# Sesquiliear versus bilinear -
# what is the real scalar product?

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**Structured perturbations and distance problems in matrix computations**

Bedlewo, March 26-30, 2007

Work supported by DFG Research Center
MATHEON – Mathematics for key technologies

# The motivation

# Perturbation analysis of symplectic matrices

**Motivation**: perturbation analysis for symplectic matrices;

**Definition**: $S \in \mathbb{C}^{2n \times 2n}$ is called **symplectic** if

$$S^* J S = J, \quad \text{where } J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

- the spectrum of symplectic matrices is symmetric with the respect to the unit circle: if $\lambda$ is an eigenvalue, then so is $\overline{\lambda}^{-1}$;
- the pairing degenerates for unimodular eigenvalues (i.e., eigenvalues on the unit circle);
- unimodular eigenvalues have **signs** as additional invariants;

# Perturbation analysis of symplectic matrices

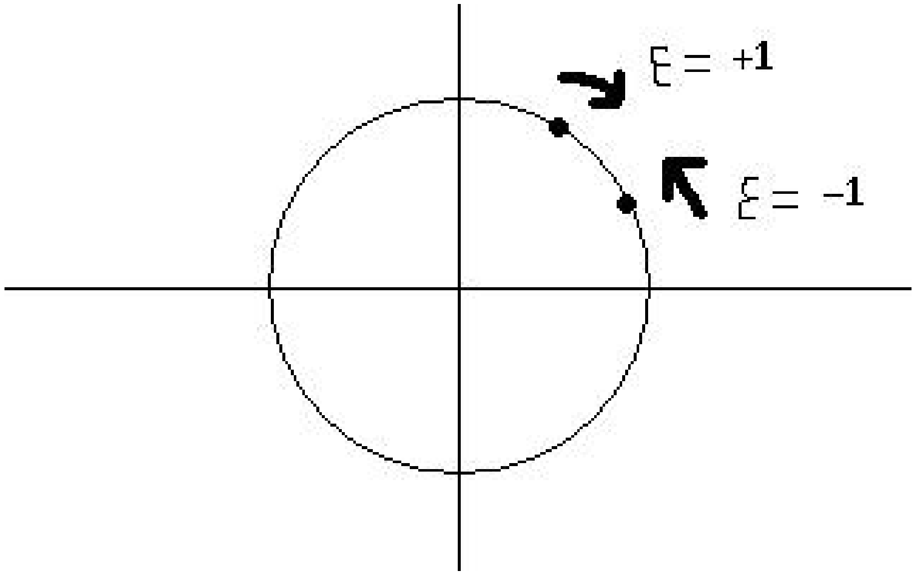
**Example**:

$$S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- $S_1$ and $S_2$ are similar as matrices (viewed as unstructured matrices)
- $S_1$ and $S_2$ are not similar via a symplectic similarity transformation (view-ed as structured matrices);
- we say the eigenvalue $1$ of $S_1$ has sign $\varepsilon = +1$ and the eigenvalue $1$ of $S_2$ has sign $\varepsilon = -1$;

**Signs are crucial in the investigation of structured perturbations!**

# What happens under structured perturbations?




- let $S \in \mathbb{C}^{2n \times 2n}$ be symplectic;
- let $S$ have two close unimodular eigenvalues with opposite signs;
- if $S$ is perturbed and the two eigenvalues meet, they generically form a Jordan block; then they may split off as a pair of nonunimodular reciprocal eigenvalues;

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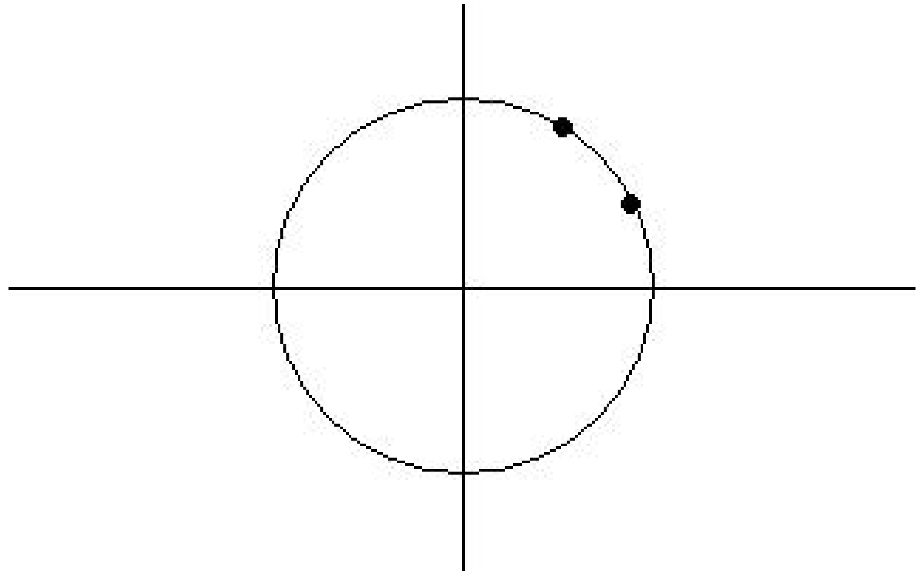

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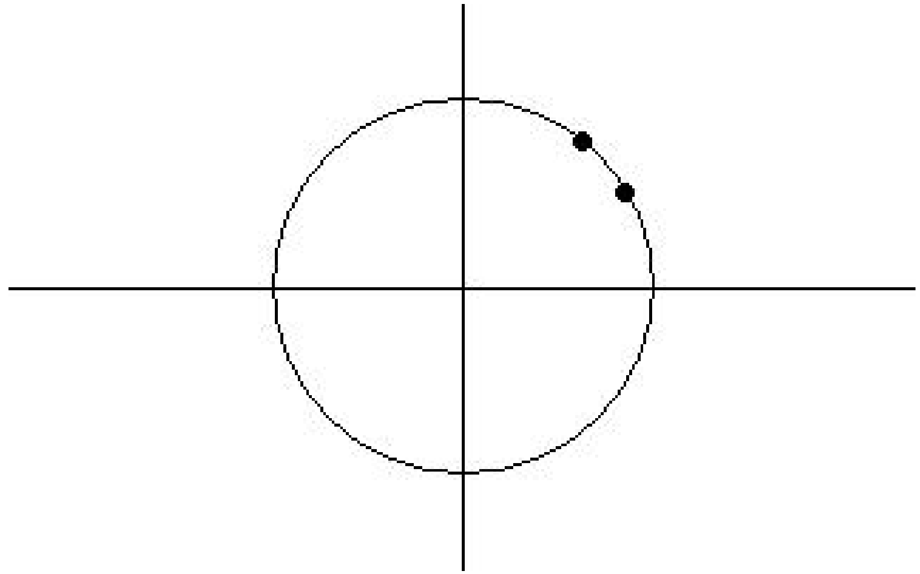

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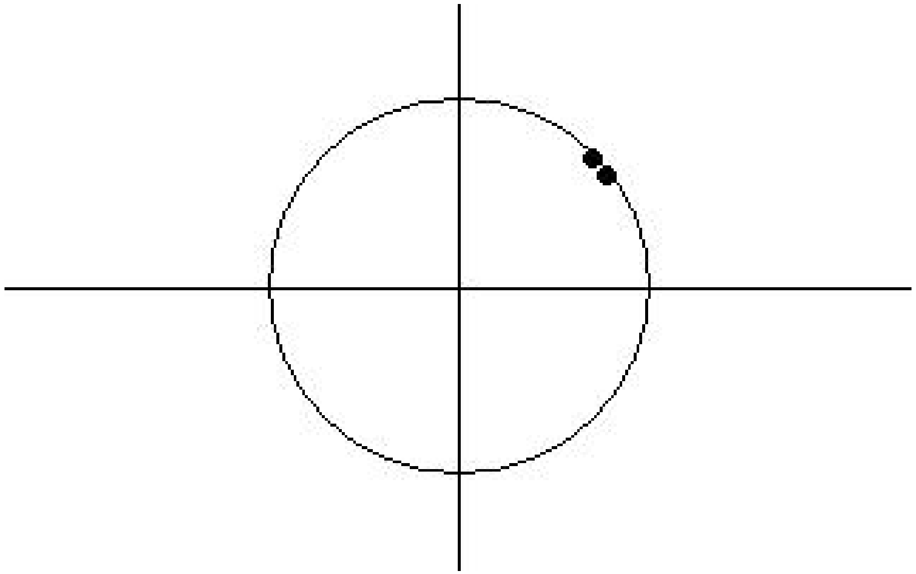

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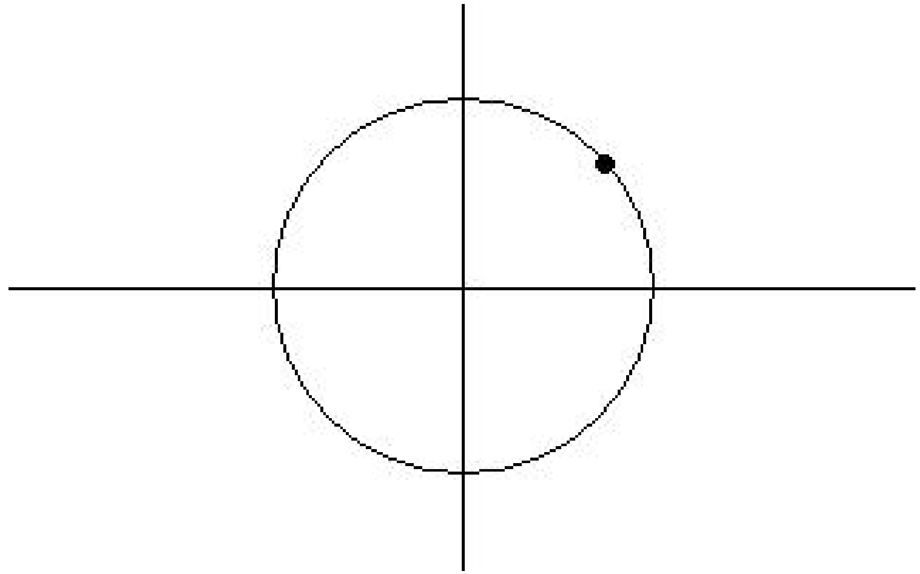

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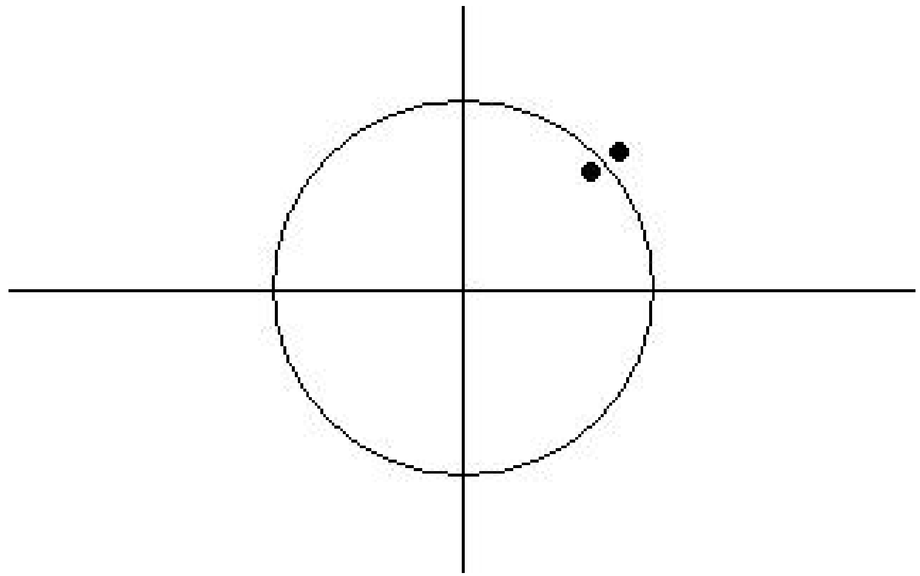

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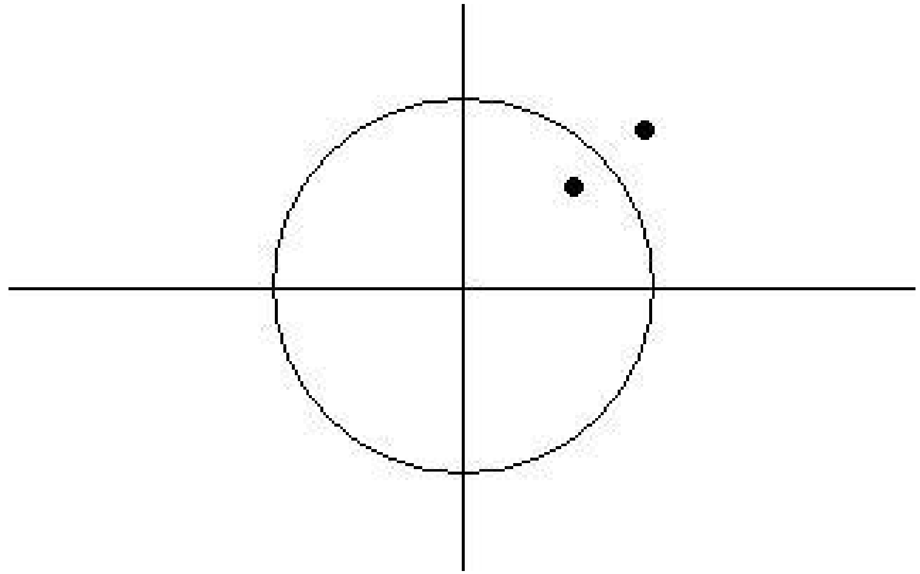

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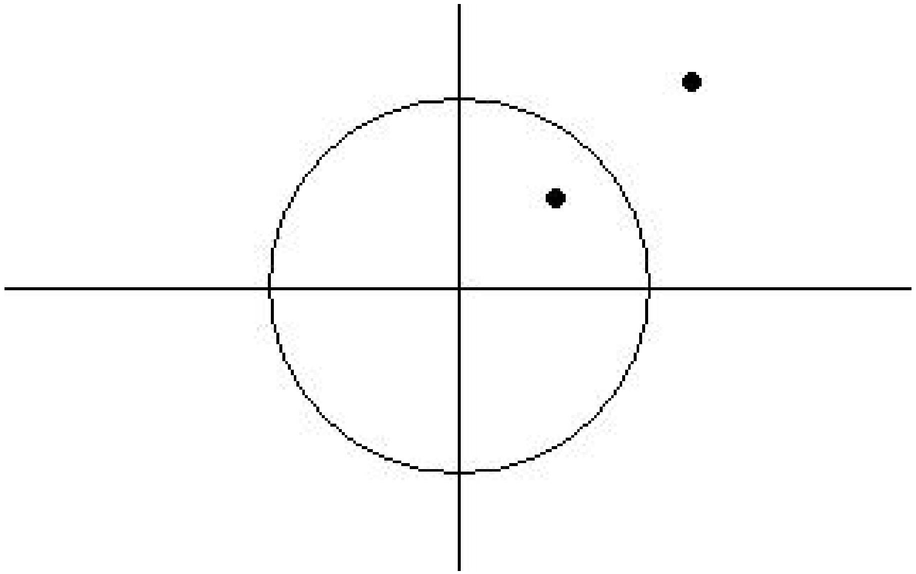

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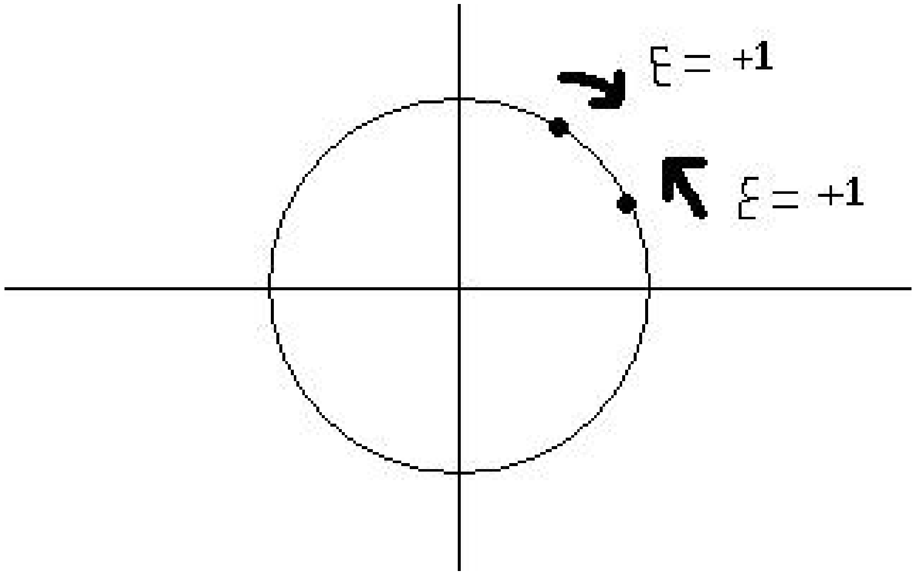


- let $S \in \mathbb{C}^{2n \times 2n}$ be symplectic;
- let $S$ have two close unimodular eigenvalues with equal signs;
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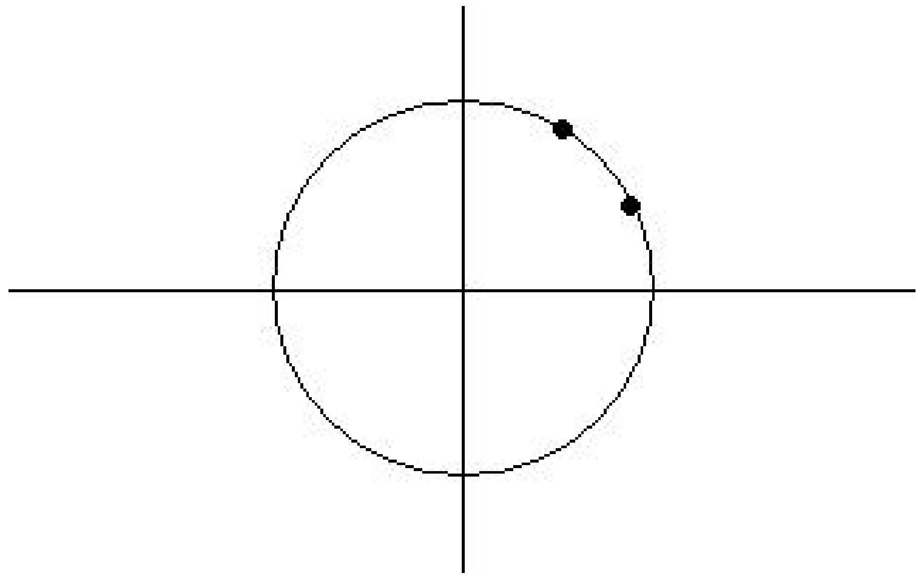

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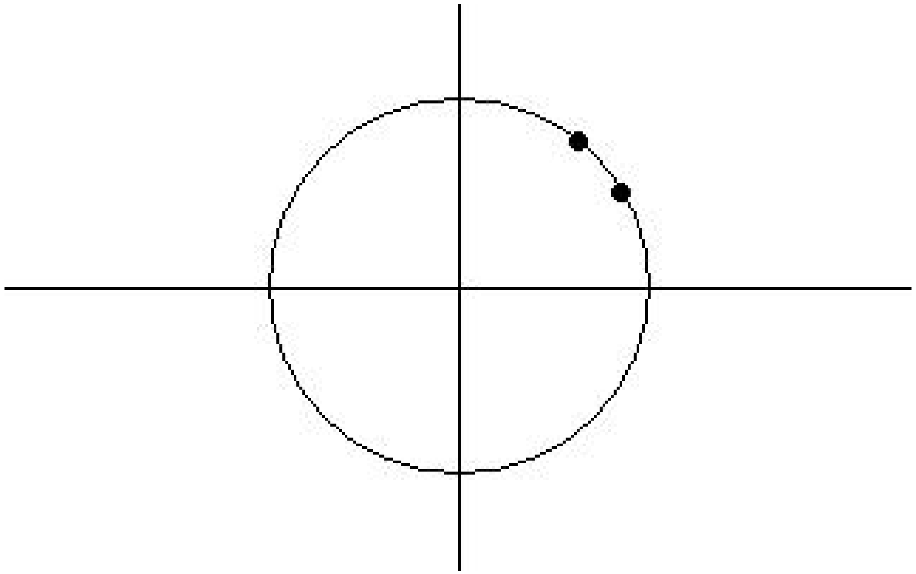

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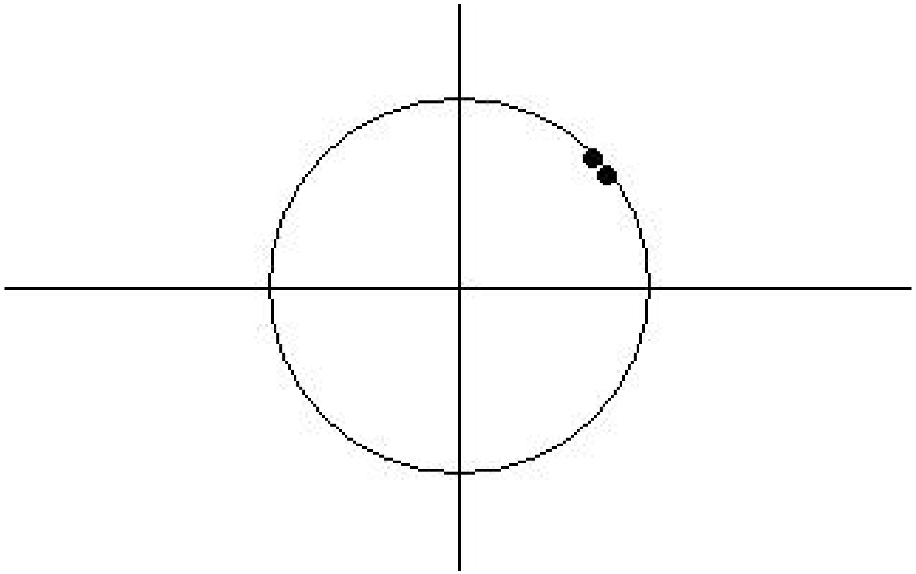

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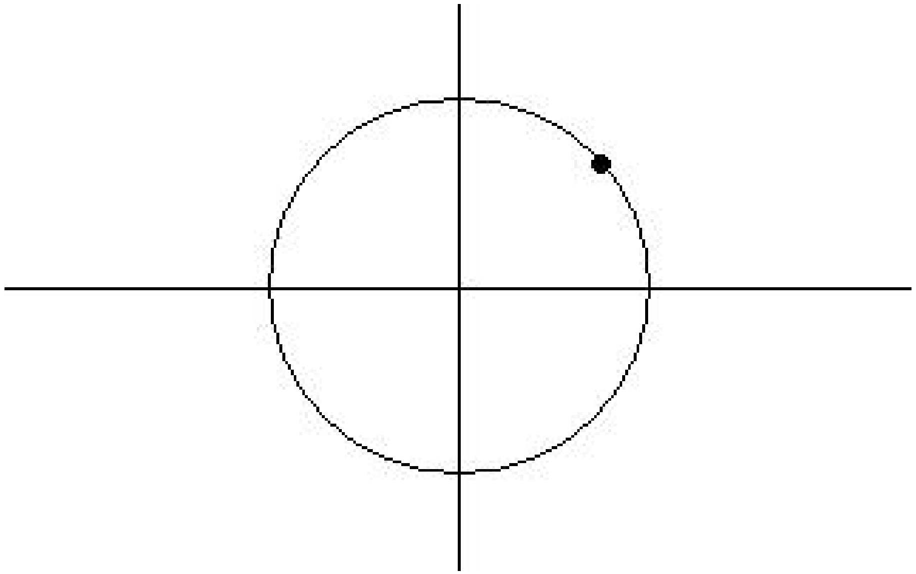

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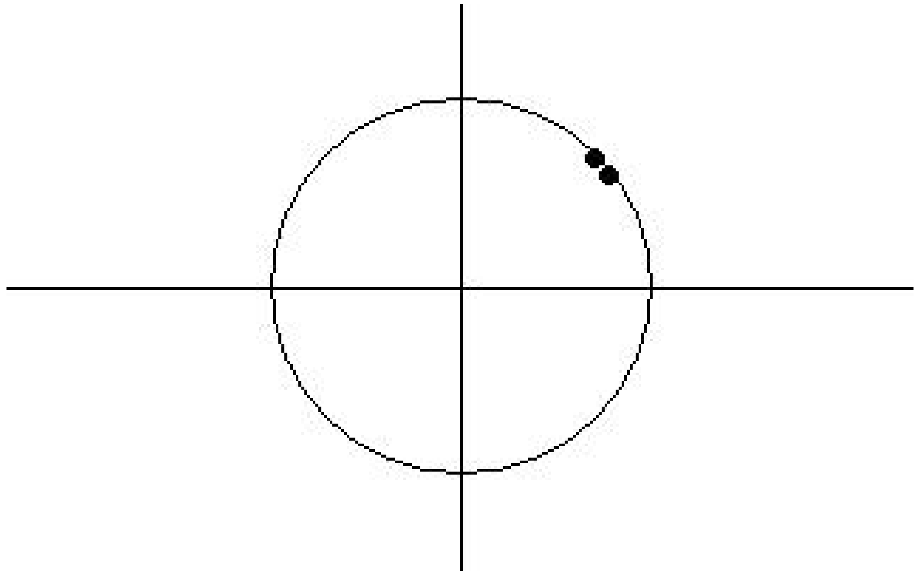

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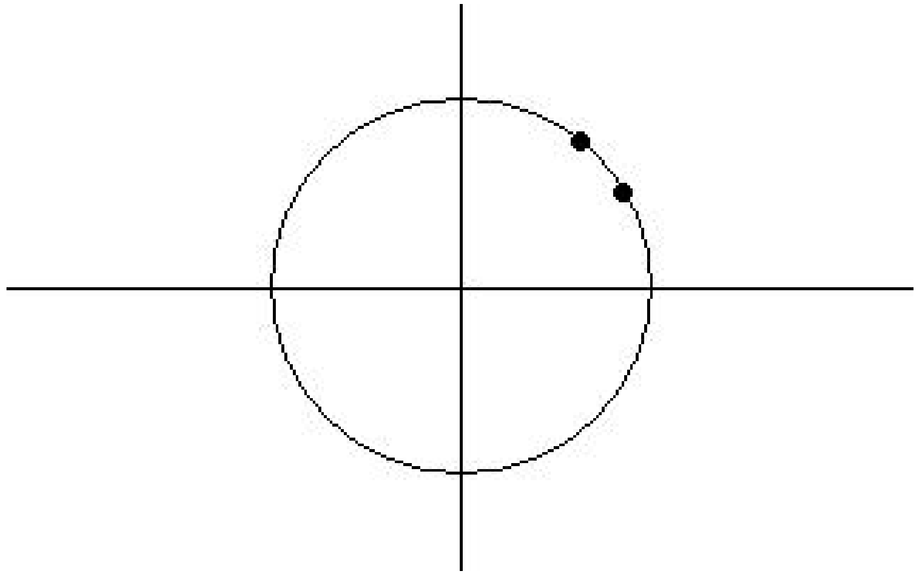

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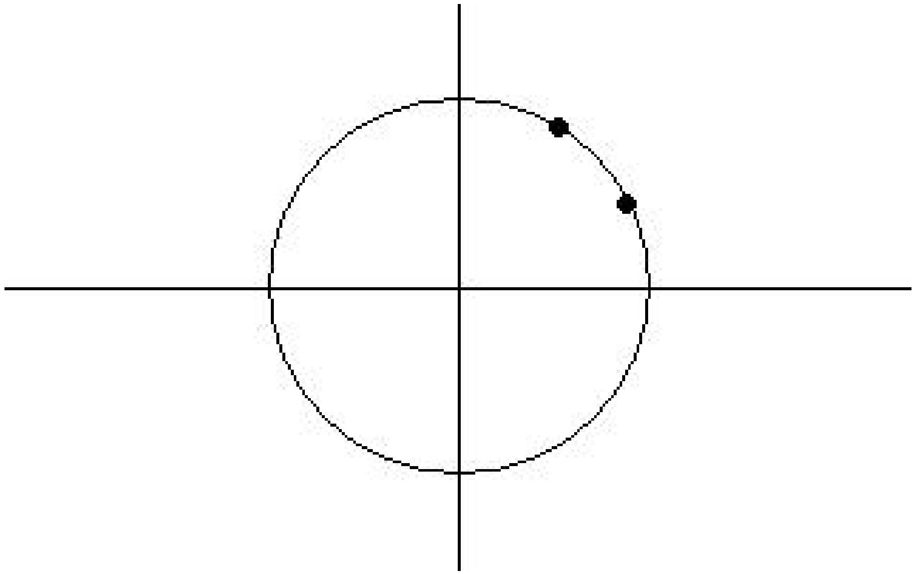

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# Sesquilinear vesus bilinear

**Canonical forms are crucial in the investigation of these signs!**

$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ can be interpreted as a matrix that either induces a sesquilinear form or a bilinear form.

- **sesquilinear form**:
  unimodular eigenvalues of symplectic matrices $(S^* J S = J)$ have signs;
- **bilinear form**:
  unimodular eigenvalues of symplectic matrices $(S^T J S = J)$ do **not** have signs;

**Question**: How does the real case fit in here? **Or, more presicely ...**

# The question

# Indefinite inner products

**Indefinite inner products**:

1) $H = \pm H^T$ invertible defines a (skew-)symmetric bilinearform on $\mathbb{C}^n$:

$$[x, y]_H := y^T H x \quad \text{for all } x, y \in \mathbb{C}^n$$

2) $H = \pm H^*$ invertible defines a (skew-)Hermitian sesquilinearform on $\mathbb{C}^n$:

$$[x, y]_H := y^* H x \quad \text{for all } x, y \in \mathbb{C}^n$$

Restricted to $\mathbb{R}^n$ and for $H \in \mathbb{R}^{n \times n}$ both inner products are identical, so:

**Question**: **What is the real scalar product?** $y^T H x$ or $y^* H x$?

# Indefinite inner products

**The adjoint**: For $X \in \mathbb{F}^{n\times n}$ let $X^\star$ be the matrix satisfying

$$[v, Xw]_H = [X^\star v, w]_H \quad \text{for all } v, w \in \mathbb{F}^n.$$

We have $X^\star = H^{-1}X^T H$ resp. $X^\star = H^{-1}X^* H$.

**Structured matrices in indefinite inner products**:

| structured matrix | adjoint | $y^T H x$ | $y^* H x$ |
|---|---|---|---|
| $A$ **$H$-selfadjoint** | $A^\star = A$ | $A^T H = HA$ | $A^* H = HA$ |
| $S$ **$H$-skew-adjoint** | $S^\star = -S$ | $S^T H = -HS$ | $S^* H = -HS$ |
| $U$ **$H$-unitary** | $U^\star = U^{-1}$ | $U^T HU = H$ | $U^* HU = H$ |

# Indefinite inner products

**Spectral symmetries**:

| structured matrix | $y^T Hx$ | $y^* Hx$ | $y^T Hx$ |
|---|---|---|---|
| field | $\mathbb{F} = \mathbb{C}$ | $\mathbb{F} = \mathbb{C}$ | $\mathbb{F} = \mathbb{R}$ |
| $H$-selfadjoints | $\lambda$ | $\lambda, \overline{\lambda}$ | $\lambda, \overline{\lambda}$ |
| $H$-skew-adjoints | $\lambda, -\lambda$ | $\lambda, -\overline{\lambda}$ | $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ |
| $H$-unitaries | $\lambda, \lambda^{-1}$ | $\lambda, \overline{\lambda}^{-1}$ | $\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$ |

**Question**: **What is the real scalar product?** $y^T Hx$ or $y^* Hx$?

**Answer**: a mixture of $y^T Hx$ and $y^* Hx$!

# Indefinite inner products

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**Answer**: a mixture of $y^T Hx$ and $y^* Hx$! (**Answer complete?**)

# Canonical forms

# Canonical forms

**Transformations that preserve structure**:

- for bilinear forms: $(H, A) \mapsto (P^T H P,\ P^{-1} A P)$, $\quad P$ invertible;
- for sesquilinear forms: $(H, A) \mapsto (P^* H P,\ P^{-1} A P)$, $\quad P$ invertible;

$$A \text{ is} \left\{ \begin{array}{l} H\text{-selfadjoint} \\ H\text{-skew-adjoint} \\ H\text{-unitary} \end{array} \right\} \Leftrightarrow P^{-1}AP \text{ is} \left\{ \begin{array}{l} P^\star HP\text{-selfadjoint} \\ P^\star HP\text{-skew-adjoint} \\ P^\star HP\text{-unitary} \end{array} \right\}$$

Here $P^\star = P^T$ or $P^\star = P^*$, respectively.

# Canonical forms

Canonical forms for $H$-selfadjoint, -skew-adjoints, -unitaries are well-known:

- Gohberg/Lancaster/Rodman (1983)
- Sergeichuk (1988)
- Gohberg/Reichstein (1991)
- Lancaster/Rodman (1995)
- Lin/Mehrmann/Xu (1999)
- Faßbender/Mackey/Mackey/Xu (1999)
- Rodman (2006)

in terms of Hermitian or (skew)-symmetric pairs:

- Weierstraß, Kronecker
- Thompson (1976,1991)
- Lancaster/Rodman (2005,2005)

# Canonical forms

$H$-**Indecomposability**:

$A \in \mathbb{F}^{n \times n}$ is called $H$-**decomposable**, if there exists $P \in \mathbb{F}^{n \times n}$ invertible such that

$$P^{-1}AP = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad P^\star HP = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad A_j, H_j \in \mathbb{F}^{n_j \times n_j}, n_j > 0$$

Otherwise $A$ is called $H$-**indecomposable**.

**Clear**: Any $A \in \mathbb{F}^{n \times n}$ can be decomposed as

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_k, \quad P^\star HP = H_1 \oplus \cdots \oplus H_k,$$

where each $A_j$ is $H_j$-indecomposable.

# Canonical forms: $H$-selfadjoints

**Case 1**: $\mathbb{F} = \mathbb{C}$, $H^T = \pm H$ defines a bilinear form;

Let $A \in \mathbb{C}^{n\times n}$ be $H$-indecomposable and $H$-selfadjoint. Then there exists $P \in \mathbb{C}^{n\times n}$ invertible such that

$$P^{-1}AP = \mathcal{J}_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad P^T HP = F_n = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}.$$

**Canonical form** for $H$-selfadjoints in general:

$$P^{-1}AP = \mathcal{J}_{n_1}(\lambda_1) \oplus \cdots \oplus \mathcal{J}_{n_k}(\lambda_k), \quad P^T AP = F_{n_1} \oplus \cdots \oplus F_{n_k}$$

# Canonical forms: $H$-selfadjoints

**Case 2**: $\mathbb{F} = \mathbb{C}$, $H^* = \pm H$ defines a sesquilinear form;

Let $A \in \mathbb{C}^{n \times n}$ be $H$-indecomposable and $H$-selfadjoint. Then there exists $P \in \mathbb{C}^{n \times n}$ invertible such that either

1) $P^{-1}AP = \mathcal{J}_n(\lambda)$, and $P^T HP = \varepsilon F_n$, where $\lambda \in \mathbb{R}$ and $\varepsilon = \pm 1$; or

2) $P^{-1}AP = \begin{bmatrix} \mathcal{J}_{n/2}(\mu) & 0 \\ 0 & \mathcal{J}_{n/2}(\overline{\mu}) \end{bmatrix}$, $P^T HP = \begin{bmatrix} F_{n/2} & 0 \\ 0 & F_{n/2} \end{bmatrix}$, where $\mu \notin \mathbb{R}$;

additional invariants: **signs**

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

There is no $P$ such that $P^{-1}AP = A$ and $P^* H_1 P = H_2$.

# Canonical forms: real matrices

**Question**: How do we construct canonical forms for real matrices?

**Idea**: decompose

$$P^{-1}AP = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad P^T H P = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix},$$

where $\sigma(A_1) \subseteq \mathbb{R}$ and $\sigma(A_2) \subseteq \mathbb{C} \setminus \mathbb{R}$.

**Problem**: How do we display real canonical forms for $(A_2, H_2)$?

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**Solution**: $\mathbb{C} \sim \mathcal{M}_{\mathbb{C}} := \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

via the field isomorphism $\quad \phi : \mathbb{C} \to \mathcal{M}_{\mathbb{C}}, \quad (\alpha + i\beta) \mapsto \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

# Canonical forms: real matrices

**Question**: How do we construct canonical forms for real matrices?

**Real Jordan blocks** associated with a pair of conjugate complex eigenvalues:

$$\mathcal{J}_n(\alpha,\beta) := \phi(\mathcal{J}_n(\alpha+i\beta)) = \begin{bmatrix} \alpha & \beta & 1 & 0 & & & & 0 \\ -\beta & \alpha & 0 & 1 & & & & \\ & & \alpha & \beta & \ddots & & & \\ & & -\beta & \alpha & & & & \\ & & & & \ddots & & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & \alpha & \beta \\ 0 & & & & & & -\beta & \alpha \end{bmatrix}.$$

# Canonical forms: real matrices

**Aim**: Canonical forms for real $H$-structured matrices $A$

**Question**: Can we find $P$ invertible such that

$$P^{-1}AP = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & \phi(\mathcal{A}_2) \end{bmatrix}, \quad P^T HP = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & \phi(\mathcal{H}_2) \end{bmatrix},$$

where

- $\sigma(A_1) \subseteq \mathbb{R}$
- $A_1$ is $H_1$-structured
- $\sigma(A_2) \subseteq \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{A}_2$ is $\mathcal{H}_2$-structured (as a complex matrix)

# Canonical forms: real matrices

**Example**: an $H$-skew-adjoint matrix for $H = H^T$:

$$S = \phi(\mathcal{S}) = \phi\left(\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

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$S$ is $H$-skew-adjoint ($-S^T H = HS$) for any $H$ having the form

$$H = \begin{bmatrix} h_1 & 0 & h_3 & h_4 \\ 0 & h_1 & -h_4 & h_3 \\ h_3 & -h_4 & h_2 & 0 \\ h_4 & h_3 & 0 & h_2 \end{bmatrix}$$

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**Observation**: $\mathcal{H}$ is Hermitian and $\mathcal{S}$ is $\mathcal{H}$-skew-adjoint: $\mathcal{S}^*\mathcal{H} = -\mathcal{H}\mathcal{S}$

# Canonical forms: real matrices

**Theorem**: Let $H \in \mathbb{R}^{n\times n}$ be invertible (skew-)symmetric and let $S \in \mathbb{R}^{n\times n}$ be $H$-skew-adjoint. Then there exists $P \in \mathbb{R}^{n\times n}$ invertible such that

$$P^{-1}SP = \begin{bmatrix} S_1 & 0 \\ 0 & \phi(\mathcal{S}_2) \end{bmatrix}, \quad P^T HP = \begin{bmatrix} H_1 & 0 \\ 0 & \phi(\mathcal{H}_2) \end{bmatrix},$$

where

1) $\sigma(S_1) \subseteq \mathbb{R}$ and $\sigma(\mathcal{S}_2) \subseteq \mathbb{C} \setminus \mathbb{R}$;
2) $\mathcal{H}_2$ is (skew-)Hermitian and $\mathcal{S}_2$ is $\mathcal{H}_2$-skew-adjoint: $\mathcal{S}_2^* \mathcal{H}_2 = -\mathcal{H}_2 \mathcal{S}_2$.

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**Question**: **What is the real scalar product?** $y^T Hx$ or $y^* Hx$?

**Answer**: $y^* Hx$!

# Canonical forms: real matrices

**Theorem**: Let $H \in \mathbb{R}^{n\times n}$ be invertible (skew-)symmetric and let $S \in \mathbb{R}^{n\times n}$ be $H$-skew-adjoint. Then there exists $P \in \mathbb{R}^{n\times n}$ invertible such that

$$P^{-1}SP = \begin{bmatrix} S_1 & 0 \\ 0 & \phi(\mathcal{S}_2) \end{bmatrix}, \quad P^T HP = \begin{bmatrix} H_1 & 0 \\ 0 & \phi(\mathcal{H}_2) \end{bmatrix},$$

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**Example**: an $H$-selfadjoint matrix for $H = H^T$:

$$A = \phi(\mathcal{A}) = \phi\left(\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

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$A$ is $H$-selfadjoint ($A^T H = HA$) for any $H$ having the form

$$H = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ h_2 & -h_1 & h_4 & -h_3 \\ h_3 & h_4 & h_5 & h_6 \\ h_4 & -h_3 & h_6 & -h_5 \end{bmatrix}$$

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**Observation**: $\mathcal{H}$ is symmetric and $\mathcal{A}$ is $\mathcal{H}$-selfadjoint: $\mathcal{A}^T\mathcal{H} = \mathcal{H}\mathcal{A}$

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**Theorem**: Let $H \in \mathbb{R}^{n\times n}$ be invertible (skew-)symmetric and let $A \in \mathbb{R}^{n\times n}$ be $H$-selfadjoint. Then there exists $P \in \mathbb{R}^{n\times n}$ invertible such that

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# The background

# Normal matrices

**Definition**:

- A matrix $N \in \mathbb{F}^{n\times n}$ is called **$H$-normal** if $N^\star N = NN^\star$.
- A matrix $X \in \mathbb{F}^{n\times n}$ is called **polynomially $H$-normal** if there exists a polynomial $p \in \mathbb{F}[t]$ such that $X^\star = p(X)$.

- $p$ is unique if it is of minimal degree and normalized.
- $H$ pos. definite: $N$ is $H$-normal $\Leftrightarrow$ $N$ is polynomially $H$-normal
- $H$ indefinite: $X$ is $H$-normal $\Leftarrow$ $X$ is polynomially $H$-normal

  $\not\Rightarrow$
- **canonical forms** are not available for $H$-normal matrices, but for polynomially $H$-normal matrices (M. 2006)

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**Examples**:

- $H$-selfadjoint matrices are polynomially $H$-normal with $p(t) = t$;
- $H$-skew-adjoint matrices are polynomially $H$-normal with $p(t) = -t$;
- $H$-unitary matrices are polynomially $H$-normal ($U^{-1} = p(U)$).

# Normal matrices

## Spectral symmetries:

| structured matrix | $y^T H x$ | $y^* H x$ | $y^T H x$ |
|---|---|---|---|
| field | $\mathbb{F} = \mathbb{C}$ | $\mathbb{F} = \mathbb{C}$ | $\mathbb{F} = \mathbb{R}$ |
| $H$-selfadjoints | $\lambda$ | $\lambda, \overline{\lambda}$ | $\lambda, \overline{\lambda}$ |
| $H$-skew-adjoints | $\lambda, -\lambda$ | $\lambda, -\overline{\lambda}$ | $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ |
| $H$-unitaries | $\lambda, \lambda^{-1}$ | $\lambda, \overline{\lambda}^{-1}$ | $\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$ |
| polynomially $H$-normals | $\lambda, p(\lambda)$ | $\lambda, \overline{p(\lambda)}$ | $\lambda, p(\lambda), \overline{\lambda}, \overline{p(\lambda)}$ |

# Normal matrices

**Theorem**: (**Essential decomposition**) Let $H \in \mathbb{R}^{n\times n}$ be (skew-)symmetric and let $X \in \mathbb{R}^{n\times n}$ be polynomially $H$-normal ($X^\star = p(X)$). Then there exists $P \in \mathbb{R}^{n\times n}$ invertible such that

$$P^{-1}XP = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & \phi(\mathcal{X}_2) & 0 \\ 0 & 0 & \phi(\mathcal{X}_3) \end{bmatrix}, \quad P^T H P = \begin{bmatrix} H_1 & 0 & 0 \\ 0 & \phi(\mathcal{H}_2) & 0 \\ 0 & 0 & (I \otimes F_2)\phi(\mathcal{H}_3) \end{bmatrix}$$

such that

1) $\sigma(X_1) \subseteq \mathbb{R}$; and $X_1$ is polynomially $H_1$-normal;
2) $\sigma(\mathcal{X}_2) \subseteq \{\lambda \in \mathbb{C} \setminus \mathbb{R} \,:\, p(\lambda) \neq \lambda\}$,
   $\mathcal{H}_2$ is (skew-)Hermitian and $\mathcal{X}_2$ is polynomially $\mathcal{H}_2$-normal;
3) $\sigma(\mathcal{X}_3) \subseteq \{\lambda \in \mathbb{C} \setminus \mathbb{R} \,:\, p(\lambda) = \lambda\}$,
   $\mathcal{H}_3$ is (skew-)symmetric and $\mathcal{X}_3$ is polynomially $\mathcal{H}_3$-normal;

# Normal matrices

**Observation**:

- For $H$-selfadjoints, we have $p(t) = t$, so we always have $p(\lambda) = \lambda$;
  $\Rightarrow$ the blocks $\mathcal{X}_2, \mathcal{H}_2$ in the essential decompositions do not appear;
- For $H$-skew-adjoints, we have $p(t) = -t$, so we have $p(\lambda) = \lambda$ iff $\lambda = 0$, but this value is real;
  $\Rightarrow$ the blocks $\mathcal{X}_3, \mathcal{H}_3$ in the essential decompositions do not appear;
- For $H$-unitaries, we have $p(\lambda) = \lambda^{-1}$ for all eigenvalues $\lambda$, so $p(\lambda) = \lambda$ iff $\lambda = \pm 1$, but those values are real;
  $\Rightarrow$ the blocks $\mathcal{X}_3, \mathcal{H}_3$ in the essential decompositions do not appear;

# The answer

# Conclusions

**Question**: **What is the real scalar product?** $y^T Hx$ or $y^* Hx$?

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**Answer**: Real polynomially $H$-normal matrices can be decomposed into three parts:

1) part I (real eigenvalues), where the real scalar product is a mixture of $y^T Hx$ and $y^* Hx$;
2) part II (complex eigenvalues), where the real scalar product is $y^T Hx$;
3) part III (complex eigenvalues), where the real scalar product is $y^* Hx$;

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**Question**: **What is the real scalar product?** $y^THx$ or $y^*Hx$?

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**Answer complete?**

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2) part II (complex eigenvalues), where the real scalar product is $y^T Hx$;

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**Answer complete? Yes, from this particular point of view!**

# Conclusions

**References**:


- C.M. *On classification of normal matrices in indefinite inner product spaces*. Electron. J. Linear Algebra, 15:50–83, 2006.
- C.M. *Essential decomposition of polynomially normal matrices in real indefinite inner product spaces*. Electron. J. Linear Algebra, 15:84–106, 2006.


**Thank you for your attention!**