Singular value-like decompositions in indefinite inner product spaces

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The Singular Value Decomposition

Theorem: Let $A \in \mathbb{C}^{m\times n}$. Then there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that

$$U^*AV = \left[\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} \sigma_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \sigma_r & 0 \\ \hline 0 & \dots & 0 & 0 \end{array}\right]$$

where $\sigma_1 \geq \cdots \geq \sigma_r > 0$. The parameters $\sigma_1, \dots, \sigma_r$ are uniquely defined and are called the (nonzero) **singular values** of A.
Moreover,

$$AA^* = \left[\begin{array}{c|c} \Sigma^2 & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times m} \quad \text{and} \quad A^*A = \left[\begin{array}{c|c} \Sigma^2 & 0 \\ \hline 0 & 0 \end{array}\right]_{n\times n}$$

The Singular Value Decomposition

Aspects: of the singular value decomposition:

- allows computation of the polar decomposition;
- displays eigenvalues of the Hermitian matrices AA^* and A^*A;
- allows numerical computation of the rank of a matrix;
- allows construction of optimal low-rank approximations;
- useful tool in Numerical Linear Algebra;

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Generalized Singular Value Decompositions

Problem: Given $A \in \mathbb{C}^{m\times n}$, compute a canonical form that displays

- the Jordan canonical form of $A^{[*]}A$ and $AA^{[*]}$, where $A^{[*]} = H^{-1}A^*H$ is the adjoint with respect to a Hermitian sesquilinear form $[\cdot,\cdot] = (H\cdot,\cdot)$; ($A^{[*]}A$ and $AA^{[*]}$ are selfadjoint with respect to $[\cdot,\cdot]$);

- the Jordan canonical form of A^TA and AA^T; (these are complex symmetric matrices);

- the Jordan canonical form of $A^{[T]}A$ and $AA^{[T]}$, where $A^{[T]}$ is the adjoint with respect to a complex symmetric or complex skew-symmetric bilinear form $[\cdot,\cdot]$;

Generalized Singular Value Decompositions

General formulation of the problem: let $A \in \mathbb{C}^{m\times n}$ and $\star \in \{*, T\}$;

- allow two inner products given by $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$;
- compute a canonical form for the triple $(A, G, \hat{G})$ via

  $(A_{\mathsf{CF}}, G_{\mathsf{CF}}, \hat{G}_{\mathsf{CF}}) = (Y^\star A X, X^\star G X, Y^\star \hat{G} Y)$, where X, Y are nonsingular;
- let this form display the eigenvalues of
  - the matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^\star G^{-1} A$;
  - the matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^\star$;

This makes sense, because

$$Y^{-1}\hat{\mathcal{H}}Y = \hat{G}_{\mathsf{CF}}^{-1} A_{\mathsf{CF}}^\star G_{\mathsf{CF}}^{-1} A_{\mathsf{CF}} \quad \text{and} \quad X^{-1}\mathcal{H}X = G_{\mathsf{CF}}^{-1} A_{\mathsf{CF}} \hat{G}_{\mathsf{CF}}^{-1} A_{\mathsf{CF}}^\star$$

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Then $G = H^{-1}$, $\hat{G} = H$, $\star = *$: $\rightsquigarrow$ forms for $A^{[*]}A = \hat{\mathcal{H}}$ and $AA^{[*]} = H\mathcal{H}H^{-1}$;

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Then $G = I$, $\hat{G} = I$, $\star = *$: $\rightsquigarrow$ SVD if we require $X^* G X = I$, $Y^* \hat{G} Y = I$;

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Then $G = I$, $\hat{G} = I$, $\star = T$: $\rightsquigarrow$ forms for $\hat{\mathcal{H}} = A^TA$ and $\mathcal{H} = AA^T$.

Generalized Singular Value Decompositions

Different cases:

	G	$\hat{G}$	$\hat{\mathcal{H}} = \hat{G}^{-1}A^\star G^{-1}A$	$\mathcal{H} = G^{-1}A\hat{G}^{-1}A^\star$
1)	Hermitian	Hermitian	$\hat{G}$-selfadjoint	G-selfadjoint
2)	symmetric	symmetric	$\hat{G}$-selfadjoint	G-selfadjoint
3)	symmetric	skew-symmetric	$\hat{G}$-skewadjoint	G-skewadjoint
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- 1) Bolshakov/Reichstein 1995
- 2)–4) this talk

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The nonsingular case

Assumption: $A \in \mathbb{C}^{n\times n}$ is nonsingular (and thus so are $\mathcal{H}$ and $\hat{\mathcal{H}}$)

- **Standard SVD**: $U^*AV = \Sigma \in \mathbb{C}^{n\times n} \Rightarrow V^*A^*AV = U^*AA^*U = \Sigma^2$.

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Theorem: Let $\hat{G}$ be complex symmetric and let $\hat{\mathcal{H}}$ be $\hat{G}$-selfadjoint and nonsingular. Then there exists a unique square root $\mathcal{S}$ of $\hat{H}$ satisfying

$$\sigma(\mathcal{S}) \subseteq \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [0, \pi)\}.$$

This square root is a polynomial in $\hat{H}$ and is therefore $\hat{G}$-selfadjoint.

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- **Idea**: start with a square root $\mathcal{S}$ of $\hat{\mathcal{H}} = \hat{G}^{-1}A^TG^{-1}A$;

- observe: $\mathcal{H}$ and $\hat{\mathcal{H}}$ are similar:

$$\hat{\mathcal{H}} = \underbrace{\hat{G}^{-1}A^T}_{\mathcal{A}}\underbrace{G^{-1}A}_{\mathcal{B}} \quad \text{and} \quad \mathcal{H} = \underbrace{G^{-1}A}_{\mathcal{B}}\underbrace{\hat{G}^{-1}A^T}_{\mathcal{A}}$$

  (Flanders 1951: $\mathcal{A}\mathcal{B}$ and $\mathcal{B}\mathcal{A}$ have the same nonzero elementary divisors)

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  (Flanders 1951: $\mathcal{AB}$ and $\mathcal{BA}$ have the same nonzero elementary divisors)

- then $(\mathcal{H}; G)$ and $(\hat{\mathcal{H}}, \hat{G})$ have the same canonical forms;

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- **Standard SVD**: $U^* A V = \Sigma \in \mathbb{C}^{n\times n} \Rightarrow V^* A^* A V = U^* A A^* U = \Sigma^2$.

- **Idea**: start with a square root $\mathcal{S}$ of $\hat{\mathcal{H}} = \hat{G}^{-1} A^T G^{-1} A$;

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  (Flanders 1951: $\mathcal{AB}$ and $\mathcal{BA}$ have the same nonzero elementary divisors)

- then $(\mathcal{H}; G)$ and $(\hat{\mathcal{H}}, \hat{G})$ have the same canonical forms;

- construct X, Y such that $X^{-1}\mathcal{H}X = (Y^{-1}\hat{\mathcal{H}}Y)^T$ and $X^T G X = Y^T \hat{G} Y$ and show that $X^T A Y$ is basically $X^{-1}\mathcal{S}X$

The nonsingular case

Theorem: Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and let $G, \hat{G} \in \mathbb{C}^{n\times n}$ be complex symmetric and nonsingular. Then there exist nonsingular matrices $X, Y \in \mathbb{C}^{n\times n}$ such that

$$\begin{array}{rcl} X^T A Y & = & \mathcal{J}_{\xi_1}(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}(\mu_m), \\ X^T G X & = & R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}, \\ Y^T \hat{G} Y & = & R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}, \end{array} \quad , \qquad R_\xi = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}_{\xi\times\xi}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi)$, and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Moreover, for the $\hat{G}$-symmetric matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^T G^{-1} A$ and for the G-symmetric matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^T$ we have that

$$\begin{array}{rcl} Y^{-1} \hat{\mathcal{H}} Y & = & \mathcal{J}_{\xi_1}^2(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}^2(\mu_m), \\ X^{-1} \mathcal{H} X & = & \mathcal{J}_{\xi_1}^2(\mu_1)^T \oplus \cdots \oplus \mathcal{J}_{\xi_m}^2(\mu_m)^T. \end{array}$$

The singular case

Now: $A \in \mathbb{C}^{m \times n}$ may be singular (and so may be $\mathcal{H}$ and $\hat{\mathcal{H}}$)

Problems:

- a square root of $\mathcal{H}$ and/or $\hat{\mathcal{H}}$ need not exist
- $\mathcal{H}$ and $\hat{\mathcal{H}}$ need not be similar;

The singular case

Now: $A \in \mathbb{C}^{m\times n}$ may be singular (and so may be $\mathcal{H}$ and $\hat{\mathcal{H}}$)

Example

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; \hat{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathcal{H} = \hat{G}^{-1} A^T G^{-1} A = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \hat{\mathcal{H}} = G^{-1} A \hat{G}^{-1} A^T = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The singular case

Theorem: Let $A \in \mathbb{C}^{m\times n}$ and let $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$ be symmetric and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{C}^{m\times m}$ and $\in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} X^T A Y &= A_{ns} \oplus A_{z,1} \oplus \cdots \oplus A_{z,k}, \\ X^T G X &= G_{ns} \oplus G_{z,1} \oplus \cdots \oplus G_{z,k}, \\ Y^T \hat{G} Y &= \hat{G}_{ns} \oplus \hat{G}_{z,1} \oplus \cdots \oplus \hat{G}_{z,k}. \end{aligned}$$

Moreover, for the $\hat{G}$-selfadjoint matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^T \hat{G}^{-1} A \in \mathbb{C}^{n\times n}$ and for the G-selfadjoint matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^T \in \mathbb{C}^{m\times m}$ we have that

$$\begin{aligned} Y^{-1} \hat{\mathcal{H}} Y &= \hat{\mathcal{H}}_{ns} \oplus \hat{\mathcal{H}}_{z,1} \oplus \cdots \oplus \hat{\mathcal{H}}_{z,k}, \\ X^{-1} \mathcal{H} X &= \mathcal{H}_{ns} \oplus \mathcal{H}_{z,1} \oplus \cdots \oplus \mathcal{H}_{z,k}, \end{aligned}$$

where A_{ns}, $\hat{\mathcal{H}}_{ns}$ and $\mathcal{H}_{ns}$ are nonsingular and ...

The singular case

... either

1) $A_{z,j} = 0_{m_0 \times n_0}$, $G_{z,j} = I_{m_0}$, $\hat{G}_{z,j} = I_{n_0}$, $\hat{\mathcal{H}}_{z,j} = 0_{n_0}$, $\mathcal{H}_{z,j} = 0_{m_0}$

$\ldots$

The singular case

... or

2) $$A_{z,j} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}_{2p\times 2p}, \quad G_{z,j} = \hat{G}_{z,j} = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}_{2p\times 2p},$$

$$\hat{\mathcal{H}}_{z,j} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}^2_{2p\times 2p}, \quad \mathcal{H}_{z,j} = \left(\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}^2_{2p\times 2p} \right)^T$$

for some $p \in \mathbb{N}$ . . .

The singular case

... or

3) $A_{z,j} = \begin{bmatrix} 0 \\ I_p \end{bmatrix}_{p+1\times p}$, $G_{z,j} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}_{p+1\times p+1}$, $\hat{G}_{z,j} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}_{p\times p}$

$$\hat{\mathcal{H}}_{z,j} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}_{p\times p}, \quad \mathcal{H}_{z,j} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}^T_{p+1\times p+1}$$

for some $p \in \mathbb{N}$ ...

The singular case

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4) $A_{z,j} = \begin{bmatrix} 0 & I_p \end{bmatrix}_{p \times p+1}$, $G_{z,j} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}_{p \times p}$, $\hat{G}_{z,j} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}_{p+1 \times p+1}$

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The singular case

Uniqueness: The canonical form is unique up to permutation of blocks.

Idea of proof:

- reduction to a staircase-like form;
- extract the part of A corresponding to the nonsingular parts of $\mathcal{H}$ and $\hat{\mathcal{H}}$ and apply the theorem for the nonsingular case;
- extract the other blocks from the remaining part

The singular case

Example

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \hat{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathcal{H} = \hat{G}^{-1} A^T G^{-1} A = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \hat{\mathcal{H}} = G^{-1} A \hat{G}^{-1} A^T = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

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The singular values: $*$ and T case

Special case: $G = I_m$, $\hat{G} = I_n$. The singular values of $A \in \mathbb{C}^{m \times n}$ are:

***-case**: $\sigma_1, \ldots, \sigma_{\min(m,n)} \geq 0$ (related to the eigenvalues of A^*A and AA^*);

T-case: $\mathcal{J}_{\xi_1}(\mu_1)$, ..., $0_{m_0 \times n_0}$, $\mathcal{J}_{2p_1}(0)$, ..., $\begin{bmatrix} 0 \\ I_{q_1} \end{bmatrix}$, ... ,$\begin{bmatrix} 0 & I_{r_1} \end{bmatrix}$, ...,
where $\arg(\mu_j) \in [0, \pi)$ and the "values" are related to the Jordan blocks of A^TA and AA^T.

Uniqueness: Singular values are unique both in the $*$-case and T-case!

Generalized Singular Value Decompositions

Different cases:

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- 1) Bolshakov/Reichstein 1995
- 2)–4) this talk

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The nonsingular case

Assumption: $A \in \mathbb{C}^{n\times n}$ is nonsingular (and thus so are $\mathcal{H}$ and $\hat{\mathcal{H}}$)

- **Problem**: squares of $\hat{G}$-selfadjoints and $\hat{G}$-skewadjoint are always $\hat{G}$-selfadjoint
  $\Rightarrow$ $\hat{\mathcal{H}}$ does not have a G-structured square root

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- **Solution**: take the G-selfadjoint forth root of $\hat{\mathcal{H}}^2$ instead.

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  $\Rightarrow \hat{\mathcal{H}}$ does not have a G-structured square root
- **Solution**: take the $\hat{G}$-selfadjoint forth root of $\hat{\mathcal{H}}^2$ instead;
- continue as in the case of G, $\hat{G}$ being both symmetric;
- in the general case ($A \in \mathbb{C}^{m\times n}$) do a staircase-like reduction.

The general case

Theorem: Let $A \in \mathbb{C}^{m \times 2n}$, let $G \in \mathbb{C}^{m \times m}$ be symmetric nonsingular, and let $\hat{G} \in \mathbb{C}^{2n \times 2n}$ be skew-symmetric and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{C}^{m \times m}$ and $\in \mathbb{C}^{2n \times 2n}$ such that

$$\begin{aligned} X^T A Y &= A_{ns,1} \oplus \cdots \oplus A_{ns,\ell} \oplus A_{z,1} \oplus \cdots \oplus A_{z,k}, \\ X^T G X &= G_{ns,1} \oplus \cdots \oplus G_{ns,\ell} \oplus G_{z,1} \oplus \cdots \oplus G_{z,k}, \\ Y^T \hat{G} Y &= \hat{G}_{ns,1} \oplus \cdots \oplus \hat{G}_{ns,\ell} \oplus \hat{G}_{z,1} \oplus \cdots \oplus \hat{G}_{z,k}. \end{aligned}$$

Moreover, for the $\hat{G}$-skewadjoint matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^T \hat{G}^{-1} A \in \mathbb{C}^{2n \times 2n}$ and for the G-skewadjoint matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^T \in \mathbb{C}^{m \times m}$ we have that

$$\begin{aligned} Y^{-1} \hat{\mathcal{H}} Y &= \hat{\mathcal{H}}_{ns,1} \oplus \cdots \oplus \hat{\mathcal{H}}_{ns,\ell} \oplus \hat{\mathcal{H}}_{z,1} \oplus \cdots \oplus \hat{\mathcal{H}}_{z,k}, \\ X^{-1} \mathcal{H} X &= \mathcal{H}_{ns,1} \oplus \cdots \oplus \mathcal{H}_{ns,\ell} \oplus \mathcal{H}_{z,1} \oplus \cdots \oplus \mathcal{H}_{z,k}, \end{aligned}$$

where ...

The general case

... the "singular values" of A are of the forms

1) $$A_{ns,j} = \begin{bmatrix} \mathcal{J}_\xi(\mu) & 0 \\ 0 & \mathcal{J}_\xi(\mu) \end{bmatrix}, \quad G_{ns,j} = \begin{bmatrix} 0 & R_\xi \\ R_\xi & 0 \end{bmatrix}, \quad \hat{G}_{ns,j} = \begin{bmatrix} 0 & R_\xi \\ -R_\xi & 0 \end{bmatrix},$$

where $\mu \neq 0$, $\arg(\mu) \in [0, \pi/2)$ and

$$\hat{\mathcal{H}} = \begin{bmatrix} -\mathcal{J}_\xi^2(\mu) & 0 \\ 0 & \mathcal{J}_\xi^2(\mu) \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} \mathcal{J}_\xi^2(\mu) & 0 \\ 0 & -\mathcal{J}_\xi^2(\mu) \end{bmatrix}^T$$

The general case

... the "singular values" of A are of the forms

2) $A_{z,j} = 0_{m_0 \times 2n_0}, \;\; G_{z,j} = I_{m_0}, \;\; \hat{G}_{z,j} = \begin{bmatrix} 0 & I_{n_0} \\ -I_{n_0} & 0 \end{bmatrix},$

$$\hat{\mathcal{H}}_{z,j} = 0_{n_0}, \quad \mathcal{H}_{z,j} = 0_{m_0}$$

The general case

... the "singular values" of A are of the forms

3) $A_{z,j} = \mathcal{J}_{2p}(0), \quad G_{z,j} = \begin{bmatrix} 0 & R_p \\ R_p & 0 \end{bmatrix}, \quad \hat{G}_{z,j} = \begin{bmatrix} 0 & R_p \\ -R_p & 0 \end{bmatrix}$

and $\hat{\mathcal{H}}_{z,j} = \begin{bmatrix} -I_p & 0 \\ 0 & I_p \end{bmatrix} \mathcal{J}_{2p}^2(0), \quad \mathcal{H}_{z,j} = \begin{bmatrix} -I_{p+1} & 0 \\ 0 & I_{p-1} \end{bmatrix} \mathcal{J}_{2p}^2(0)^T,$

for some $p \in \mathbb{N}$ ...

The general case

... the "singular values" of A are of the forms

4) $\quad A_{z,j} = \begin{bmatrix} 0 \\ I_{2p} \end{bmatrix}, \quad G_{z,j} = R_{2p+1}, \quad \hat{G}_{z,j} = \begin{bmatrix} 0 & R_p \\ -R_p & 0 \end{bmatrix}$

and $\quad \hat{\mathcal{H}}_{z,j} = \begin{bmatrix} -I_p & 0 \\ 0 & I_p \end{bmatrix} \mathcal{J}_{2p}(0), \quad \mathcal{H}_{z,j} = \begin{bmatrix} I_{p+1} & 0 \\ 0 & -I_p \end{bmatrix} \mathcal{J}_{2p+1}(0)^T,$

for some $p \in \mathbb{N}$ . . .

The general case

... the "singular values" of A are of the forms

5) $\quad A_{z,j} = \begin{bmatrix} 0 & 0 \\ 0 & I_q \\ 0 & 0 \\ I_q & 0 \end{bmatrix}, \quad G_{z,j} = \begin{bmatrix} 0 & R_{2q+1} \\ R_{2q+1} & 0 \end{bmatrix}, \quad \hat{G}_{z,j} = \begin{bmatrix} 0 & R_q \\ -R_q & 0 \end{bmatrix}$

and $\quad \hat{\mathcal{H}}_{z,j} = \begin{bmatrix} -\mathcal{J}_q(0) & 0 \\ 0 & \mathcal{J}_q(0) \end{bmatrix}, \quad \mathcal{H}_{z,j} = \begin{bmatrix} -\mathcal{J}_{q+1}(0) & 0 \\ 0 & \mathcal{J}_{q+1}(0) \end{bmatrix}^T,$

for some odd $q \in \mathbb{N}$ ...

The general case

... the "singular values" of A are of the forms

6) and 7) "transposed" versions of 4) and 5)

Generalized Singular Value Decompositions

Different cases:

	G	$\hat{G}$	$\hat{\mathcal{H}} = \hat{G}^{-1}A^\star G^{-1}A$	$\mathcal{H} = G^{-1}A\hat{G}^{-1}A^\star$
1)	Hermitian	Hermitian	$\hat{G}$-selfadjoint	G-selfadjoint
2)	symmetric	symmetric	$\hat{G}$-selfadjoint	G-selfadjoint
3)	symmetric	skew-symmetric	$\hat{G}$-skewadjoint	G-skewadjoint
4)	skew-symmetric	skew-symmetric	$\hat{G}$-selfadjoint	G-selfadjoint

- 1) Bolshakov/Reichstein 1995
- 2)–4) this talk

Generalized Singular Value Decompositions

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1)	Hermitian	Hermitian	$\hat{G}$-selfadjoint	G-selfadjoint
2)	symmetric	symmetric	$\hat{G}$-selfadjoint	G-selfadjoint
3)	symmetric	skew-symmetric	$\hat{G}$-skewadjoint	G-skewadjoint
4)	skew-symmetric	skew-symmetric	$\hat{G}$-selfadjoint	G-selfadjoint

- 1) Bolshakov/Reichstein 1995
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The general case

Theorem: Let $A \in \mathbb{C}^{2m\times 2n}$, let $G \in \mathbb{C}^{2m\times 2m}$ and $\hat{G} \in \mathbb{C}^{2n\times 2n}$ be skew-symmetric and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{C}^{2m\times 2m}$ and $\in \mathbb{C}^{2n\times 2n}$ such that

$$\begin{aligned} X^T A Y &= A_{ns,1} \oplus \cdots \oplus A_{ns,\ell} \oplus A_{z,1} \oplus \cdots \oplus A_{z,k}, \\ X^T G X &= G_{ns,1} \oplus \cdots \oplus G_{ns,\ell} \oplus G_{z,1} \oplus \cdots \oplus G_{z,k}, \\ Y^T \hat{G} Y &= \hat{G}_{ns,1} \oplus \cdots \oplus \hat{G}_{ns,\ell} \oplus \hat{G}_{z,1} \oplus \cdots \oplus \hat{G}_{z,k}. \end{aligned}$$

Moreover, for the $\hat{G}$-selfadjoint matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^T \hat{G}^{-1} A \in \mathbb{C}^{2n\times 2n}$ and for the G-selfadjoint matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^T \in \mathbb{C}^{2m\times 2m}$ we have that

$$\begin{aligned} Y^{-1} \hat{\mathcal{H}} Y &= \hat{\mathcal{H}}_{ns,1} \oplus \cdots \oplus \hat{\mathcal{H}}_{ns,\ell} \oplus \hat{\mathcal{H}}_{z,1} \oplus \cdots \oplus \hat{\mathcal{H}}_{z,k}, \\ X^{-1} \mathcal{H} X &= \mathcal{H}_{ns,1} \oplus \cdots \oplus \mathcal{H}_{ns,\ell} \oplus \mathcal{H}_{z,1} \oplus \cdots \oplus \mathcal{H}_{z,k}, \end{aligned}$$

where ...

The general case

... the "singular values" of A are of the forms

1) $\quad A_{ns,j} = \begin{bmatrix} \mathcal{J}_\xi(\mu) & 0 \\ 0 & \mathcal{J}_\xi(\mu) \end{bmatrix}, \quad G_{ns,j} = \begin{bmatrix} 0 & R_\xi \\ -R_\xi & 0 \end{bmatrix}, \quad \hat{G}_{ns,j} = \begin{bmatrix} 0 & -R_\xi \\ R_\xi & 0 \end{bmatrix},$

where $\mu \neq 0$, $\arg(\mu) \in [0, \pi)$ and

$$\hat{\mathcal{H}} = \begin{bmatrix} \mathcal{J}_\xi^2(\mu) & 0 \\ 0 & \mathcal{J}_\xi^2(\mu) \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} \mathcal{J}_\xi^2(\mu) & 0 \\ 0 & \mathcal{J}_\xi^2(\mu) \end{bmatrix}^T$$

The general case

... the "singular values" of A are of the forms

2) $A_{z,j} = 0_{m_0 \times 2n_0}$, $G_{z,j} = \begin{bmatrix} 0 & I_{m_0} \\ -I_{m_0} & 0 \end{bmatrix}$, $\hat{G}_{z,j} = \begin{bmatrix} 0 & I_{n_0} \\ -I_{n_0} & 0 \end{bmatrix}$,

$$\hat{\mathcal{H}}_{z,j} = 0_{n_0}, \quad \mathcal{H}_{z,j} = 0_{m_0}$$

The general case

... the "singular values" of A are of the forms

3) $A_{z,j} = J_{2p}(0), \quad G_{z,j} = \begin{bmatrix} 0 & R_p \\ -R_p & 0 \end{bmatrix}, \quad \hat{G}_{z,j} = \begin{bmatrix} 0 & R_p \\ -R_p & 0 \end{bmatrix}$

and $\hat{\mathcal{H}}_{z,j} = \Sigma \mathcal{J}_{2p}^2(0), \quad \mathcal{H}_{z,j} = \hat{\Sigma} \mathcal{J}_{2p}^2(0)^T,$

for some $p \in \mathbb{N}$ and some signature matrices $\Sigma, \hat{\Sigma}$ ...

The general case

... the "singular values" of A are of the forms

4) $$A_{z,j} = \begin{bmatrix} 0 & 0 \\ 0 & I_q \\ 0 & 0 \\ I_q & 0 \end{bmatrix}, \quad G_{z,j} = \begin{bmatrix} 0 & R_{q+1} \\ -R_{q+1} & 0 \end{bmatrix}, \quad \hat{G}_{z,j} = \begin{bmatrix} 0 & R_q \\ -R_q & 0 \end{bmatrix}$$

and $$\hat{\mathcal{H}}_{z,j} = \begin{bmatrix} \mathcal{J}_q(0) & 0 \\ 0 & \mathcal{J}_q(0) \end{bmatrix}, \quad \mathcal{H}_{z,j} = \begin{bmatrix} \mathcal{J}_{q+1}(0) & 0 \\ 0 & \mathcal{J}_{q+1}(0) \end{bmatrix}^T,$$

for some $q \in \mathbb{N}$ ...

The general case

... the "singular values" of A are of the forms

5) $A_{z,j} = \begin{bmatrix} 0 & 0 & 0 & I_q \\ 0 & I_q & 0 & 0 \end{bmatrix}$, $G_{z,j} = \begin{bmatrix} 0 & R_q \\ -R_q & 0 \end{bmatrix}$, $\hat{G}_{z,j} = \begin{bmatrix} 0 & R_{q+1} \\ -R_{q+1} & 0 \end{bmatrix}$

and $\hat{\mathcal{H}}_{z,j} = \begin{bmatrix} \mathcal{J}_{q+1}(0) & 0 \\ 0 & \mathcal{J}_{q+1}(0) \end{bmatrix}$, $\mathcal{H}_{z,j} = \begin{bmatrix} \mathcal{J}_q(0) & 0 \\ 0 & \mathcal{J}_q(0) \end{bmatrix}^T$,

for some $q \in \mathbb{N}$ ...

Thank you for your attention!