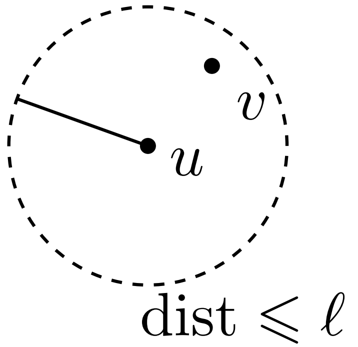
auxilliary coloring

let $\ell \in \{1, 2, \ldots\}$

A vertex coloring ψ of G is distance-ℓ coloring if

$\forall$ u, v in G if $\mathrm{dist}_G(u, v) \leqslant \ell$ then $\psi(u) \neq \psi(v)$

$\overline{\chi_\ell}(G)$ min number of colors in such a coloring of G



proof strategy

Lemma $\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \chi_p(H_2)$ Wrong!

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A vertex coloring ψ of G is distance-ℓ coloring if

$\forall_{u,\, v \text{ in } G}$ if $\mathrm{dist}_G(u, v) \leqslant \ell$ then $\psi(u) \neq \psi(v)$

- every path connecting two distinct vertices of the same color contains $> \ell$ colors
- distance-p coloring is p-centered

proof strategy

G planar $\longrightarrow$ $G \subseteq H \boxtimes P \boxtimes K_3$

proof strategy

G planar → $G \subseteq H \boxtimes P \boxtimes K_3$

$$\begin{aligned}\chi_p(G) &\leqslant \chi_p(H \boxtimes P \boxtimes K_3) \leqslant \chi_p(H \boxtimes P) \cdot \overline{\chi_p}(K_3) \\ &\leqslant \chi_p(H) \cdot \overline{\chi_p}(P) \cdot \overline{\chi_p}(K_3) \\ &\leqslant \chi_p(H) \cdot (p+1) \cdot 3 \\ &= \mathcal{O}(p^2 \log p) \cdot (p+1) \cdot 3 \\ &= \mathcal{O}(p^3 \log p)\end{aligned}$$

proof strategy

G planar $\longrightarrow$ $G \subseteq H \boxtimes P \boxtimes K_3$

monotonicity

$$\chi_p(G) \leqslant \chi_p(H \boxtimes P \boxtimes K_3) \leqslant \chi_p(H \boxtimes P) \cdot \overline{\chi_p}(K_3)$$

$$\leqslant \chi_p(H) \cdot \overline{\chi_p}(P) \cdot \overline{\chi_p}(K_3)$$

$$\leqslant \chi_p(H) \cdot (p+1) \cdot 3$$

$$= \mathcal{O}(p^2 \log p) \cdot (p+1) \cdot 3$$

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monotonicity lemma

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lemma

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G planar $\longrightarrow$ $G \subseteq H \boxtimes P \boxtimes K_3$

monotonicity — lemma

$$\chi_p(G) \leqslant \chi_p(H \boxtimes P \boxtimes K_3) \leqslant \chi_p(H \boxtimes P) \cdot \overline{\chi_p}(K_3)$$

lemma $\leqslant \chi_p(H) \cdot \overline{\chi_p}(P) \cdot \overline{\chi_p}(K_3)$

H planar
$\mathrm{tw}(H) \leqslant 3$

$$\leqslant \chi_p(H) \cdot (p+1) \cdot 3$$

$$= \mathcal{O}(p^2 \log p) \cdot (p+1) \cdot 3$$

$$= \mathcal{O}(p^3 \log p)$$

proof of the lemma

$$\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \overline{\chi_p}(H_2)$$

Proof

$\psi_1 \quad \equiv \quad p$-centered coloring of H_1

$\psi_2 \quad \equiv \quad$ distance-p coloring of H_2

proof of the lemma

$$\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \overline{\chi_p}(H_2)$$

Proof

$\psi_1 \equiv$ p-centered coloring of H_1

$\psi_2 \equiv$ distance-p coloring of H_2

Goal

$\phi((v,w)) = (\psi_1(v), \psi_2(w))$ for every (v,w) in $H_1 \boxtimes H_2$

is a p-centered coloring of $H_1 \boxtimes H_2$

proof of the lemma

$$\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \overline{\chi_p}(H_2)$$

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is a p-centered coloring of $H_1 \boxtimes H_2$

consider $G \subseteq H_1 \boxtimes H_2$ connected subgraph $\longrightarrow$ $G_1 = \pi_1(G)$, $G_2 = \pi_2(G)$ are connected subgraphs of H_1, H_2

proof of the lemma

$$\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \overline{\chi_p}(H_2)$$

Proof

$\psi_1 \quad \equiv \quad p$-centered coloring of H_1

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either $|\psi_1(G_1)| > p$, or

there is a ψ_1-color used exactly once in G_1

proof of the lemma

$$\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \overline{\chi_p}(H_2)$$

Proof

$\psi_1 \equiv$ p-centered coloring of H_1

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Goal

$\phi((v,w)) = (\psi_1(v), \psi_2(w))$ for every (v,w) in $H_1 \boxtimes H_2$

is a p-centered coloring of $H_1 \boxtimes H_2$

consider $G \subseteq H_1 \boxtimes H_2$ connected subgraph → $G_1 = \pi_1(G)$, $G_2 = \pi_2(G)$ are connected subgraphs of H_1, H_2

either $|\psi_1(G_1)| > p$, or there is a ψ_1-color used exactly once in G_1 → $|\phi(G)| > p$ as desired

proof of the lemma

$$\chi_p(H_1 \boxtimes H_2) \leqslant \chi_p(H_1) \cdot \overline{\chi_p}(H_2)$$

Proof

$\psi_1 \equiv$ p-centered coloring of H_1

$\psi_2 \equiv$ distance-p coloring of H_2

Goal

$\phi((v,w)) = (\psi_1(v), \psi_2(w))$ for every (v,w) in $H_1 \boxtimes H_2$

is a p-centered coloring of $H_1 \boxtimes H_2$

consider $G \subseteq H_1 \boxtimes H_2$ connected subgraph → $G_1 = \pi_1(G)$, $G_2 = \pi_2(G)$ are connected subgraphs of H_1, H_2

either $|\psi_1(G_1)| > p$, or → $|\phi(G)| > p$ as desired

there is a ψ_1-color used exactly once in G_1 → ...

proof of the lemma

there is a ψ_1-color used exactly once in G_1

say at vertex v_1 in G_1

fix a (v_1, w) in G

proof of the lemma

there is a ψ_1-color used exactly once in G_1

say at vertex v_1 in G_1

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either (v_1, w) has a unique ϕ-color in G, or → as desired

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Q a path in $G \subseteq H_1 \boxtimes H_2$

$\pi_2(Q)$ is a lazy walk in H_2
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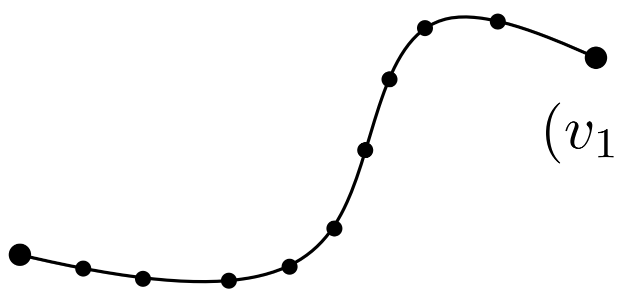

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(v_1, w)

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$|\psi_2(\pi_2(Q))| > p$

so $|\phi(G)| > p$

as desired

nonrepetitive sequences

square of a word w: $$w^2 = ww$$ $$(ab)^2 = abab$$

A word is nonrepetitive if it contains no square

Example square is nonrepetitive

repetition has a repetition

nonrepetitive sequences

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A word is nonrepetitive if it contains no square

Example square is nonrepetitive

repetition has a repetition

Theorem (Thue 1906)

There is an infinite nonrepetitve word on 3 symbols

abcabacabcbacbcacbabcabacabcb. . .

nonrepetitive colorings

A coloring of vertices of G is nonrepetitive if
for every path in G, the color sequence along the path

is nonrepetivive

$\pi(G)$ minimum number of colors in
a nonrepetitive coloring of G

Theorem (Thue 1906)
For every path P, $\pi(P) \leqslant 3$

Question (Alon, Grytczuk, Hałuszczak, Riordan 2002)
Is there a constant c such that
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Theorem (Dujmović, Esperet, Joret, Walczak, Wood 2019)
For every planar graph G, $\pi(G) \leqslant 768$

nonrepetitive colorings

$$
\begin{aligned}
\pi(G) \quad & \leqslant \pi(H \boxtimes P \boxtimes K_3) \\
& \leqslant \pi^*(H \boxtimes P \boxtimes K_3) \\
& \leqslant \pi^*(H \boxtimes P) \cdot 3 \\
& \leqslant \pi^*(H) \cdot 4 \cdot 3 \\
& \leqslant 4^3 \cdot 4 \cdot 3 = 768
\end{aligned}
$$

nonrepetitive colorings

monotonicity

$$\begin{aligned}\pi(G) &\leqslant \pi(H \boxtimes P \boxtimes K_3)\\ &\leqslant \pi^*(H \boxtimes P \boxtimes K_3)\\ &\leqslant \pi^*(H \boxtimes P) \cdot 3\\ &\leqslant \pi^*(H) \cdot 4 \cdot 3\\ &\leqslant 4^3 \cdot 4 \cdot 3 = 768\end{aligned}$$

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$\pi \leqslant \pi^*$

$\pi^*(G)$ minimum number of colors in a strongly nonrepetitive coloring of G

A coloring of vertices of G is strongly nonrepetitive if for every lazy walk $(v_1, \ldots, v_{2t})$ in G, if the color sequence is a repetition then

$\exists i$ st $v_i = v_{t+i}$

nonrepetitive colorings

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$$\begin{aligned}\pi(G) &\leqslant \pi(H \boxtimes P \boxtimes K_3)\\ &\leqslant \pi^*(H \boxtimes P \boxtimes K_3)\\ &\leqslant \pi^*(H \boxtimes P) \cdot 3\\ &\leqslant \pi^*(H) \cdot 4 \cdot 3\\ &\leqslant 4^3 \cdot 4 \cdot 3 = 768\end{aligned}$$

$\pi \leqslant \pi^*$

(Kündgen, Pelsmajer 2008)
$\pi^*(H) \leqslant 4^k$
when $\mathrm{tw}(H) \leqslant k$

$\pi^*(G)$ minimum number of colors in a strongly nonrepetitive coloring of G

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