Polar decompositions and indefinite inner products

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Outline

1. Introduction: Polar decompositions and indefinite inner products

2. Applications of H-polar decompositions

3. Construction of H-polar decompositions

4. H-polar decompositions and H-normal matrices

5. H-polar decompositions and Krein spaces

H-structured matrices

H-adjoint of X is the matrix $X^{[*]}$ satisfying

$$[x, Xy] = [X^{[*]}x, y] \quad \text{for all } x, y \in \mathbb{C}^n$$

$$\Leftrightarrow \; y^*X^*Hx = y^*HX^{[*]}x \quad \text{for all } x, y \in \mathbb{C}^n \; \Leftrightarrow \; X^{[*]} = H^{-1}X^*H.$$

A is **H-selfadjoint**	$A^{[*]} = A$	$A^*H = HA$
S is **H-skew-adjoint**	$S^{[*]} = -S$	$S^*H = -HS$
U is **H-unitary**	$U^{[*]} = U^{-1}$	$U^*HU = H$

H-selfadjoint matrices

Example: $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\quad A_1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\lambda \in \mathbb{R}$, $\quad A_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

- A_1 and A_2 are H-selfadjoint: $A_1^* H = H A_1$ and $A_2^* H = H A_2$;
- H-selfadjoint matrices need not be diagonalizable;
- the spectrum of H-selfadjoint matrices need not be real;
- **but**: the spectrum of H-selfadjoint matrices is symmetric with respect to the real axis;
- canonical forms for H-selfadjoint matrices are known;

H-unitary matrices

Example: $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad a \neq 0.$

- U_1 and U_2 are H-unitary: $U_1^* H U_1 = H$ and $U_2^* H U_2 = H$;
- H-unitary matrices satisfy $[Ux, Uy] = [x, y]$ for all x, y;
- the spectrum of H-unitary matrices is symmetric with respect to the unit circle;
- canonical forms for H-unitary matrices are known;

H-polar decompositions

polar decomposition of a matrix $X \in \mathbb{C}^{n\times n}$:

$$X = U \cdot A, \quad U \text{ is unitary, } A \text{ is Hermitian, } A \geq 0$$

H-**polar decomposition** of a matrix $X \in \mathbb{C}^{n\times n}$:

$$X = U \cdot A, \quad U \text{ is } H\text{-unitary, } A \text{ is } H\text{-selfadjoint}$$

Note: there is no *extra* assumption on A (like $HA \geq 0$ or $\sigma(A) \subseteq \mathbb{C}_+$).

Applications:
- H-Procrustes problems
- linear optics
- matrix sign function

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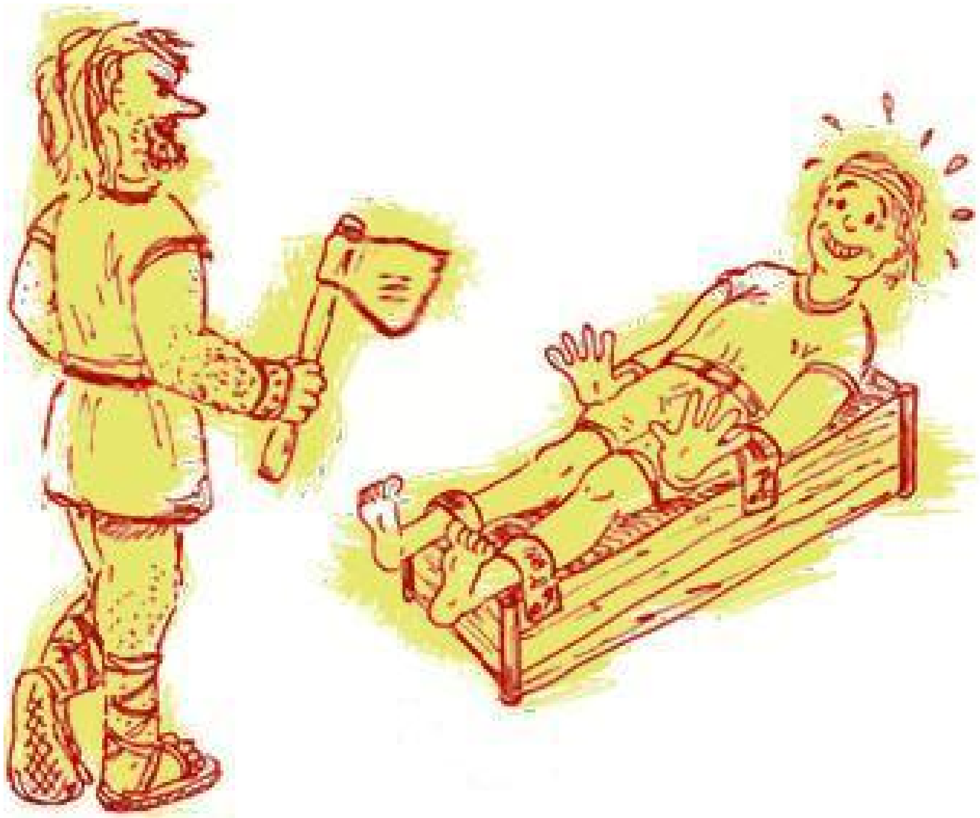
Applications: H-Procrustes problems

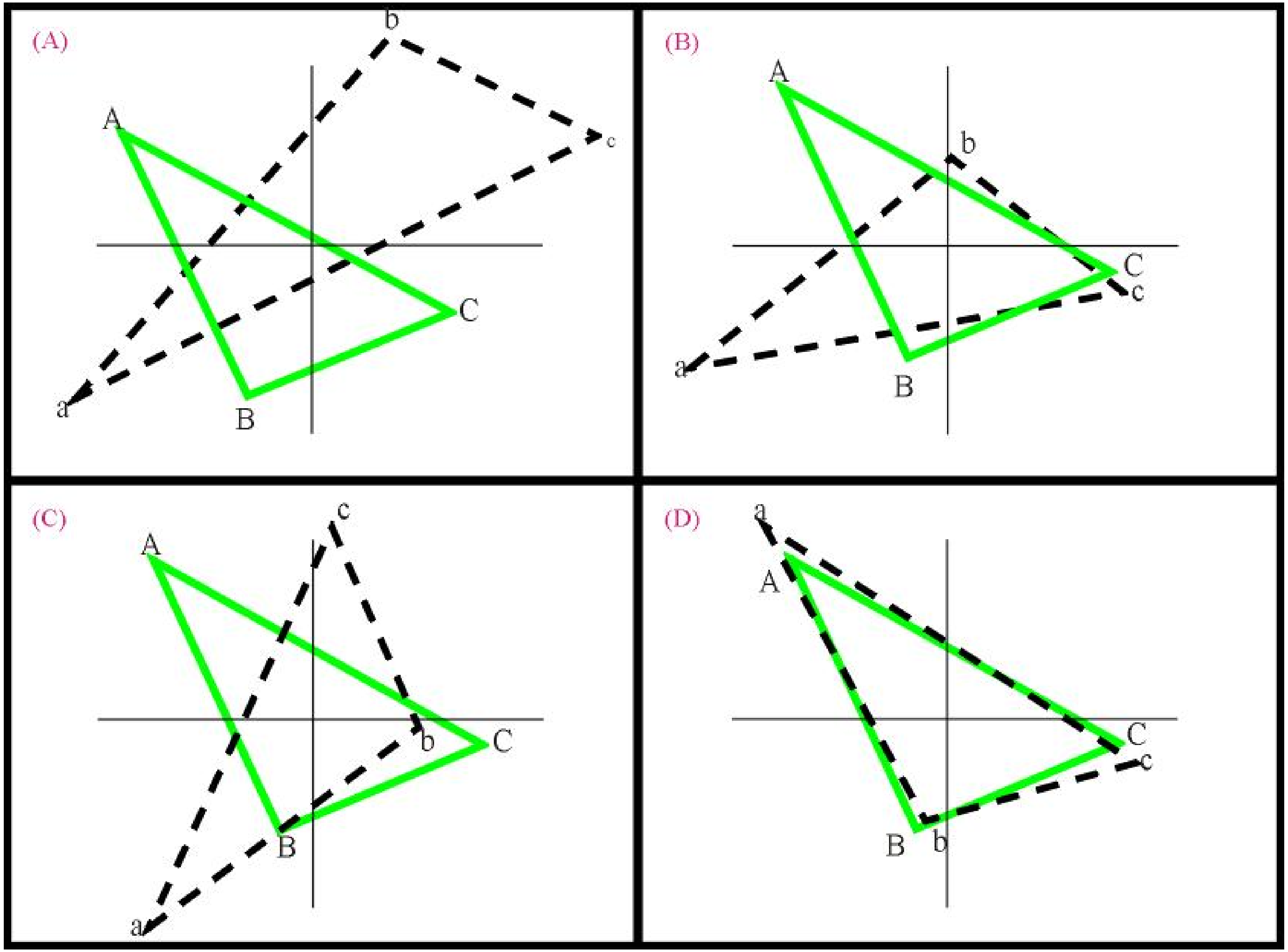


Procrustes = "the one who stretches" (thief in Greek Mythology)

Applications: H-Procrustes problems

Procrustes Analysis: a method for comparing two sets of data:

- based on matching corresponding points (landmarks) from each of the two data sets;
- minimize the sum of squared deviations between landmarks;
- do this by translation, reflection, rotation, and scaling;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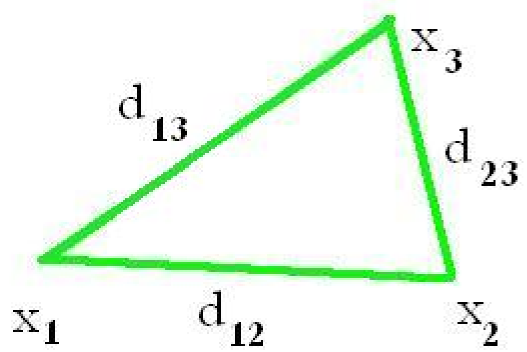
Applications: H-Procrustes problems

Specific application: MDS (multidimensional scaling) in psychology:

- test persons estimate the similarity or dissimilarity of N objects by pairwise comparison $\rightsquigarrow$ **proximities** p_{kl}, $k, l = 1, \ldots, N$;

- proximities are transformed into **distances** $d_{kl} = f(p_{kl}) \geq 0$ (triangle inequality must be satisfied);

- points $x_k \in \mathbb{R}^n$ are constructed such that $\|x_k - x_l\| = d_{kl}$;

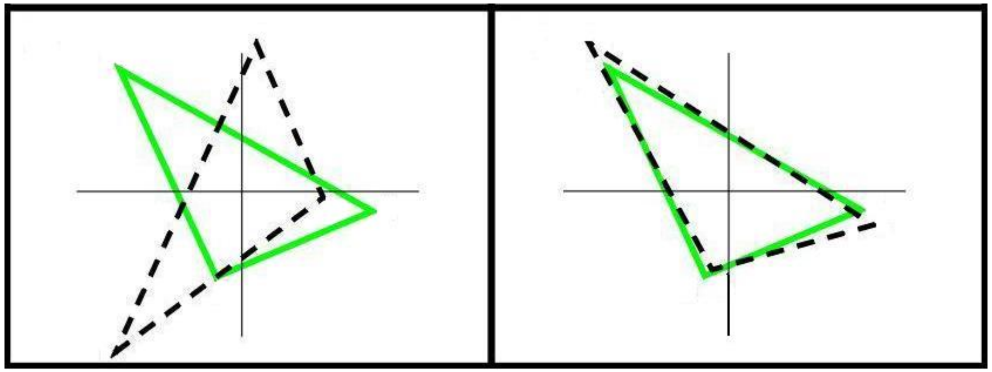


Applications: H-Procrustes problems

Specific application: MDS (multidimensional scaling) in psychology:

- comparison of two constellations $X = [x_1, \ldots, x_N]$ and $Y = [y_1, \ldots, y_N]$: solve an **orthogonal Procrustes problem** first:

$$\text{find } U \text{ orthogonal such that } \sum_{k=1}^{N} \|Ux_k - y_k\|^2 \text{ is minimized}$$



Applications: H-Procrustes problems

Specific application: MDS (multidimensional scaling) in psychology:

- comparison of two constellations $X = [x_1, \ldots, x_N]$ and $Y = [y_1, \ldots, y_N]$: solve an **orthogonal Procrustes problem** first:

  $$\text{find } U \text{ orthogonal such that } \sum_{k=1}^{N} \|Ux_k - y_k\|^2 \text{ is minimized}$$

- **Solution**: if

  $$Y^*X = UA$$

  is a polar decomposition of Y^*X, then U does the job!

Applications: H-Procrustes problems

Specific application: MDS (multidimensional scaling) in psychology:

- test persons estimate the similarity or dissimilarity of N objects by pairwise comparison $\rightsquigarrow$ **proximities** p_{kl}, $k, l = 1, \ldots, N$;
- Kintzel (2004) suggested: do not transform the proximities, but interpret them as pseudo-Euclidean distances;
- construct an indefinite inner product $[\cdot, \cdot]_H = (H\cdot, \cdot)$;
- construct points $x_k \in \mathbb{R}^n$ such that $[x_k - x_l, x_k - x_l]_H = p_{kl}$;

Applications: H-Procrustes problems

Specific application: MDS (multidimensional scaling) in psychology:

- comparison of two constellations $X = [x_1, \ldots, x_N]$ and $Y = [y_1, \ldots, y_N]$: solve an **H-orthogonal Procrustes problem** first:

$$\text{find } U \ H\text{-unitary such that } \sum_{k=1}^{N} [Ux_k - y_k, Ux_k - y_k] \text{ is optimal}$$

- **Solution** (Kintzel 2004): if

$$Y^* XH = UA, \quad HA \geq 0$$

is an H-polar decomposition of $Y^* XH$ with an H-nonnegative H-selfadjoint factor A, then U does the job!

Applications: Linear Optics

beams of light: vectors $I = \left[\begin{array}{cccc} i & q & u & v \end{array} \right]^T$.

$i > 0$	intesity
q/i, u/i, v/i	state of polarizations
$p = \sqrt{q^2 + u^2 + v^2}/i \in [0, 1]$	degree of polarization

$I = \left[\begin{array}{cccc} i & q & u & v \end{array} \right]^T$ must satisfy the inequality $i \geq \sqrt{q^2 + u^2 + v^2}$.

Applications: Linear Optics

Let $[\cdot,\cdot]$ be the inner product induced by $H = \text{diag}(1,-1,-1,-1)$.

$I = \begin{bmatrix} i & q & u & v \end{bmatrix}^T$ satisfies the inequality $i \geq \sqrt{q^2+u^2+v^2}$ if and only if

$$[I,I] = i^2 - q^2 - v^2 - u^2 \geq 0 \quad \text{and} \quad i > 0.$$

orthochronous Lorentz group: Matrices $U = (u_{ij}) \in \mathbb{R}^{4\times 4}$ that are H-unitary and satisfy $u_{11} > 0$.

Clear: $[Ux, Ux] = [x, x]$ for all $x \in \mathbb{R}^4$; also $(Ux)_1 > 0$ if $x_1 > 0$.

Applications: Linear Optics

$M \in \mathbb{R}^{4\times 4}$ satisfies the **Stokes criterion** if for all $\left[\begin{array}{cccc} i_0 & q_0 & u_0 & v_0 \end{array}\right]^T$ we have

$$i_0 \geq \sqrt{q_0^2 + u_0^2 + v_0^2} \quad \Longrightarrow \quad i \geq \sqrt{q^2 + u^2 + v^2},$$

where $\left[\begin{array}{cccc} i & q & u & v \end{array}\right]^T = M \left[\begin{array}{cccc} i_0 & q_0 & u_0 & v_0 \end{array}\right]^T$.

Such matrices transform one beam of light into another.

Example: Matrices from the orthochronous Lorentz group satisfy Stokes criterion.

Applications: Linear Optics

Criterion: Let U_1, U_2 from the orthochronous Lorentz group.

$M \in \mathbb{R}^{4\times 4}$ satisfies Stokes criterion $\Leftrightarrow$ $U_1 M U_2$ satisfies Stokes criterion.

Test whether Stokes criterion is satisfied or not:

- compute H-polar decomposition $M = UA$ with U from the orthochronous Lorentz group;
- check if H-selfadjoint factor A satisfies Stokes criterion.
  (This is easier than checking it for M directly!)

Applications: Matrix sign function

Matrix sign function: Let $Z \in \mathbb{R}^{n \times n}$ be such that $\sigma(Z) \cap i\mathbb{R} = \emptyset$. If

$$Z = S \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} S^{-1}, \quad \text{where}$$

- J_+: Jordan blocks corresponding to the (say) k eigenvalues in the open right half plane;

- J_-: Jordan blocks corresponding to the $n-k$ eigenvalues in the open left half plane;

then

$$\text{sign}(Z) := S \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} S^{-1}.$$

Applications: Matrix sign function

Usage: computation of solutions of continuous-time algebraic Riccati equations

Theorem (Byers, 1984): $X \in \mathbb{C}^{n\times n}$ nonsingular, $X = UA$ polar decomposition, then

$$\mathrm{sign}\left(\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & U \\ U^{-1} & 0 \end{bmatrix}.$$

Theorem (Higham, Mackey, Mackey, Tisseur, 2004): $X \in \mathbb{C}^{n\times n}$ nonsingular, $X = UA$ an H-polar decomposition such that $\sigma(A) \subseteq \mathbb{C}^+$, then

$$\mathrm{sign}\left(\begin{bmatrix} 0 & X \\ X^{[*]} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & U \\ U^{-1} & 0 \end{bmatrix}.$$

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How to construct H-polar decompositions

Observations for polar decompositions $X = UA$:

- $X^{[*]} = A^{[*]}U^{[*]} = AU^{-1}$
- $X^{[*]}X = AU^{-1}UA = A^2$
- $\mathrm{Ker}\ X = \mathrm{Ker}\ A$.

Construction of H-polar decompositions:

i) compute H-selfadjoint square root A of $X^{[*]}X$ s.t. $\mathrm{Ker}\ X = \mathrm{Ker}\ A$;

ii) compute H-unitary U such that $X = UA$

An example

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad X^{[*]} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

i) computation of H-selfadjoint factor A:

$$X^{[*]}X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{take, e.g., } A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

ii) computation of H-unitary factor U:

$$U = XA^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note: $HA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is NOT positive semidefinite.

Another example

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad X^{[*]} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X^{[*]}X = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \stackrel{?}{=} A^2$$

- If A would be such an H-selfadjoint square root, then $\sigma(A) \subseteq \{-i, i\}$.

- The spectrum of H-selfadjoint matrices is symmetric w.r.t. real axis $\Rightarrow \ \sigma(A) = \{-i, i\}$.

- There is no H-selfadjoint square root for $X^{[*]}X$.

When do H-polar decompositions exist?

Construction of H-polar decompositions:

i) compute H-selfadjoint square root A of $X^{[*]}X$ s.t. $\mathrm{Ker}\ X = \mathrm{Ker}\ A$;

ii) compute H-unitary U such that $X = UA$

Two questions:

i) When does $X^{[*]}X$ have an H-selfadjoint square root?;

ii) When does there exist an H-unitary polar factor U?

When do H-polar decompositions exist?

Question: i) When does $X^{[*]}X$ have an H-selfadjoint square root?

Theorem [Bolshakov, van der Mee, Ran, Reichstein, Rodman, 1996]: Let $X \in \mathbb{C}^{n\times n}$. Then $X^{[*]}X$ has an H-selfadjoint square root A satisfying $\mathrm{Ker}\ X = \mathrm{Ker}\ A$ if and only if

a) each Jordan block $\mathcal{J}_p(\lambda)$ associated with $\lambda < 0$ in the canonical form for $(X^{[*]}X, H)$ occurs an even number (say $2m$) of times such that there are exactly m blocks with sign $\varepsilon = +1$;

b) several conditions on eigenvalue $\lambda = 0$ are satisfied.

Note: $\lambda < 0$: e.g., $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $A(\varepsilon) = \begin{bmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{bmatrix}$;

$A(1)$ and $A(-1)$ are NOT H-unitarily similar $\rightsquigarrow$ additional invariant $\varepsilon = \pm 1$

When do H-polar decompositions exist?

Question: i) When does $X^{[*]}X$ have an H-selfadjoint square root?

Theorem [Bolshakov, van der Mee, Ran, Reichstein, Rodman, 1996]: Let $X \in \mathbb{C}^{n\times n}$. Then $X^{[*]}X$ has an H-selfadjoint square root A satisfying $\mathrm{Ker}\ X = \mathrm{Ker}\ A$ if and only if

a) each Jordan block $\mathcal{J}_p(\lambda)$ associated with $\lambda < 0$ in the canonical form for $(X^{[*]}X, H)$ occurs an even number (say $2m$) of times such that there are exactly m blocks with sign $\varepsilon = +1$;

b) several conditions on eigenvalue $\lambda = 0$ are satisfied.

Note: $\lambda = 0$: one set of conditions comes from $X^{[*]}X = A^2$; a second set of conditions comes from $\mathrm{Ker}\ X = \mathrm{Ker}\ A$.

When do H-polar decompositions exist?

Question: ii) When does there exist an H-unitary polar factor U?

Theorem [Bolschakov, Reichstein, 1995]: Let $X, Y \in \mathbb{C}^{n\times n}$. Then the following are equivalent:

a) $X = U_0 Y$ for an injective H-isometry $U_0 : \mathrm{Im}\ Y \to \mathrm{Im}\ X$

(i.e. $[U_0 x, U_0 y] = [x, y]$ for all $x, y \in \mathrm{Im}\ Y$);

b) $X^{[*]}X = Y^{[*]}Y$ and $\mathrm{Ker}\ X = \mathrm{Ker}\ Y$

Idea of proof: define $U_0 : \mathrm{Im}\ Y \to \mathrm{Im}\ X$ by $U_0(Yv) = Xv$

When do H-polar decompositions exist?

Question: ii) When does there exist an H-unitary polar factor U?

Theorem [Witt]: $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathbb{C}^n$ subspaces of dimension m, $U_0 : \mathcal{V}_1 \to \mathcal{V}_2$ injective H-isometry, then there exists H-unitary U such that

$$U|_{\mathcal{V}_1} = U_0.$$

Corollary: X has an H-polar decomposition iff $X^{[*]}X$ has an H-selfadjoint square root A such that $\mathrm{Ker}\ X = \mathrm{Ker}\ A$.

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Normal matrices

Definition: $X \in \mathbb{C}^{n\times n}$ is called ***H*-normal** if $X^{[*]}X = XX^{[*]}$.

Remark: $H = I$. Let $X = UA$ be the polar decomposition of X. Then

$$X \text{ is normal } \iff UA = AU.$$

Questions: Let X be H-normal.

1) Does X have an H-polar decomposition?

2) Does X have an H-polar decomposition with commuting factors?

Normal matrices

Definition: $X \in \mathbb{C}^{n\times n}$ is called **H-normal** if $X^{[*]}X = XX^{[*]}$.

Problem: canonical forms for H-normal matrices are known for special cases only:

- when H has at most one negative eigenvalue (Gohberg/Reichstein,1990);
- when H has at most two negative eigenvalues (Holtz/Strauss, 1995);
- when X is block-Toeplitz H-normal (Gohberg/Reichstein, 1991/93);
- when X satisfies $X^{[*]} = p(X)$ for some polynomial p (M., 2006);

Polar decompositions of normal matrices

Answers to 1): (existence of H-pd's)

- Bolshakov, van der Mee, Ran, Reichstein, Rodman, 1996: invertible H-normals have H-pd's;

- Bolshakov, van der Mee, Ran, Reichstein, Rodman, 1996: H-normals have H-pd's if H has at most one negative eigenvalue;

- Lins, Meade, M., Rodman, 2001: H-normals have H-pd's if H has at most two negative eigenvalues;

Polar decompositions of normal matrices

Theorem [M., Ran, Rodman, 2004] Let X be H-normal. Then X admits an H-polar decomposition.

Proof:

- requires six pages;
- induction on dim$(\mathrm{Ker}\ X)$;
- basic idea: construct an H-selfadjoint square root A of $X^{[*]}X$ satisfying $\mathrm{Ker}\ X = \mathrm{Ker}\ A$ from an H-polar decomposition of a smaller submatrix;

A criterion for the existence of polar decomps

Corollary [conjectured by Kintzel, 2002] $X \in \mathbb{C}^{n\times n}$ admits an H-polar decomposition $X = UA$ if and only if $XX^{[*]}$ and $X^{[*]}X$ are H-unitarily similar.

Proof: „$\Rightarrow$“: $XX^{[*]} = UAAU^{-1} = UX^{[*]}XU^{-1}$, where U is H-unitary

„$\Leftarrow$“: assume $UXX^{[*]}U^{-1} = X^{[*]}X$; then $\tilde{X} = UX$ is H-normal:

$$\tilde{X}\tilde{X}^{[*]} = UXX^{[*]}U^{-1} = X^{[*]}X = X^{[*]}U^{[*]}UX = \tilde{X}^{[*]}\tilde{X}.$$

$\tilde{X}$ has an H-polar decomposition $\tilde{X} = \hat{U}\hat{A} \;\Rightarrow\; X = U^{-1}\hat{U}\hat{A}$

Consequence: H-normal matrices are the prototype of matrices allowing an H-polar decomposition.

Polar decompositions of normal matrices

Questions: Let X be H-normal.

1) Does X have an H-polar decomposition?

2) Does X have an H-polar decomposition with commuting factors?

Answers:

1) Yes! (And H-normal matrices are the prototype of matrices allowing an H-polar decomposition.)

2) ?

The generalized Toeplitz decomposition

Generalized Toeplitz decomposition: $X = A_X + S_X$, where

$A_X = \frac{1}{2}(X + X^{[*]})$ is H-selfadjoint, $S_X = \frac{1}{2}(X - X^{[*]})$ is H-skew-adjoint.

$$\begin{aligned} X^{[*]} &= A_X^{[*]} + S_X^{[*]} = A_X - S_X \\ XX^{[*]} &= A_X^2 - A_X S_X + S_X A_X - S_X^2 \\ X^{[*]}X &= A_X^2 + A_X S_X - S_X A_X - S_X^2 \end{aligned}$$

Lemma: X is H-normal if and only if $A_X S_X = S_X A_X$.

The exponential map

H-unitary matrices	Lie group
H-skew-adjoint matrices	Lie algebra

Lie theory: the matrix exponential $\exp$ maps the Lie algebra into the Lie group, i.e.,

$$S \text{ is } H\text{-skew-adjoint} \Rightarrow \exp(S) \text{ is } H\text{-unitary.}$$

Moreover: A is H-selfadjoint $\Rightarrow \exp(A)$ is H-selfadjoint.

Polar decompositions with commuting factors

Theorem [Lins, Meade, M., Rodman, 2001]: Let X be an invertible H-normal. Then X admits an H-polar decomposition with commuting factors.

Idea of proof:

- show: invertible H-normals X have H-normal logarithm $Y = \log X$;
- generalized Toeplitz decomp.: $Y = A_Y + S_Y$; A_Y and S_Y commute;
- $X = \exp(Y) = \exp(A_Y + S_Y) = \exp(A_Y)\exp(S_Y)$
- $\exp(A_Y)$ is H-selfadjoint, $\exp(S_Y)$ is H-unitary and both matrices commute.

What about the singular case?

Example: $X = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X^{[*]} = \begin{bmatrix} -i & 0 \\ 0 & 0 \end{bmatrix};$

$$XX^{[*]} = X^{[*]}X = 0 \overset{!}{=} A^2 \text{ with } \operatorname{Ker} X = \operatorname{Ker} A$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \qquad a \in \mathbb{R} \setminus \{0\}$$

$$\Rightarrow U = \begin{bmatrix} 0 & u_{12} \\ ia^{-1} & u_{22} \end{bmatrix} \quad \Rightarrow \quad U^{[*]} = \begin{bmatrix} \overline{u}_{22} & \overline{u}_{12} \\ -ia^{-1} & 0 \end{bmatrix}$$

$$\overset{UU^{[*]}=I}{\Rightarrow} \quad U = \begin{bmatrix} 0 & ia \\ ia^{-1} & b \end{bmatrix}, \qquad b \in \mathbb{R}.$$

Unfortunately: $AU \neq UA$ for all choices of A and U.

What about the singular case?

Example: $X = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X^{[*]} = \begin{bmatrix} -i & 0 \\ 0 & 0 \end{bmatrix};$

$$XX^{[*]} = X^{[*]}X = 0 \overset{!}{=} A^2 \text{ with } \mathrm{Ker}\ X = \mathrm{Ker}\ A$$

$$\Rightarrow \ A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad a \in \mathbb{R} \setminus \{0\}$$

$$\Rightarrow U = \begin{bmatrix} 0 & u_{12} \\ ia^{-1} & u_{22} \end{bmatrix} \quad \Rightarrow \quad U^{[*]} = \begin{bmatrix} \overline{u}_{22} & \overline{u}_{12} \\ -ia^{-1} & 0 \end{bmatrix}$$

$$\overset{UU^{[*]}=I}{\Rightarrow} \quad U = \begin{bmatrix} 0 & ia \\ ia^{-1} & b \end{bmatrix}, \quad b \in \mathbb{R}.$$

Reason: $X^{[*]} = AU^{-1} = U^{-1}A$ but $\mathrm{Ker}\ X^{[*]} \neq \mathrm{Ker}\ A = \mathrm{Ker}\ X$.

Polar decompositions with commuting factors

Theorem [M., Ran, Rodman, 2004]: The following statements are equivalent:

i) X admits an H-polar decomposition with commuting factors;

ii) X is H-normal and $\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]}$;

iii) there is an H-unitary V such that $X = VX^{[*]}$.

Idea of proof: $i) \Rightarrow ii)$: let $X = UA$, i.e., $X^{[*]} = AU^{-1}$:

$$XX^{[*]} = UAAU^{-1} = AU^{-1}UA = X^{[*]}X;$$

$$\mathrm{Ker}\ X = \mathrm{Ker}\ A = \mathrm{Ker}\ X^{[*]} \text{ is clear;}$$

Polar decompositions with commuting factors

Theorem [M., Ran, Rodman, 2004]: The following statements are equivalent:

i) X admits an H-polar decomposition with commuting factors;

ii) X is H-normal and $\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]}$;

iii) there is an H-unitary V such that $X = VX^{[*]}$.

Idea of proof: $ii) \Rightarrow iii)$:

this is Witt's theorem;

Polar decompositions with commuting factors

Theorem [M., Ran, Rodman, 2004]: The following statements are equivalent:

i) X admits an H-polar decomposition with commuting factors;

ii) X is H-normal and $\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]}$;

iii) there is an H-unitary V such that $X = VX^{[*]}$.

Idea of proof: $iii) \Rightarrow i)$: X and V commute:

$$XV = VX^{[*]}V = V(VX^{[*]})^{[*]}V = VXV^{[*]}V = VX;$$

construct H-unitary square root U of V that commutes with X;

Polar decompositions with commuting factors

Theorem [M., Ran, Rodman, 2004]: The following statements are equivalent:

i) X admits an H-polar decomposition with commuting factors;

ii) X is H-normal and $\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]}$;

iii) there is an H-unitary V such that $X = VX^{[*]}$.

Idea of proof: $iii) \Rightarrow i)$:

choose $X = UA$ with $A := U^{-1}X$, then A is H-selfadjoint:

$$(U^{-1}X)^{[*]} = X^{[*]}U = V^{-1}XU = V^{-1}UX = U^{-2}UX = U^{-1}X.$$

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Krein spaces

Convention: all operators considered in the following are bounded.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let H be an invertible selfadjoint operator.

- indefinite inner product: $[x, y] = \langle Hx, y\rangle$, $x, y \in \mathcal{H}$;
- $(\mathcal{H}, [\cdot,\cdot])$ is a **Krein space**;
- $(\mathcal{H}, [\cdot,\cdot])$ is a **Pontryagin space** if the H-invariant subspace corresponding to negative part of the spectrum of H has dimension $\kappa < \infty$.

Polar decompositions in Krein spaces

- **H-polar decomposition**: for $X : \mathcal{H} \to \mathcal{H}$:

  $X = UA$ where A is H-selfadjoint and $U : \mathrm{Im}\ A \to \mathrm{Im}\ X$ is an injective H-isometry;

- **H-unitary polar decomposition**: if U above is H-unitary;

- **as before**: X has an H-polar decomposition iff $X^{[*]}X$ has an H-selfadjoint square root A s.t. $\mathrm{Ker}\ X = \mathrm{Ker}\ A$.

Extension of H-isometries

Main problem: extension of H-isometries to H-unitary operators.

Theorem [van der Mee, Ran, Rodman, 2002]: Let $\mathcal{H}$ be a Pontryagin space and $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{H}$ be closed subspaces, and let $U_0 : \mathcal{V}_1 \to \mathcal{V}_2$ a continuous and injective H-isometry. If codim $\mathcal{V}_1 =$codim $\mathcal{V}_2$, then U_0 can be extended to an H-unitary operator $U : \mathcal{H} \to \mathcal{H}$.

Remark: The theorem also holds for Krein spaces if $\mathcal{V}_1 = \mathcal{V}_2$.

Polar decompositions of normal operators

Theorem [M., Ran, Rodman, 2004]: Let $X : \mathcal{H} \to \mathcal{H}$ be H-normal s.t.

1) either X is invertible or 0 is an isolated point of the spectrum with finite algebraic multiplicity;
2) $\sigma(X)$ does not surround zero;
3) either $\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]}$ or $\mathcal{H}$ is a Pontryagin space.

Then X admits an H-unitary polar decomposition.

Proof: as in the finite dimensional case; this provides an H-polar decomposition $X = U_0 A$ with an H-isometry $U_0 : \mathrm{Im}\ A \to \mathrm{Im}\ X$;

then use „$\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]} \Rightarrow \mathrm{Im}\ A = \mathrm{Im}\ X$" for Krein spaces or „$\mathrm{codim}\ \mathrm{Im}\ A = \mathrm{codim}\ \mathrm{Im}\ X$" for Pontryagin spaces.

Conclusions

- theory of existence of H-polar decompositions now complete;
- normal matrices are the prototypes of matrices having H-polar decompositions;
- main problem in infinite dimensions: extension of H-isometries;
- important question: how to compute H-polar decompositions numerically (Kintzel 2004, Higham/Mackey/Mackey/Tisseur 2004/05).