# ESSENTIAL DECOMPOSITION OF POLYNOMIALLY NORMAL MATRICES IN REAL INDEFINITE INNER PRODUCT SPACES*

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**Abstract.** Polynomially normal matrices in real indefinite inner product spaces are studied, i.e., matrices whose adjoint with respect to the indefinite inner product is a polynomial in the matrix. The set of these matrices is a subset of indefinite inner product normal matrices that contains all selfadjoint, skew-adjoint, and unitary matrices, but that is small enough such that all elements can be completely classified. The essential decomposition of a real polynomially normal matrix is introduced. This is a decomposition into three parts, one part having real spectrum only and two parts that can be described by two complex matrices that are polynomially normal with respect to a sesquilinear and bilinear form, respectively. In the paper, the essential decomposition is used as a tool in order to derive a sufficient condition for existence of invariant semidefinite subspaces and to obtain canonical forms for real polynomially normal matrices. In particular, canonical forms for real matrices that are selfadjoint, skewadjoint, or unitary with respect to an indefinite inner product are recovered.

**Key words.** Indefinite inner products, normal matrices, selfadjoint matrices, skewadjoint matrices, unitary matrices, essential decomposition

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## 1. Introduction.

Let $H \in \mathbb{R}^{n\times n}$ be invertible and (skew-)symmetric. Then $H$ induces a nondegenerate (skew-)symmetric bilinear form on $\mathbb{R}^n$ via $[x, y] := y^T\mathcal{H}x$ for $x, y \in \mathbb{R}^n$. This form can be extended to $\mathbb{C}^n$ either as a (skew-)Hermitian sesquilinear form via $[x, y] := y^*Hx$ for $x, y \in \mathbb{C}^n$ or as a (skew-)symmetric bilinear form via $[x, y] := y^THx$ for $x, y \in \mathbb{C}^n$. In the paper, we will use both extensions in order to obtain canonical forms for several classes of real matrices that are normal with respect to the real indefinite inner product induced by $H$.

In the following let $\mathbb{F}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$, and for $M \in \mathbb{F}^{m\times n}$ let $M^\star$ denote either $M^T$, the transpose, or $M^*$, the conjugate transpose of $M$, respectively. Moreover, let $H \in \mathbb{F}^{n\times n}$ be invertible and satisfy $H^\star = \pm H$. Then $H$ induces a nondegenerate (skew-)symmetric bilinear form (in the case $\star = T$) or a nondegenerate (skew-)Hermitian sesquilinear form (in the case $\star = *$) via $[x, y] := y^\star Hx$ for $x, y \in \mathbb{F}^n$. For a matrix $M \in \mathbb{F}^{n\times n}$, the *H-adjoint* of $M$ is defined to be the unique matrix $M^{[\star]}$ satisfying

$$[x, My] = [M^{[\star]}x, y] \qquad \text{for all } x, y \in \mathbb{F}^n.$$

The matrix $M \in \mathbb{F}^{n\times n}$ is called *$\mathcal{H}$-selfadjoint*, *$\mathcal{H}$-skew-adjoint*, or *$\mathcal{H}$-unitary*, respectively, if $M^{[\star]} = M$, $M^{[\star]} = -M$, or $M^{[\star]} = M^{-1}$, respectively. $H$-selfadjoint, $H$-skewadjoint, and $H$-unitary matrices have been widely discussed in the literature, see [1, 6, 7, 13, 18] and the references therein, both from the viewpoint of theory and the viewpoint of numerical analysis. In particular, the case $\mathbb{F} = \mathbb{R}$ and

$$H := J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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has been intensively studied. Canonical forms for $H$-selfadjoint and $H$-skewadjoint matrices have been developed in various sources, see [4, 6, 7, 13] for the case $\mathbb{F} = \mathbb{C}$ and $H$ being Hermitian, and [4, 5, 6, 7, 13] for the case $\mathbb{F} = \mathbb{R}$ and $H$ being symmetric or skew-symmetric. These canonical forms are obtained under transformations of the form

$$(M, H) \mapsto (P^{-1}MP, P^{\star}HP), \quad P \in \mathbb{F}^{n\times n} \text{ nonsingular.} \tag{1.1}$$

that correspond to a change of bases $x \mapsto Px$ in the space $\mathbb{F}^n$. It is easy to check that $M$ is $H$-selfadjoint, $H$-skewadjoint, or $H$-unitary, respectively, if and only if $P^{-1}MP$ is $P^{\star}HP$-selfadjoint, $P^{\star}HP$-skewadjoint, or $P^{\star}HP$-unitary, respectively.

Canonical forms for the case $\mathbb{F} = \mathbb{C}$ and $H$ being symmetric or skew-symmetric have been developed in [15], but have been implicitly known by the canonical forms for pairs of complex symmetric or skew-symmetric matrices given in [22]. (Observe that, for example, for symmetric $H$, a matrix $M \in \mathbb{C}^{n\times n}$ is $H$-selfadjoint if and only if $HM$ is symmetric. Thus, a canonical form for the pair $(M, H)$ under transformations of the form (1.1) can be easily obtained from the canonical form for the pair $(HM, H)$ of symmetric matrices under simultaneous congruence.)

Canonical forms for $H$-unitary matrices have been developed in [9] for the case of sesquilinear forms on $\mathbb{C}^n$. For $\mathbb{F} = \mathbb{R}$ and the case of skew-symmetric bilinear forms, they can be obtained from [21, Theorem 5]. For the case $\mathbb{F} = \mathbb{R}$ and symmetric $H$, a canonical form is given in [20] in general and in [2] for the special case that $M$ is diagonalizable (over the complex field). In addition, canonical forms for $H$-unitary matrices for some particular choices of $H$ have been developed in [14, 19] under similarity transformations that leave $H$ invariant.

Also, attempts have been made to obtain a more general theory by investigating $H$-normal matrices, i.e., matrices $M$ satisfying $M^{[\star]}M = MM^{[\star]}$. (It is easy to see that $H$-normality is invariant under transformations of the form (1.1) as well.) However, it has been observed in [8] that the problem of classifying $H$-normal matrices is wild and so far, canonical forms have been obtained for some special cases only, see [8, 11, 12]. In [9] and [10], canonical forms for a subclass of $H$-normal matrices, the so-called *block-Toeplitz $H$-normal matrices*, have been obtained for the case $\mathbb{F} = \mathbb{C}$ and $H$ induces a sesquilinear form. However, it has been explained in [15] that it does not make sense to generalize this concept to bilinear forms, because even $H$-selfadjoint matrices fail to be block-Toeplitz $H$-normal in general.

Therefore, the class of *polynomially $H$-normal matrices* has been studied in [17] and [15]. Recall that a matrix $X \in \mathbb{F}^{n\times n}$ is polynomially $H$-normal if there exists a polynomial $p \in \mathbb{F}[t]$ such that $X^{\star} = p(X)$. It easy to check that $H$-selfadjoint, $H$-skewadjoint, and $H$-unitary matrices are polynomially $H$-normal. For $H$-selfadjoint and $H$-skewadjoint matrices this is trivial and for $H$-unitary matrices $U$, this follows because the inverse of a matrix is a polynomial of the matrix, i.e., $U^{\star} = U^{-1} = p(U)$ for some polynomial $p \in \mathbb{F}[t]$. On the other hand, polynomial $H$-normality implies $H$-normality, because any square matrix $M$ commutes with any polynomial of $M$. In [15] canonical forms for complex polynomially $H$-normal matrices have been developed both for the case of sesquilinear forms and bilinear forms.

It is the aim of this paper to extend the results of [15] to the real case. This could be done by starting "from scratch", i.e., decomposing polynomially $H$-normal matrices into indecomposable blocks (for the concept of decomposability see Section 2) and then reducing these blocks towards canonical form. Instead, we introduce a special representation of real polynomially $H$-normal matrices in this paper, the so-called *essential decomposition.* In this representation a real polynomially $H$-normal matrix is decomposed into three parts: the *real part*, i.e., a part with real spectrum only, the *complex sesquilinear part* that can be described with the help of a complex matrix that is polynomially $H$-normal matrix with respect to a sesquilinear form, and the *complex bilinear part* that can be described with the help of a complex matrix that is polynomially $H$-normal with respect to a bilinear form. It is then shown that canonical forms can be obtained by computing canonical forms for all three parts of the essential decomposition separately. In particular, the canonical forms of the two latter parts are implicitly given by corresponding canonical forms for the complex case.

Although the essential decomposition has been designed having in mind the computation of canonical forms for real polynomially normal matrices in indefinite inner products, it is of independent interest and appears to be a convenient tool in the investigation of real polynomially normal matrices. Instead of proving results by starting from canonical forms, one may use the essential decomposition to reduce the problem to the corresponding problem in the complex cases. We give an example for this strategy by using the essential decomposition for the proof of existence of semidefinite invariant subspaces for polynomially normal matrices of special type.

The remainder of the paper is organized as follows. In Section 2, we discuss some basic properties of real polynomially $H$-normal matrices. In Section 3, we recall the well-known algebra isomorphism that identifies the complex numbers with a set of particular $2 \times 2$-matrices and discuss several properties of this isomorphism. In Section 4, we state and prove the main result of this paper, i.e., existence of the essential decomposition. Then, we show in Section 5 how this result can be applied to obtain canonical forms for real $H$-selfadjoint, $H$-skewadjoint, and $H$-unitary matrices. In Section 6, the essential decomposition is used in order to prove existence of semidefinite invariant subspaces for some polynomially $H$-normal matrices.

Throughout the paper, we use the following notation. $\mathbb{N}$ is the set of natural numbers (excluding zero). If it is not explicitly stated otherwise, $H \in \mathbb{F}^{n\times n}$ always denotes an invertible matrix satisfying $H^\star = \pm H$ and induces a bilinear, respectively, sesquilinear form $[\cdot,\cdot]$. A matrix $A = A_1 \oplus \cdots \oplus A_k$ denotes a block diagonal matrix $A$ with diagonal blocks $A_1, \ldots, A_k$ (in that order) and $e_i$ is the $i$-th unit vector in $\mathbb{F}^n$. The spectrum of a matrix $A \in \mathbb{F}^{n\times n}$ is denoted by $\sigma(A)$. If $A = [a_{ij}]_{i,j} \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{k\times l}$, then $A \otimes B$ denotes the Kronecker product

$$A \otimes B := [a_{ij}B]_{i,j} \in \mathbb{F}^{(nk)\times(ml)}.$$

The symbols $R_n$, $\Sigma_n$, and $\mathcal{J}_n(\lambda)$ denote the $n \times n$ reverse identity, the $n \times n$ reverse identity with alternating signs, and the upper triangular Jordan block of size $n$ associated with the eigenvalue $\lambda$, respectively, i.e.,

$$R_n = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}, \quad \Sigma_n = \begin{bmatrix} 0 & & (-1)^0 \\ & \iddots & \\ (-1)^{n-1} & & 0 \end{bmatrix}, \quad \mathcal{J}_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}.$$

Finally, we use the abbreviation $M^{-\star} := (M^\star)^{-1} = (M^{-1})^\star$.

**2. Preliminaries.** In this section, we collect some basic results for polynomially $H$-normal matrices. We start with the following proposition. Recall that the sizes of

Jordan blocks associated with an eigenvalue $\lambda$ of a matrix $A$ are also called the *partial multiplicities* of $\lambda$.

PROPOSITION 2.1. *Let $X \in \mathbb{F}^{n\times n}$ be such that $X^{[\star]} = \widetilde{p}(X)$ for some $\widetilde{p} \in \mathbb{F}[t]$.*

*1) There is a unique polynomial $p \in \mathbb{F}[t]$ of minimal degree so that $X^{[\star]} = p(X)$.*

*2) $p'(\lambda) \neq 0$ for all eigenvalues $\lambda \in \mathbb{C}$ of $X$ having partial multiplicities larger than one.*

*3) If $[\cdot,\cdot]$ is a sesquilinear form, then $\overline{p}\big(p(X)\big) = X$. If $[\cdot,\cdot]$ is a bilinear form, then $p\big(p(X)\big) = X$.*

*Proof.* See [15]. ◻

DEFINITION 2.2. Let $X \in \mathbb{F}^{n\times n}$ be such that $X^{[\star]} = \widetilde{p}(X)$ for some $\widetilde{p} \in \mathbb{F}[t]$. Then the unique polynomial $p \in \mathbb{F}[t]$ of minimal degree such that $X^{[\star]} = p(X)$ is called the *$H$-normality polynomial* of $X$.

An important notion in the context of classification of matrices that are structured with respect to indefinite inner products is the notion of $H$-decomposability. A matrix $X \in \mathbb{F}^{n\times n}$ is called *$H$-decomposable* if there exists a nonsingular matrix $P \in \mathbb{F}^{n\times n}$ such that

$$P^{-1}XP = X_1 \oplus X_2, \quad P^\star HP = H_1 \oplus H_2,$$

where $X_1, H_1 \in \mathbb{C}^{m\times m}$ and $0 < m < n$. Clearly, any matrix $X$ can always be decomposed as

$$P^{-1}XP = X_1 \oplus \cdots \oplus X_k, \quad P^\star HP = H_1 \oplus \cdots \oplus H_k, \tag{2.1}$$

where $X_j$ is $H_j$-indecomposable, $j = 1,\dots,k$. Thus, it remains to classify indecomposable matrices.

PROPOSITION 2.3. *Let $H \in \mathbb{R}^{n\times n}$ be nonsingular and symmetric or skew-symmetric. Furthermore, let $X \in \mathbb{R}^{n\times n}$ be an $H$-indecomposable polynomially $H$-normal matrix. Then $\sigma(X) \subseteq \mathbb{R}$ or $\sigma(X) \cap \mathbb{R} = \emptyset$.*

*Proof.* Clearly, for any real matrix $X$ there exists a similarity transformation with a nonsingular transformation matrix $P \in \mathbb{R}^{n\times n}$ such that $\widetilde{X} := P^{-1}XP = X_1 \oplus X_2$, where $\sigma(X_1) \subseteq \mathbb{R}$ and $\sigma(X_2) \cap \mathbb{R} = \emptyset$, for instance, let $\widetilde{X}$ be the real Jordan canonical form of $X$ (see, e.g., Section 3). Since $X$ is polynomially $H$-normal, say with $H$-normality polynomial $p \in \mathbb{R}[t]$, it follows that $\sigma\big(p(X_1)\big) \subseteq \mathbb{R}$, because $p$ has real coefficients. We claim that $\sigma\big(p(X_2)\big) \cap \mathbb{R} = \emptyset$. Indeed, assume that $\mu \in \sigma\big(p(X_2)\big) \cap \mathbb{R}$. Then there exists an eigenvalue $z \in \mathbb{C} \setminus \mathbb{R}$ of $X_2$ such that $\mu = p(z)$. But then, Proposition 2.1 item 3) implies $z = p\big(p(z)\big)$ which identifies $z$ as a real number, a contradiction. Hence, the claim follows. Now set $\widetilde{H} := P^T HP$. Then the identity

$$\big(p(X_1) \oplus p(X_2)\big)^T \widetilde{H} = p(\widetilde{X})^T \widetilde{H} = \widetilde{H}\widetilde{X} = \widetilde{H}(X_1 \oplus X_2)$$

together with the information on the spectra of $X_j$ and $p(X_j)$, $j = 1,2$ implies that $\widetilde{H} = H_1 \oplus H_2$ has a block structure conformable with $\widetilde{X}$. (Here, we used that the Sylvester equation $AY = YB$ has only the trivial solution $Y = 0$ if the spectra of $A$ and $B$ are disjoint.) But then, the $H$-indecomposability assumption on $X$ implies $\widetilde{X} = X_1$ or $\widetilde{X} = X_2$ and the assertion follows. ◻

In view of Proposition 2.3, it is sufficient to develop canonical forms for polynomially $H$-normal matrices that have either real spectrum only or nonreal spectrum only. We start with the case of a real spectrum. The following result has been proved in [15].

THEOREM 2.4 ([15, Theorem 9.1]). *Let $\delta = \pm 1$ be such that $H^T = \delta H$ and $X \in \mathbb{R}^{n\times n}$ be polynomially $H$-normal with $H$-normalily polynomial $p \in \mathbb{R}[t]$. If $\sigma(X) \subseteq \mathbb{R}$, then there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$P^{-1}XP = X_1 \oplus \cdots \oplus X_p, \quad P^T HP = H_1 \oplus \cdots \oplus H_p, \tag{2.2}$$

*where $X_j$ is $H_j$-indecomposable, and $X_j$ and $H_j$ have one of the following forms:*

*i) blocks associated with $\lambda_j \in \mathbb{R}$ satisfying $p(\lambda_j) = \lambda_j$ and $p'(\lambda_j) = 1$ if $n_j > 1$:*

$$\text{if } \delta = +1 : X_j = \mathcal{J}_{n_j}(\lambda_j), \quad H_j = \varepsilon_j R_{n_j}, \tag{2.3}$$

$$\text{if } \delta = -1 : X_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & p\big(\mathcal{J}_{m_j}(\lambda_j)\big)^T \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{m_j} \\ -I_{m_j} & 0 \end{bmatrix}, \tag{2.4}$$

*where $\varepsilon_j = \pm 1$ and $n_j \in \mathbb{N}$ if $\delta = +1$ and $n_j = 2m_j \in \mathbb{N}$ is even if $\delta = -1$;*

*ii) blocks associated with $\lambda_j \in \mathbb{R}$ satisfying $p(\lambda_j) = \lambda_j$ and $p'(\lambda_j) = -1$:*

$$X_j = T(\lambda_j, 1, a_{j,2}, \ldots, a_{j,n_j-1}), \quad H_j = \varepsilon_j \Sigma_{n_j}, \tag{2.5}$$

*where $n_j > 1$ is odd if $\delta = 1$ and even if $\delta = -1$, $a_{j,2}, \ldots, a_{j,n_j-1} \in \mathbb{R}$, $a_{j,k} = 0$ for odd $k$, and $\varepsilon_j = \pm 1$;*

*iii) blocks associated with $\lambda_j \in \mathbb{R}$ satisfying $p(\lambda_j) = \lambda_j$ and satisfying $p'(\lambda_j) = -1$ if $m_j > 1$:*

$$X_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & p\big(\mathcal{J}_{m_j}(\lambda_j)\big)^T \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{m_j} \\ \delta I_{m_j} & 0 \end{bmatrix}, \tag{2.6}$$

*where $m_j \in \mathbb{N}$ is even if $\delta = +1$ and odd if $\delta = -1$;*

*iv) blocks associated with a pair $(\lambda_j, \mu_j) \in \mathbb{R} \times \mathbb{R}$ with $\mu_j = p(\lambda_j) < \lambda_j = p(\mu_j)$:*

$$X_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & p\big(\mathcal{J}_{m_j}(\lambda_j)\big)^T \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{m_j} \\ \delta I_{m_j} & 0 \end{bmatrix}, \tag{2.7}$$

*where $m_j \in \mathbb{N}$.*

*The form (2.2) is unique up to permutation of blocks and the nonzero parameters $a_{j,k}$ in (2.5) are uniquely determined by $\lambda_j$ and the coefficients of $p$ and can be computed from the identity $T(\lambda_j, -1, a_{j,2}, 0, a_{j,4}, 0, \ldots) = p\big(T(\lambda_j, 1, a_{j,2}, 0, a_{j,4}, 0, \ldots)\big)$.*

Observe that both cases ii) and iii) describe Jordan blocks associated with eigenvalues $\lambda_j$ satisfying $p(\lambda_j) = \lambda_j$ and satisfying $p'(\lambda_j) = -1$ if the corresponding partial multiplicity is larger than one. Then the theorem tells us that if $\delta = 1$ then even-sized Jordan blocks must occur in pairs while odd-sized blocks need not. Analogously, odd-sized Jordan blocks must occur in pairs, but even-sized blocks need not, if $\delta = -1$.

Theorem 2.4 settles the case that the matrix under consideration has real spectrum only and it remains to investigate the case of nonreal spectrum only. This will be done in the following sections.

**3. Relating real and complex matrices.** Assume that $X \in \mathbb{R}^{n\times n}$ is a polynomially $H$-normal matrix with nonreal spectrum. Instead of developing a canonical form for such matrices directly, it is our aim to construct these forms by applying the known results for the complex case obtained in [15]. As a tool, we use the well-known algebra isomorphism that relates complex matrices with real matrices of double size that have a special structure. Indeed, it is well known that the set

$$\mathcal{M}_{\mathbb{C}} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

equipped with the usual matrix addition and matrix multiplication is a field that is isomorphic to the field $\mathbb{C}$ of complex numbers. The corresponding field isomorphism

$$\phi : \mathbb{C} \to \mathcal{M}_{\mathbb{C}}, \quad (\alpha + i\beta) \mapsto \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

can be easily extended to an algebra isomorphism (that we will also denote by $\phi$) from the matrix algebra $\mathbb{C}^{n\times n}$ onto the matrix algebra $\mathcal{M}_{\mathbb{C}}^{n\times n}$ consisting of $n \times n$ matrices with entries in $\mathcal{M}_{\mathbb{C}}$ by

$$\phi\big((z_{ij})\big) := \big(\phi(z_{ij})\big) = \left(\begin{bmatrix} \operatorname{Re} z_{ij} & \operatorname{Im} z_{ij} \\ -\operatorname{Im} z_{ij} & \operatorname{Re} z_{ij} \end{bmatrix}\right)_{j,k=1}^{n}, \quad (z_{ij}) \in \mathbb{C}^{n\times n}.$$

If scalar multiplication in $\mathcal{M}_{\mathbb{C}}^{n\times n}$ is restricted to multiplication by diagonal matrices from $\mathcal{M}_{\mathbb{C}}$ (which are images of real numbers under $\phi$), then we can (and do) canonically identify $\mathcal{M}_{\mathbb{C}}^{n\times n}$ with a subalgebra of $\mathbb{R}^{2n\times 2n}$.

With the help of the isomorphism $\phi$ the real Jordan canonical form of a real matrix can be conveniently described. Indeed, recall that a Jordan block associated with a pair of conjugate nonreal eigenvalues $\alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$ has the form

$$\mathcal{J}_n(\alpha,\beta) := I_n \otimes \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} + \mathcal{J}_n(0) \otimes I_2 = \begin{bmatrix} \alpha & \beta & 1 & 0 & & & 0 \\ -\beta & \alpha & 0 & 1 & & & \\ & & \alpha & \beta & \ddots & & \\ & & -\beta & \alpha & & & \\ & & & & \ddots & 1 & 0 \\ & & & & & 0 & 1 \\ & & & & & \alpha & \beta \\ 0 & & & & & -\beta & \alpha \end{bmatrix}. \tag{3.1}$$

Clearly $\mathcal{J}_n(\alpha,\beta)$ is in the range of $\phi$, because $\phi\big(\mathcal{J}_n(\alpha + i\beta)\big) = \mathcal{J}_n(\alpha,\beta)$ for all $\alpha, \beta \in \mathbb{R}$. Other properties of $\phi$ are listed in the following remark and can be verified by straightforward calculations.

Remark 3.1. Let $Z \in \mathbb{C}^{n\times n}$ and $M \in \mathcal{M}_{\mathbb{C}}^{n\times n}$.

a) If $\lambda \in \mathbb{C}$ is an eigenvalue of $Z$, then $\lambda, \overline{\lambda}$ are eigenvalues of $\phi(Z)$.
b) $\phi(Z)^T = \phi(Z^*)$ and $\phi^{-1}(M^T) = \phi^{-1}(M)^*$.
c) $(I_n \otimes R_2)\, \phi(Z)(I_n \otimes R_2) = \phi\big(\overline{Z}\big)$.
d) $\phi\big(R_n Z R_n\big) = R_{2n}\, \phi\big(\overline{Z}\big) R_{2n}$.
e) If $T \in \mathbb{C}^{n\times n}$ is upper triangular Toeplitz, then $\phi(T) = R_{2n}\, \phi(T)^T R_{2n}$ .
f) $p\big(\phi(Z)\big) = \phi\big(p(Z)\big)$ for any polynomial $p \in \mathbb{R}[t]$.

Note that each Jordan block $\mathcal{J}_n(\alpha,\beta) = \phi\big(\mathcal{J}_n(\alpha + i\beta)\big)$ in (3.1) is similar to the block $\mathcal{J}_n(\alpha,-\beta) = \phi\big(\mathcal{J}_n(\alpha - i\beta)\big)$ and thus, $\mathcal{J}_n(\alpha,\beta)$ can be represented by a complex matrix either having the eigenvalue $\alpha + i\beta$ or $\alpha - i\beta$. This observation easily generalizes to the following lemma.

Lemma 3.2. *If $X \in \mathbb{R}^{2n\times 2n}$ has no real eigenvalues and if $\sigma_1 \subseteq \sigma(X)$ satisfies $\sigma_1 \cap \overline{\sigma_1} = \emptyset$ and $\sigma_1 \cup \overline{\sigma_1} = \sigma(X)$, then there exists a nonsingular $P \in \mathbb{R}^{2n\times 2n}$ and a matrix $\mathcal{X} \in \mathbb{C}^{n\times n}$ such that $\sigma(\mathcal{X}) = \sigma_1$ and $P^{-1}XP = \phi(\mathcal{X})$.*

Let us assume that $M \in \mathcal{M}_{\mathbb{C}}^{n\times n}$ is a polynomially $H$-normal matrix with $H$-normality polynomial $p$, that is, the identity $p(M)^T H = HM$ holds true. By, Remark 3.1 items b) and f), we immediately obtain that also $p(M)^T \in \mathcal{M}_{\mathbb{C}}^{n\times n}$. The question arises if also $H$ is contained in $\mathcal{M}_{\mathbb{C}}^{n\times n}$, or, equivalently, if the Sylvester equation $p(M)^T H = HM$ has a solution in $\mathcal{M}_{\mathbb{C}}^{n\times n}$. A sufficient condition is given in the following result.

PROPOSITION 3.3. *Let $A, B \in \mathbb{C}^{n\times n}$ such that $\sigma(A) \cap \sigma(\overline{B}) = \emptyset$. If $Y \in \mathbb{R}^{2n\times 2n}$ satisfies $\phi(A)Y = Y\phi(B)$, then $Y \in \mathcal{M}_{\mathbb{C}}^{n\times n}$.*

*Proof.* Define the permutation matrix $\Pi = [e_1, e_{n+1}, e_2, e_{n+2}, \dots, e_n, e_{2n}]$, where $e_j$ denotes the $j$th unit column vector of length $2n$. Then for any $Z \in \mathbb{C}^{n\times n}$ the transformation with $\Pi$ has the following effect:

$$\Pi^{-1}\phi(Z)\Pi = \begin{bmatrix} \mathrm{Re}(Z) & \mathrm{Im}(Z) \\ -\mathrm{Im}(Z) & \mathrm{Re}(Z) \end{bmatrix},$$

where $\mathrm{Re}(Z) := \frac{1}{2}(Z + \overline{Z})$ and $\mathrm{Im}(Z) := \frac{1}{2}(Z - \overline{Z})$. Moreover,

$$Q^{-1}\phi(Z)Q = \begin{bmatrix} Z & 0 \\ 0 & \overline{Z} \end{bmatrix}, \qquad \text{where } Q := \frac{1}{\sqrt{2}}\,\Pi \begin{bmatrix} -iI_n & I_n \\ I_n & -iI_n \end{bmatrix}.$$

Next consider

$$\Pi^{-1}Y\Pi = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, \tag{3.2}$$

where $Y_j \in \mathbb{R}^{n\times n}$, $j = 1, 2, 3, 4$. Then

$$Q^{-1}YQ = \begin{bmatrix} Y_1 + Y_4 + i(Y_2 + Y_3) & Y_2 + Y_3 + i(Y_1 - Y_4) \\ Y_2 + Y_3 + i(Y_4 - Y_1) & Y_1 + Y_4 + i(Y_3 + Y_2) \end{bmatrix}$$

and $\phi(A)Y = Y\phi(B)$ implies

$$\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} Q^{-1}YQ = Q^{-1}YQ \begin{bmatrix} B & 0 \\ 0 & \overline{B} \end{bmatrix} \tag{3.3}$$

Since $\sigma(A)\cap\sigma(\overline{B}) = \emptyset$, the Sylvester equation $AX = X\overline{B}$ has only the trivial solution $X = 0$. Thus, we obtain from (3.3) that $Y_2+Y_3+i(Y_1-Y_4) = 0$ which implies $Y_3 = -Y_2$ and $Y_4 = Y_1$. Inserting this in (3.2), it follows that $Y = \phi(Y_1 + iY_2) \in \mathcal{M}_{\mathbb{C}}^{n\times n}$. $\square$

**4. Essential decomposition of polynomially $H$-normal matrices.** In this section, we prove the main result of the paper that shows the existence of a decomposition of a real polynomially $H$-normal matrix $X$ that we will call *essential decomposition.* As a first step, we recall that in view of Proposition 2.3, it remains to investigate polynomially $H$-normal matrices $X$ that have nonreal spectrum only. Since any real matrix with nonreal spectrum only is similar to a matrix in $\mathcal{M}_{\mathbb{C}}^{n\times n}$, we may assume without loss of generality that $X = \phi(\mathcal{X})$, where $\mathcal{X} \in \mathbb{C}^{\frac{n}{2}\times\frac{n}{2}}$. In addition, we know that $X$ is polynomially $H$-normal, i.e., $p(X)^T H = HX$ for some polynomial $p \in \mathbb{F}[t]$. It is natural to ask if this property is inherited by $\mathcal{X}$, i.e., we ask whether there exists some complex (skew-)Hermitian matrix $\mathcal{H}$ such that $p(\mathcal{X})^*\mathcal{H} = \mathcal{H}\mathcal{X}$. Since

$$\begin{aligned} &\phi\big(p(\mathcal{X})^*\mathcal{H}\big) = \phi\big(p(\mathcal{X})^*\big)\phi(\mathcal{H}) = \phi\big(p(\mathcal{X})\big)^T\phi(\mathcal{H}) = p\big(\phi(\mathcal{X})\big)^T\phi(\mathcal{H}) = p(X)^T\phi(\mathcal{H}) \\ &\text{and} \quad \phi(\mathcal{H}\mathcal{X}) = \phi(\mathcal{H})\phi(\mathcal{X}) = \phi(\mathcal{H})X, \end{aligned}$$

we obtain that the answer is affirmative if $H$ is in the range of $\phi$, that is, $H = \phi(\mathcal{H})$ for some $\mathcal{H} \in \mathbb{C}^{n\times n}$. (It is easy to check that in this case $\mathcal{H}$ is (skew-)Hermitian if and only if $H$ is (skew-)symmetric.) By Proposition 3.3, we know that a sufficient condition is given by $\sigma(\mathcal{X})\cap\sigma\big(p(\mathcal{X})\big)=\emptyset$. The following example illustrates this fact.

EXAMPLE 4.1. Let $H\in\mathbb{R}^{4\times 4}$ be nonsingular and consider the matrix

$$S=\left[\begin{array}{cc}0&1\\-1&0\end{array}\right]\oplus\left[\begin{array}{cc}0&1\\-1&0\end{array}\right]=\phi\Bigg(\underbrace{\left[\begin{array}{cc}i&0\\0&i\end{array}\right]}_{=:\,\mathcal{S}}\Bigg).$$

Assume that $-S^TH=HS$, that is, $S$ is $H$-skewadjoint, or, equivalently, $S$ is polynomially $H$-normal with $H$-normality polynomial $p(t)=-t$. Since the only eigenvalue of $\mathcal{S}$ is $i$, we have $\sigma(\mathcal{S})\cap\sigma\big(p(\mathcal{S})\big)=\emptyset$. Thus, $H\in\mathcal{M}_{\mathbb{C}}^{2\times 2}$ by Proposition 3.3. Indeed, a straightforward computation reveals that $S$ is $H$-skewadjoint if and only if $H$ has the form

$$H=\left[\begin{array}{cccc}h_1&0&h_3&h_4\\0&h_1&-h_4&h_3\\h_3&-h_4&h_2&0\\h_4&h_3&0&h_2\end{array}\right]=\phi\Bigg(\underbrace{\left[\begin{array}{cc}h_1&h_3+ih_4\\h_3-ih_4&h_2\end{array}\right]}_{=:\,\mathcal{H}}\Bigg),$$

where $h_1,h_2,h_3,h_4\in\mathbb{R}$. It is easy to check that $\mathcal{S}$ is skewadjoint with respect to the sesquilinear form induced by the Hermitian matrix $\mathcal{H}$.

Unfortunately, the trick in Example 4.1 does not work if $\sigma(\mathcal{X})\cap\sigma\big(p(\mathcal{X})\big)\neq\emptyset$. We illustrate this with the help of another example.

EXAMPLE 4.2. Let $H\in\mathbb{R}^{4\times 4}$ be nonsingular and consider the matrix

$$A=\left[\begin{array}{cc}0&1\\-1&0\end{array}\right]\oplus\left[\begin{array}{cc}0&1\\-1&0\end{array}\right]=\phi\Bigg(\underbrace{\left[\begin{array}{cc}i&0\\0&i\end{array}\right]}_{=:\,\mathcal{A}}\Bigg).$$

Assume that $A$ is $H$-selfadjoint, or, equivalenty, that $A$ is polynomially $H$-normal with $H$-normality polynomial $p(t)=t$. Then a straightforward computation reveals that this is the case if and only if $H$ has the form

$$H=\left[\begin{array}{cccc}h_1&h_2&h_3&h_4\\h_2&-h_1&h_4&-h_3\\h_3&h_4&h_5&h_6\\h_4&-h_3&h_6&-h_5\end{array}\right],$$

where $h_1,h_2,h_3,h_4,h_5,h_6\in\mathbb{R}$. Thus, since $H\neq 0$, we obtain that $H$ is not in the range of $\phi$. However, observe that

$$(I_2\otimes R_2)H=\left[\begin{array}{cccc}h_2&-h_1&h_4&-h_3\\h_1&h_2&h_3&h_4\\h_4&-h_3&h_6&-h_5\\h_3&h_4&h_5&h_6\end{array}\right]=\phi\Bigg(\underbrace{\left[\begin{array}{cc}h_2-ih_1&h_4-ih_3\\h_4-ih_3&h_6-ih_5\end{array}\right]}_{=:\,\mathcal{H}}\Bigg),$$

It is interesting to note that $\mathcal{H}$ is not Hermitian, but complex symmetric, and that $\mathcal{A}$ is selfadjoint with respect to the bilinear form induced by $\mathcal{H}$ as it follows easily from a straightforward computation.

Examples 4.1 and 4.2 suggest the following strategy for the investigation of a matrix $X=\phi(\mathcal{X})$. If $\sigma(\mathcal{X})\cap\sigma\big(p(\mathcal{X})\big)=\emptyset$ then we interpret $\mathcal{X}$ as a polynomially normal matrix with respect to a sesquilinear form. If this is not the case, then we will try to interpret $\mathcal{X}$ as a polynomially normal matrix with respect to a bilinear form along the lines of Example 4.2. This is the main idea that leads to the *essential decomposition* of polynomially $H$-normal matrices.

THEOREM 4.3 (Essential decomposition). *Let $\delta = \pm 1$ be such that $H^T = \delta H$ and let $X \in \mathbb{R}^{n\times n}$ be a polynomially $H$-normal matrix with $H$-normality polynomial $p \in \mathbb{R}[t]$. Then there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$P^{-1}XP = X_1 \oplus X_2 \oplus X_3, \qquad P^T HP = H_1 \oplus H_2 \oplus H_3, \tag{4.1}$$

*where for $j = 1, 2, 3$ the matrices $X_j$ and $H_j$ have the same size $n_j \times n_j$ and satisfy the following conditions:*

*(1)* $\sigma(X_1) \subseteq \mathbb{R}$*;*

*(2)* $X_2 = \phi(\mathcal{X}_2)$ *and* $H_2 = \phi(\mathcal{H}_2)$*, where* $\mathcal{H}_2^* = \delta\mathcal{H}_2$ *and where*

$$p(\mathcal{X}_2)^*\mathcal{H}_2 = \mathcal{H}_2\mathcal{X}_2,$$

*i.e., $\mathcal{X}_2$ is polynomially $\mathcal{H}_2$-normal with respect to the sesquilinear form induced by $\mathcal{H}_2$; moreover, $\mathcal{X}_2$ satisfies*

$$\sigma(\mathcal{X}_2) \subseteq \{\lambda \in \mathbb{C}\setminus\mathbb{R} \,|\, p(\lambda) \neq \overline{\lambda}\}, \quad \sigma(\mathcal{X}_2) \cap \sigma\big(p(\mathcal{X}_2)\big) = \emptyset,$$

*(3)* $X_3 = \phi(\mathcal{X}_3)$ *and* $(I \otimes R_2)H_3 = \phi(\mathcal{H}_3)$*, where* $\mathcal{H}_3^T = \delta\mathcal{H}_3$ *and where*

$$p(\mathcal{X}_3)^T\mathcal{H}_3 = \mathcal{H}_3\mathcal{X}_3,$$

*i.e., $\mathcal{X}_3$ is polynomially $\mathcal{H}_3$-normal with respect to the bilinear form induced by $\mathcal{H}_3$; moreover, $\mathcal{X}_3$ satisfies*

$$\sigma(\mathcal{X}_3) \subseteq \{\lambda \in \mathbb{C}\setminus\mathbb{R} \,|\, p(\lambda) = \overline{\lambda}\}, \quad \sigma(\mathcal{X}_3) \cap \sigma\big(\overline{p(\mathcal{X}_3)}\big) = \emptyset.$$

*Furthermore, the decomposition (4.1) is unique up to equivalence of the factors $X_j$, $H_j$ in the sense $(X_j, H_j) \sim (P^{-1}X_jP, P^T H_jP)$ for some nonsingular $P \in \mathbb{R}^{n_j\times n_j}$.*

*Proof.* Clearly $\mathbb{C} = \mathbb{R}\,\dot\cup\,\{\lambda \in \mathbb{C}\setminus\mathbb{R} \,|\, p(\lambda) \neq \overline{\lambda}\}\,\dot\cup\,\{\lambda \in \mathbb{C}\setminus\mathbb{R} \,|\, p(\lambda) = \overline{\lambda}\}$. Thus, the spectrum of $X$ can be split analogously into three disjoint parts and there exists a nonsingular matrix $P$ such that $P^{-1}XP = X_1 \oplus \widetilde{X}_2 \oplus \widetilde{X}_3$ where

$$\sigma(X_1)\subseteq\mathbb{R}, \quad \sigma(\widetilde{X}_2)\subseteq\{\lambda \in \mathbb{C}\setminus\mathbb{R} \,|\, p(\lambda) \neq \overline{\lambda}\}, \quad \sigma(\widetilde{X}_3)\subseteq\{\lambda \in \mathbb{C}\setminus\mathbb{R} \,|\, p(\lambda) = \overline{\lambda}\}. \tag{4.2}$$

Observe that if $\lambda \in \sigma(X)$ is from one of the spectra in (4.2), then $p(\lambda)$ is from the same spectrum. (This follows easily from the property $p\big(p(\lambda)\big) = \lambda$ which holds for all eigenvalues $\lambda \in \mathbb{C}$ of $X$.) Consequently, the identity

$$p(X_1 \oplus \widetilde{X}_2 \oplus \widetilde{X}_3)^T(P^T HP) = (P^T HP)(X_1 \oplus \widetilde{X}_2 \oplus \widetilde{X}_3)$$

implies that $P^T HP$ has a block structure $H_1 \oplus \widetilde{H}_2 \oplus \widetilde{H}_3$ corresponding to $P^{-1}XP$, where we used that the Sylvester equation $AY = YB$ only has the trivial solution $Y = 0$ when the spectra of $A$ and $B$ are disjoint. Clearly, the decomposition of $X$ into the three parts $X_1$, $\widetilde{X}_2$, and $\widetilde{X}_3$ is then unique in the sense of the theorem.

Next, we focus our attention on the blocks $\widetilde{X}_2$ and $\widetilde{H}_2$. Since $\widetilde{X}_2$ is polynomially $\widetilde{H}_2$-normal and since $\widetilde{X}_2$ has no eigenvalues satisfying $p(\lambda) = \overline{\lambda}$, it follows from Proposition 2.1, item 3) that the eigenvalues of $\widetilde{X}_2$ occur in pairs $(\lambda, \mu)$, where $p(\lambda) = \mu$ and

$p(\mu) = \lambda$. Since $\widetilde{X}_2$ is furthermore real and, hence, all eigenvalues occur in conjugate pairs, we obtain that

$$\sigma(\widetilde{X}_2) = \bigcup_{j=1}^{m} \{\lambda_j, \overline{\lambda}_j\} \cup \bigcup_{j=m+1}^{r} \{\lambda_j, \overline{\lambda}_j, \mu_j, \overline{\mu}_j\}, \tag{4.3}$$

for some $\lambda_j, \mu_j \in \mathbb{C} \setminus \mathbb{R}$, where $p(\lambda_j) = \overline{\lambda}_j$ for $j = 1, \ldots, m$ and $p(\lambda_j) = \mu_j \neq \overline{\lambda}_j$ for $j = m+1, \ldots, r$, and $\lambda_i \neq \lambda_j, \overline{\lambda}_j$ for $i \neq j$. Setting

$$\sigma_2 := \bigcup_{j=1}^{m} \{\lambda_1\} \cup \bigcup_{j=m+1}^{r} \{\lambda_j, \overline{\mu}_j\}, \tag{4.4}$$

we obtain that $\sigma_2 \cap \overline{\sigma_2} = \emptyset$ and $\sigma_2 \cup \overline{\sigma_2} = \sigma(\widetilde{X}_2)$. Hence, by Lemma 3.2, there exists a real nonsingular matrix $\widetilde{P}_2$ such that $X_2 := \widetilde{P}_2^{-1} \widetilde{X}_2 \widetilde{P}_2 = \phi(\mathcal{X}_2)$, where $\sigma(\mathcal{X}_2) = \sigma_2$. Observe that, by construction, we also have $\sigma(\mathcal{X}_2) \cap \sigma\big(p(\mathcal{X}_2)\big) = \emptyset$. Now denote $H_2 := \widetilde{P}_2^T \widetilde{H}_2 \widetilde{P}_2$. Then $X_2$ is polynomially $H_2$-normal and in view of Proposition 3.3, we obtain from the identity

$$\phi\big(\overline{p(\mathcal{X}_2)}^T\big) H_2 = \phi\big(p(\mathcal{X}_2)\big)^T H_2 = p(X_2)^T H_2 = H_2 X_2 = H_2\, \phi(\mathcal{X}_2) \tag{4.5}$$

that $H_2 \in \mathcal{M}_{\mathbb{C}}^{m_2 \times m_2}$, where $m_2 = n_2/2$, i.e., there exists $\mathcal{H}_2 \in \mathbb{C}^{m_2 \times m_2}$ such that $H_2 = \phi(\mathcal{H}_2)$. From $H_2^T = \delta H_2$, we easily obtain $\mathcal{H}_2^* = \delta \mathcal{H}_2$. Moreover, identity (4.5) implies that

$$\phi\big(p(\mathcal{X}_2)^* \mathcal{H}_2\big) = \phi\big(p(\mathcal{X}_2)^*\big) H_2 = H_2\, \phi(\mathcal{X}_2) = \phi(\mathcal{H}_2 \mathcal{X}_2),$$

and since $\phi$ is an isomorphism, it follows that $p(\mathcal{X}_2)^* \mathcal{H}_2 = \mathcal{H}_2 \mathcal{X}_2$.

Next, consider the blocks $\widetilde{X}_3$ and $\widetilde{H}_3$. Since the spectrum of $\widetilde{X}_3$ is nonreal, there exists a real nonsingular matrix $\widetilde{P}_3$ such that $X_3 := \widetilde{P}_3^{-1} \widetilde{X}_3 \widetilde{P}_3 = \phi(\mathcal{X}_3)$ and where $\mathcal{X}_3$ is chosen such that

$$\sigma(\mathcal{X}_3) \cap \sigma(\overline{\mathcal{X}_3}) = \emptyset. \tag{4.6}$$

(For example, we may choose $\sigma(\mathcal{X}_3) \subseteq \{\lambda \,|\, \mathrm{Re}(\lambda) > 0\}$.) Denote $H_3 := \widetilde{P}_3^T \widetilde{H}_3 \widetilde{P}_3$. Then $X_3$ is polynomially $H_3$-normal and from Remark 3.1, we obtain that

$$\begin{aligned}
(I \otimes R_2) H_3\, \phi(\mathcal{X}_3) = (I \otimes R_2)\, p\big(\phi(\mathcal{X}_3)\big)^T H_3 \;&=\; (I \otimes R_2)\, \phi\big(p(\mathcal{X}_3)^*\big) H_3 \\
&= (I \otimes R_2)\, \phi\big(p(\mathcal{X}_3)^*\big)(I \otimes R_2)(I \otimes R_2) H_3 \\
&= \phi\big(p(\mathcal{X}_3)^T\big)(I \otimes R_2) H_3.
\end{aligned}$$

Recall that all eigenvalues $\lambda$ of $\widetilde{X}_3$ satisfy $p(\lambda) = \lambda$. Thus, in view of (4.6), we have $\sigma(\mathcal{X}_3) \cap \sigma\big(\overline{p(\mathcal{X}_3)}\big) = \emptyset$. Then Proposition 3.3 implies $(I \otimes R_2) H_3 \in \mathcal{M}_{\mathbb{C}}^{m_3 \times m_3}$, where $m_3 = n_3/2$, that is, $(I \otimes R_2) H_3 = \phi(\mathcal{H}_3)$ for some $\mathcal{H}_3 \in \mathbb{C}^{m_3 \times m_3}$. Moreover, we obtain from $H_3^T = \delta H_3$ that

$$\begin{aligned}
\phi(\mathcal{H}_3) = (I \otimes R_2) H_3 \;&=\; \delta (I \otimes R_2) H_3^T \;=\; \delta (I \otimes R_2)\big((I \otimes R_2) H_3\big)^T (I \otimes R_2) \\
&= \delta (I \otimes R_2) \phi(\mathcal{H}_3)^T (I \otimes R_2) \;=\; \delta (I \otimes R_2) \phi(\mathcal{H}_3^*) (I \otimes R_2) \;=\; \delta \phi(\mathcal{H}_3^T),
\end{aligned}$$

that is, $\mathcal{H}_3^T = \delta \mathcal{H}_3$. Finally, we have that

$$\phi\big(p(\mathcal{X}_3)^T \mathcal{H}_3\big) = \phi\big(p(\mathcal{X}_3)^T\big)(I \otimes R_2) H_3 = (I \otimes R_2) H_3\, \phi(\mathcal{X}_3) = \phi(\mathcal{H}_3 \mathcal{X}_3)$$

which implies $p(\mathcal{X}_3)^T\mathcal{H}_3 = \mathcal{H}_3\mathcal{X}_3$ and concludes the proof. □

The uniqueness property of Theorem 4.3 justifies the following definition.

DEFINITION 4.4. Let $X \in \mathbb{R}^{n\times n}$ be polynomially $H$-normal and let

$$P^{-1}XP = X_1 \oplus X_2 \oplus X_3, \qquad P^THP = H_1 \oplus H_2 \oplus H_3,$$

be its essential decomposition. Then $X_1$ is called *the real part of* $X$, $X_2$ is called *the complex sesquilinear part of* $X$, and $X_3$ is called *the complex bilinear part of* $X$.

In view of Theorem 4.3, it seems natural to compute the canonical form for the pair $(X, H)$ by computing the canonical forms for the pairs $(X_j, H_j)$ in the essential decomposition. The following theorem justifies that the combination of these canonical forms does indeed yield a canonical form for the pair $(X, H)$.

THEOREM 4.5. *Let $\delta = \pm 1$ be such that $H, \widetilde{H} \in \mathbb{R}^{n\times n}$ satisfy $H^T = \delta H$ and $\widetilde{H}^T = \delta\widetilde{H}$. Moreover, let $X \in \mathbb{R}^{n\times n}$ be polynomially $H$-normal and let $\widetilde{X} \in \mathbb{R}^{n\times n}$ be polynomially $\widetilde{H}$-normal such that the two pairs $(X, H)$ and $(\widetilde{X}, \widetilde{H})$ are essentially decomposed*

$$\begin{aligned} X &= X_1 \oplus X_2 \oplus X_3, \qquad & H &= H_1 \oplus H_2 \oplus H_3, \\ \widetilde{X} &= \widetilde{X}_1 \oplus \widetilde{X}_2 \oplus \widetilde{X}_3, \qquad & \widetilde{H} &= \widetilde{H}_1 \oplus \widetilde{H}_2 \oplus \widetilde{H}_3, \end{aligned}$$

*in the sense of Theorem 4.3, in particular, $X_1, H_1 \in \mathbb{R}^{n_1\times n_1}$, $\widetilde{X}_1, \widetilde{H}_1 \in \mathbb{R}^{\widetilde{n}_1\times\widetilde{n}_1}$,*

$$\begin{aligned} &X_2 = \phi(\mathcal{X}_2), \quad H_2 = \phi(\mathcal{H}_2), \quad X_3 = \phi(\mathcal{X}_3), \quad H_3 = (I \otimes R_2)\,\phi(\mathcal{H}_3), \\ &\widetilde{X}_2 = \phi(\widetilde{\mathcal{X}}_2), \quad \widetilde{H}_2 = \phi(\widetilde{\mathcal{H}}_2), \quad \widetilde{X}_3 = \phi(\widetilde{\mathcal{X}}_3), \quad \widetilde{H}_3 = (I \otimes R_2)\,\phi(\widetilde{\mathcal{H}}_3), \end{aligned}$$

*where $\mathcal{X}_j, \mathcal{H}_j \in \mathbb{C}^{m_j\times m_j}$, $\widetilde{\mathcal{X}}_j, \widetilde{\mathcal{H}}_j \in \mathbb{C}^{\widetilde{m}_j\times\widetilde{m}_j}$, $j = 2, 3$ and where all blocks satisfy the assumptions of Theorem 4.3. Assume further that the blocks $M = \mathcal{X}_2, \widetilde{\mathcal{X}}_2, \mathcal{X}_3, \widetilde{\mathcal{X}}_3$ have been chosen so that their spectra satisfy the condition*

$$\lambda \in \sigma(M) \;\Rightarrow\; \overline{\lambda} \notin \sigma(M). \tag{4.7}$$

*Then the following conditions are equivalent.*

*(1) The identities $n_1 = \widetilde{n}_1$, $m_2 = \widetilde{m}_2$, $m_3 = \widetilde{m}_3$ hold and there exist nonsingular matrices $P_1 \in \mathbb{R}^{n_1\times n_1}$, $\mathcal{P}_j \in \mathbb{C}^{m_j\times m_j}$, $j = 2, 3$ such that*
   *(1a) $P_1^{-1}\widetilde{X}_1P_1 = X_1$ and $P_1^T\widetilde{H}_1P_1 = H_1$,*
   *(1b) $\mathcal{P}_2^{-1}\widetilde{\mathcal{X}}_2\mathcal{P}_2 = \mathcal{X}_2$ and $\mathcal{P}_2^*\widetilde{\mathcal{H}}_2\mathcal{P}_2 = \mathcal{H}_2$,*
   *(1c) $\mathcal{P}_3^{-1}\widetilde{\mathcal{X}}_3\mathcal{P}_3 = \mathcal{X}_3$ and $\mathcal{P}_3^T\widetilde{\mathcal{H}}_3\mathcal{P}_3 = \mathcal{H}_3$.*

*(2) The identities $\sigma(\mathcal{X}_2) = \sigma(\widetilde{\mathcal{X}}_2)$, $\sigma(\mathcal{X}_3) = \sigma(\widetilde{\mathcal{X}}_3)$ hold and there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that $P^{-1}\widetilde{X}P = X$ and $P^T\widetilde{H}P = H$.*

*Proof.* '(1) ⇒ (2)': From (1b) and (1c), we immediately obtain $\sigma(\mathcal{X}_2) = \sigma(\widetilde{\mathcal{X}}_2)$ and $\sigma(\mathcal{X}_3) = \sigma(\widetilde{\mathcal{X}}_3)$. Let $P := P_1 \oplus \phi(\mathcal{P}_2) \oplus \phi(\mathcal{P}_3)$. Then $P$ satisfies the requirements of the theorem, because we clearly have

$$P_j^{-1}\widetilde{X}_jP_j = \phi(\mathcal{P}_j)^{-1}\,\phi(\mathcal{X}_j)\phi(\mathcal{P}_j) = \phi(\mathcal{P}_j^{-1}\widetilde{\mathcal{X}}_j\mathcal{P}_j) = \phi(\mathcal{X}_j) = X_j \quad \text{for } j = 2, 3,$$

$$P_2^T\widetilde{H}_2P_2 = \phi(\mathcal{P}_2)^T\phi(\widetilde{\mathcal{H}}_2)\,\phi(\mathcal{P}_2) = \phi(\mathcal{P}_2^*\widetilde{\mathcal{H}}_2\mathcal{P}_2) = \phi(\mathcal{H}_2) = H_2, \quad \text{and} \tag{4.8}$$

$$\begin{aligned} P_3^T\widetilde{H}_3P_3 &= \phi(\mathcal{P}_3^*)(I \otimes R_2)\,\phi(\widetilde{\mathcal{H}}_3)\phi(\mathcal{P}_3) \\ &= (I \otimes R_2)\,\phi(\mathcal{P}_3^T)(I \otimes R_2)(I \otimes R_2)\,\phi(\widetilde{\mathcal{H}}_3)\phi(\mathcal{P}_3) \\ &= (I \otimes R_2)\,\phi(\mathcal{P}_3^T\widetilde{\mathcal{H}}_3\mathcal{P}_3) = (I \otimes R_2)\,\phi(\mathcal{H}_3) = H_3. \end{aligned} \tag{4.9}$$

'(2) $\Rightarrow$ (1)': Applying the uniqueness statement of Theorem 4.3, we obtain from the existence of $P$ as in (2) that $n_1 = \widetilde{n}_1$, $m_2 = \tilde{m}_2$, and $m_3 = \tilde{m}_3$, and that there exists nonsingular matrices $P_1 \in \mathbb{R}^{n_1 \times n_1}$, $P_j \in \mathbb{R}^{2m_j \times 2m_j}$, $j = 2, 3$ such that $P_j^{-1} \widetilde{X}_j P_j = X_j$ and $P_j^T \widetilde{H}_j P_j = H_j$ for $j = 1, 2, 3$. In particular, this implies (1a). Then from (4.7), from $\sigma(\mathcal{X}_j) = \sigma(\widetilde{\mathcal{X}}_j)$, $j = 2, 3$, and from Proposition 3.3, we obtain that $P_j \in \mathcal{M}_{\mathbb{C}}^{m_j \times m_j}$, $j = 2, 3$, that is, there exist (nonsingular) matrices $\mathcal{P}_j \in \mathbb{C}^{m_j \times m_j}$ such that $P_j = \phi(\mathcal{P}_j)$, $j = 2, 3$. Then analogously to the calculations in (4.8) and (4.9), we show that

$$\phi(\mathcal{P}_j^{-1} \widetilde{\mathcal{X}}_j \mathcal{P}_j) = \phi(\mathcal{X}_j),\ j = 2, 3, \quad \phi(\mathcal{P}_2^* \widetilde{\mathcal{H}}_2 \mathcal{P}_2) = \phi(\mathcal{H}_2), \quad \phi(\mathcal{P}_3^T \widetilde{\mathcal{H}}_3 \mathcal{P}_3) = \phi(\mathcal{H}_3).$$

Then using the fact that $\phi$ is an isomorphism implies (1b) and (1c). ∎

Remark 4.6. Combining Theorem 4.3 and Theorem 4.5, the problem of computing a real canonical form for a real polynomially $H$-normal matrix $X$ is finally reduced to computing one real canonical form as in Theorem 2.4 and two complex canonical forms as in [15, Theorem 6.1] and [15, Theorem 7.1]. Since these three canonical forms are unique up to permutation of blocks, by Theorem 4.5 we obtain an analogous uniqueness statement for the real canonical form of a real polynomially $H$-normal matrix $X$ once we have specified the spectra of the matrices $\mathcal{X}_2$ and $\mathcal{X}_3$ in essential decomposition of $X$. For $\mathcal{X}_3$, this could be easily achieved, e.g., by requiring that $\sigma(\mathcal{X}_3) \subseteq \{\lambda \,|\, \mathrm{Re}(\lambda) > 0\}$. For $\mathcal{X}_2$ this is not as easy, because we have to choose a subset $\sigma_2$ as in (4.4) from a set $\sigma(\widetilde{X}_2)$ as in (4.3). In general, we cannot require $\sigma(\mathcal{X}_2) \subseteq \{\lambda \,|\, \mathrm{Re}(\lambda) > 0\}$, because $\sigma_2$ must contain pairs $\{\lambda_j, \overline{\mu}_j\} = \{\lambda_j, \overline{p(\lambda_j)}\}$ and it is not guaranteed that with $\lambda_j$ also $\overline{\mu}_j = \overline{p(\lambda_j)}$ is in the open upper half plane of the complex numbers. However, we may specify the spectrum of $\mathcal{X}_2$ as follows. Introducing the following relation of the complex numbers

$$c_1 < c_2 \ :\Leftrightarrow \ \Big(|c_1| < |c_2| \text{ or } \big(|c_1| = |c_2| \text{ and } \arg(c_1) < \arg(c_2)\big)\Big),$$

let the elements of the set in (4.3)

$$\sigma(\widetilde{X}_2) = \bigcup_{j=1}^{m} \{\lambda_j, \overline{\lambda}_j\} \cup \bigcup_{j=m+1}^{r} \{\lambda_j, \overline{\lambda}_j, \mu_j, \overline{\mu}_j\}$$

be ordered such that $\lambda_j < \overline{\lambda}_j$ for $j = 1, \ldots, m$ and $\lambda_j < \overline{\lambda}_j, \mu_j, \overline{\mu}_j$ for $j = m+1, \ldots, r$. Then we choose

$$\sigma_2 := \bigcup_{j=1}^{m} \{\lambda_1\} \cup \bigcup_{j=m+1}^{r} \{\lambda_j, \overline{\mu}_j\},$$

as the spectrum of $\mathcal{X}_2$. With this specification, the real canonical form for a real polynomially $H$-normal matrix $X$ is unique up to permutation of blocks.

**5. $H$-selfadjoint, $H$-skewadjoint, and $H$-unitary matrices.** In this section, we derive real canonical forms for real $H$-selfadjoint, $H$-skewadjoint, and $H$-unitary matrices by applying Theorem 4.3. As before, $H \in \mathbb{R}^{n \times n}$ always denotes a symmetric or skew-symmetric, nonsingular matrix.

Theorem 5.1 (Canonical forms for $H$-selfadjoint matrices). *Let $\delta = \pm 1$ be such that $H^T = \delta H$ and let $A \in \mathbb{R}^{n \times n}$ be $H$-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{R}^{n \times n}$ such that*

$$P^{-1} A P = A_1 \oplus \cdots \oplus A_p, \quad P^T H P = H_1 \oplus \cdots \oplus H_p, \tag{5.1}$$

*where $A_j$ is $H_j$-indecomposable and where $A_j$ and $H_j$ have one of the following forms:*

*a) in the case $\delta = +1$:*

*i) blocks associated with real eigenvalues $\lambda_j \in \mathbb{R}$:*

$$A_j = \mathcal{J}_{n_j}(\lambda_j), \quad H_j = \varepsilon_j R_{n_j}, \tag{5.2}$$

*where $n_j \in \mathbb{N}, \varepsilon_j = \pm 1$;*

*ii) blocks associated with a pair $\alpha_j \pm i\beta_j$ of conjugate complex eigenvalues:*

$$A_j = \mathcal{J}_{m_j}(\alpha_j, \beta_j), \quad H_j = R_{2m_j}, \tag{5.3}$$

*where $m_j \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$, and $\beta_j > 0$.*

*b) in the case $\delta = -1$:*

*i) paired blocks associated with real eigenvalues $\lambda_j \in \mathbb{R}$:*

$$A_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & \mathcal{J}_{m_j}(\lambda_j)^T \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{m_j} \\ -I_{m_j} & 0 \end{bmatrix}, \tag{5.4}$$

*where $m_j \in \mathbb{N}$;*

*ii) blocks associated with a pair $\alpha_j \pm i\beta_j$ of conjugate complex eigenvalues:*

$$A_j = \begin{bmatrix} \mathcal{J}_{m_j}(\alpha_j, \beta_j) & 0 \\ 0 & \mathcal{J}_{m_j}(\alpha_j, \beta_j)^T \end{bmatrix}, H_j = \begin{bmatrix} 0 & I_{2m_j} \\ -I_{2m_j} & 0 \end{bmatrix}, \tag{5.5}$$

*where $m_j \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$, and $\beta_j > 0$.*

*Moreover, the form (5.1) is unique up to the permutation of blocks.*

*Proof.* Since $A$ is $H$-selfadjoint, we have that $A$ is polynomially $H$-normal with $H$-normality polynomial $p(t) = t$. Without loss of generality, we may assume that $A$ and $H$ are already essentially decomposed in the sense of Theorem 4.3. Observe that $A$ has no complex sesquilinear part, due to the special structure of the polynomial $p$. Thus, we have

$$A = \widetilde{A}_1 \oplus \widetilde{A}_3, \quad H = \widetilde{H}_1 \oplus \widetilde{H}_3,$$

where $\sigma(\widetilde{A}_1) \subseteq \mathbb{R}$ and $\widetilde{A}_3 = \phi(\mathcal{A}_3)$, $\widetilde{H}_3 = (I \otimes R_2)\phi(\mathcal{H}_3)$, and where $\mathcal{A}_3$ is $\mathcal{H}_3$-selfadjoint with respect to the bilinear form induced by the complex (skew-)symmetric matrix $\mathcal{H}_3$. For the sake of uniqueness, we choose the eigenvalues of $\mathcal{A}_3$ such that $\sigma(\mathcal{A}_3) \subseteq \{\lambda \in \mathbb{C} \setminus \mathbb{R} \mid \mathrm{Im}(\lambda) > 0\}$. In view of Theorem 4.5, we may furthermore assume that the pairs $(\widetilde{A}_1, \widetilde{H}_1)$ and $(\mathcal{A}_3, \mathcal{H}_3)$ are in canonical form.

The canonical form of the pair $(\widetilde{A}_1, \widetilde{H}_1)$ can be read off Theorem 2.4. If $\delta = 1$ then only blocks of the form (2.3) occur. This gives blocks of the form (5.2). On the other hand if $\delta = -1$, then only blocks of the form (2.4) occur. This gives blocks of the form (5.4).

The canonical form of the pair $(\mathcal{A}_3, \mathcal{H}_3)$ in the case $\delta = 1$ can be read off [15, Theorem 7.2] and is

$$\mathcal{A}_3 = \mathcal{J}_{m_1}(\alpha_1 + i\beta_1) \oplus \cdots \oplus \mathcal{J}_{m_k}(\alpha_k + i\beta_k), \quad \mathcal{H}_3 = R_{m_1} \oplus \cdots \oplus R_{m_k},$$

where $m_j \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$, and $\beta_j > 0$ for $j = 1, \ldots, k$. Using $(I \otimes R_2)\phi(R_{m_j}) = R_{2m_j}$ this gives the blocks of the form (5.3) (after eventually renaming indices).

The canonical form of the pair $(\mathcal{A}_3, \mathcal{H}_3)$ in the case $\delta = -1$ can be read off [15, Theorem 8.2] and is block diagonal with diagonal blocks of the form

$$\mathcal{A}_j = \begin{bmatrix} \mathcal{J}_{m_j}(\alpha_j + i\beta_j) & 0 \\ 0 & \mathcal{J}_{m_j}(\alpha_j + i\beta_j)^T \end{bmatrix}, \quad \mathcal{H}_j = \begin{bmatrix} 0 & I_{m_j} \\ -I_{m_j} & 0 \end{bmatrix},$$

where $m_j \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$, and $\beta_j > 0$, or, equivalently,

$$\mathcal{A}_j = \begin{bmatrix} \mathcal{J}_{m_j}(\alpha_j + i\beta_j) & 0 \\ 0 & \mathcal{J}_{m_j}(\alpha_j + i\beta_j) \end{bmatrix}, \quad \mathcal{H}_j = \begin{bmatrix} 0 & R_{m_j} \\ -R_{m_j} & 0 \end{bmatrix}$$

which follows easily by applying a transformation with the transformation matrix $I_{m_j} \oplus R_{m_j}$. Using the fact that $(I \otimes R_2)\phi(R_{m_j}) = R_{2m_j}$ this gives the blocks of the form (5.5) (after renaming of indices and after applying a transformation with the transformation matrix $I_{2m_j} \oplus R_{2m_j}$). □

REMARK 5.2. The canonical form given in Theorem 5.1 is well known see, e.g., [13] for the case $\delta = +1$ and [5] for the case $\delta = -1$.

Concerning the corresponding result for $H$-skewadjoint matrices, we will need additional notation. Let $E_n$ denote the diagonal matrix with increasing powers of $i$, that is, $E_n := \operatorname{diag}(1, i, i^2, \dots, i^{n-1})$. Observe that the following identities hold:

$$E_n^{-1}\mathcal{J}_n(i\lambda)E_n = i\mathcal{J}_n(\lambda), \quad E_n^{-1}R_nE_n = i^{n-1}\Sigma_n, \quad E_nR_nE_n^{-1} = (-i)^{n-1}\Sigma_n.$$

THEOREM 5.3 (Canonical forms for $H$-skewadjoint matrices). *Let $\delta = \pm 1$ be such that $H^T = \delta H$ and let $S \in \mathbb{R}^{n \times n}$ be $H$-skewadjoint. Then there exists a nonsingular matrix $P \in \mathbb{R}^{n \times n}$ such that*

$$P^{-1}SP = S_1 \oplus \cdots \oplus S_p, \quad P^THP = H_1 \oplus \cdots \oplus H_p, \tag{5.6}$$

*where $S_j$ is $H_j$-indecomposable and where $S_j$ and $H_j$ have one of the following forms:*

*i) blocks associated with the eigenvalue $\lambda_j = 0$:*

$$S_j = \mathcal{J}_{n_j}(0), \quad H_j = \varepsilon_j\Sigma_{n_j}, \tag{5.7}$$

*where $\varepsilon_j = \pm 1$ and where $n_j \in \mathbb{N}$ is odd if $\delta = 1$ and even if $\delta = -1$;*

*ii) paired blocks associated with the eigenvalue $\lambda_j = 0$:*

$$S_j = \begin{bmatrix} \mathcal{J}_{m_j}(0) & 0 \\ 0 & -\mathcal{J}_{m_j}(0)^T \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{m_j} \\ \delta I_{m_j} & 0 \end{bmatrix}, \tag{5.8}$$

*where $m_j \in \mathbb{N}$ is even if $\delta = 1$ and odd if $\delta = -1$;*

*iii) blocks associated with a pair $(\lambda_j, -\lambda_j) \in \mathbb{R}^2$ of real nonzero eigenvalues:*

$$S_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & -\mathcal{J}_{m_j}(\lambda_j)^T \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{m_j} \\ \delta I_{m_j} & 0 \end{bmatrix}, \tag{5.9}$$

*where $m_j \in \mathbb{N}$, and $\lambda_j > 0$.*

*iv) blocks associated with a pair $(i\lambda_j, -i\lambda_j) \in i\mathbb{R}^2$ of purely imaginary eigenvalues:*

$$S_j = \mathcal{J}_{m_j}(0, \lambda_j), \quad H_j = \varepsilon_j\Sigma_{m_j} \otimes I_2, \tag{5.10}$$

*where $m_j \in \mathbb{N}$ is odd if $\delta = 1$ and even if $\delta = -1$, and $\varepsilon_j = \pm 1$, and $\lambda_j > 0$; or*

$$S_j = \mathcal{J}_{m_j}(0, \lambda_j), \quad H_j = \varepsilon_j\Sigma_{m_j} \otimes \Sigma_2, \tag{5.11}$$

*where $m_j \in \mathbb{N}$ is even if $\delta = 1$ and odd if $\delta = -1$, and $\varepsilon_j = \pm 1$, and $\lambda_j > 0$;*

*v) blocks associated with a quadruplet $\pm\alpha_j \pm i\beta_j$ of nonreal, non purely imaginary eigenvalues, where $m_j \in \mathbb{N}$ and $\alpha_j < 0 < \beta_j$:*

$$S_j = \begin{bmatrix} \mathcal{J}_{m_j}(\alpha_j, \beta_j) & 0 \\ 0 & -\mathcal{J}_{m_j}(\alpha_j, \beta_j)^T \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{2m_j} \\ \delta I_{2m_j} & 0 \end{bmatrix}. \tag{5.12}$$

*Moreover, the form (5.6) is unique up to the permutation of blocks.*

*Proof.* Since $S$ is $H$-skewadjoint, $S$ is polynomially $H$-normal with $H$-normality polynomial $p(t) = -t$. Without loss of generality, we may assume that $S$ and $H$ are essentially decomposed in the sense of Theorem 4.3. Observe that the real number $\lambda = 0$ is the only number satisfying $\lambda = p(\lambda)\,(= -\lambda)$. Consequently, $S$ has no complex bilinear part. Thus, we have

$$S = \widetilde{S}_1 \oplus \widetilde{S}_2, \quad H = \widetilde{H}_1 \oplus \widetilde{H}_2,$$

where $\sigma(\widetilde{S}_1) \subseteq \mathbb{R}$ and $\widetilde{S}_2 = \phi(\mathcal{S}_2)$, $\widetilde{H}_2 = \phi(\mathcal{H}_2)$, where $\mathcal{S}_2$ is $\mathcal{H}_2$-skewadjoint with respect to the sesquilinear form induced by the complex (skew-)Hermitian matrix $\mathcal{H}_2$. Here, we choose the eigenvalues of $\mathcal{S}_2$ such that $\sigma(\mathcal{S}_2) \subseteq \{\lambda \in \mathbb{C} \setminus \mathbb{R} \mid \mathrm{Im}(\lambda) > 0\}$. In view of Theorem 4.5, we may furthermore assume that the pairs $(\widetilde{S}_1, \widetilde{H}_1)$ and $(\mathcal{S}_2, \mathcal{H}_2)$ are in canonical form.

The canonical form of the pair $(\widetilde{S}_1, \widetilde{H}_1)$ can be read off Theorem 2.4. If $\delta = 1$ then only blocks of the forms (2.5)–(2.7) and of the form (2.3) for $n_j = 1$ occur. These give the blocks of the form (5.7)–(5.9) in (5.6). (Observe that the parameters $a_{j,2}, \ldots, a_{j,n_j-1}$ in (2.5) are zero. Indeed, this follows immediately from the identity $\widetilde{S}_1^T \widetilde{H}_1 = -\widetilde{H}_1 \widetilde{S}_1$.) On the other hand if $\delta = -1$, then only blocks of the forms (2.5)–(2.7) occur. This gives again blocks of the form (5.7)–(5.9) in (5.6). (Again, the parameters $a_{j,2}, \ldots, a_{j,n_j-1}$ in (2.5) are seen to be zero.)

The canonical form of the pair $(\mathcal{S}_2, \mathcal{H}_2)$ in the case $\delta = 1$ can be read off [15, Theorem 6.3]. (The canonical form is explicitly given for $H$-selfadjoints only, but contains implicitly the canonical form of $H$-skewadjoint matrices, because multiplying an $H$-selfadjoint matrix with the imaginary unit $i$ results in an $H$-skewadjoint matrix.) Having in mind that the spectrum of $\mathcal{S}_2$ is a subset of the open upper half plane, we see that this form consists of blocks of the form

$$\mathcal{S}_j = i\mathcal{J}_{m_j}(\lambda_j), \quad \mathcal{H}_j = \varepsilon_j R_{m_j}$$

where $\lambda_j > 0$ and $\varepsilon_j = \pm 1$, or

$$\mathcal{S}_j = \begin{bmatrix} i\mathcal{J}_{m_j}(\beta_j + i\alpha_j) & 0 \\ 0 & i\mathcal{J}_{m_j}(\beta_j + i\alpha_j)^* \end{bmatrix}, \quad \mathcal{H}_j = \begin{bmatrix} 0 & I_{m_j} \\ I_{m_j} & 0 \end{bmatrix},$$

where $\alpha_j, \beta_j > 0$. Applying a transformation with the matrix $E_{m_j}^{-1}$ or $(E_{m_j} \oplus E_{m_j})^{-1}$, respectively, we obtain the alternative representations

$$\mathcal{S}_j = \mathcal{J}_{m_j}(i\lambda_j), \quad \mathcal{H}_j = (-i)^{m_j-1}\varepsilon_j \Sigma_{m_j}, \quad \text{or} \tag{5.13}$$

$$\mathcal{S}_j = \begin{bmatrix} \mathcal{J}_{m_j}(-\alpha_j + i\beta_j) & 0 \\ 0 & -\mathcal{J}_{m_j}(-\alpha_j + i\beta_j)^* \end{bmatrix}, \quad \mathcal{H}_j = \begin{bmatrix} 0 & I_{m_j} \\ I_{m_j} & 0 \end{bmatrix}, \tag{5.14}$$

respectively. Observe that

$$\phi\big((-i)^{m_j-1}\Sigma_{m_j}\big) = \begin{cases} \pm\Sigma_{m_j} \otimes I_2 & \text{if } m_j \text{ is odd} \\ \pm\Sigma_{m_j} \otimes \Sigma_2 & \text{if } m_j \text{ is even} \end{cases}$$

Thus, we obtain from (5.13) blocks of the forms (5.10) and (5.11), respectively, (after eventually replacing $\varepsilon_j$ with $-\varepsilon_j$ and after possibly renaming some indices). Analogously, we obtain from (5.14) blocks of the forms (5.12) (after possibly renaming some indices). The case $\delta = -1$ is completely analogous and follows again from [15, Theorem 6.3]. The only difference is that each block $\mathcal{H}_j$ has to be multiplied with $-i$. After applying a transformation with the matrix $E_{m_j}^{-1}$ or $(iE_{m_j} \oplus E_{m_j})^{-1}$, respectively, we obtain

$$\mathcal{S}_j = \mathcal{J}_{m_j}(i\lambda_j), \quad \mathcal{H}_j = (-i)^{m_j}\varepsilon_j\Sigma_{m_j}, \quad \text{or} \tag{5.15}$$

$$\mathcal{S}_j = \begin{bmatrix} \mathcal{J}_{m_j}(-\alpha_j + i\beta_j) & 0 \\ 0 & -\mathcal{J}_{m_j}(-\alpha_j + i\beta_j)^* \end{bmatrix}, \quad \mathcal{H}_j = \begin{bmatrix} 0 & I_{m_j} \\ -I_{m_j} & 0 \end{bmatrix}, \tag{5.16}$$

This gives us blocks of the forms (5.11), (5.11), and (5.12), respectively (after replacing $\alpha_j$ with $-\alpha_j$, eventually replacing $\varepsilon_j$ with $-\varepsilon_j$, and after possibly renaming some indices). ∎

REMARK 5.4. Again, the result of Theorem 5.3 is not new, but can be found, e.g., in [13]. It should also be noted that the results of this and the previous subsection are related to canonical forms for pairs of real symmetric and skew-symmetric matrices under simultaneous congruence that have been obtained by Weierstraß and Kronecker, see [22] and the references therein.

In the following, let $T(a_0, a_1, \ldots, a_{n-1})$ denote the upper triangular Toeplitz matrix with first row $\begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1} \end{bmatrix}$, i.e.,

$$T(a_0, \ldots, a_{n-1}) = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ 0 & a_0 & \ddots & \vdots \\ 0 & 0 & \ddots & a_1 \\ 0 & 0 & 0 & a_0 \end{bmatrix}.$$

THEOREM 5.5 (Canonical forms for $H$-unitary matrices). *Let $H \in \mathbb{R}^{n\times n}$ and $\delta = \pm 1$ such that $H^T = \delta H$. Furthermore, let $U \in \mathbb{R}^{n\times n}$ be $H$-unitary. Then there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$P^{-1}UP = U_1 \oplus \cdots \oplus U_p, \quad P^T HP = H_1 \oplus \cdots \oplus H_p, \tag{5.17}$$

*where $U_j$ is $H_j$-indecomposable and where $U_j$ and $H_j$ have one of the following forms:*

*i) blocks associated with $\lambda_j = \eta = \pm 1$:*

$$U_j = T(\eta, 1, r_2, \ldots, r_{n_j-1}), \quad H_j = \varepsilon_j\Sigma_{n_j}, \tag{5.18}$$

*where $n_j \in \mathbb{N}$ is odd if $\delta = 1$ and even if $\delta = -1$. Moreover, $\varepsilon_j = \pm 1$, $r_k = 0$ for odd $k$ and the parameters $r_k$ for even $k$ are real and uniquely determined by the recursive formula*

$$r_2 = \frac{1}{2}\eta, \quad r_k = -\frac{1}{2}\eta\left(\sum_{\nu=1}^{\frac{k}{2}-1} r_{2\cdot\nu} r_{2\cdot(\frac{k}{2}-\nu)}\right), \quad 4 \le k \le n_j; \tag{5.19}$$

*ii) paired blocks associated with $\lambda_j = \pm 1$:*

$$U_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & \left(\mathcal{J}_{m_j}(\lambda_j)\right)^{-T} \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{m_j} \\ \delta I_{m_j} & 0 \end{bmatrix}, \tag{5.20}$$

*where $m_j \in \mathbb{N}$ is even if $\delta = 1$ and odd if $\delta = -1$.*

*iii) blocks associated with a pair* $(\lambda_j, \lambda_j^{-1}) \in \mathbb{R} \times \mathbb{R}$*, where* $\lambda_j > \lambda_j^{-1}$ *and* $m_j \in \mathbb{N}$*:*

$$(5.21)\quad U_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & \left(\mathcal{J}_{m_j}(\lambda_j)\right)^{-T} \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{m_j} \\ \delta I_{m_j} & 0 \end{bmatrix}.$$

*iv) blocks associated with a pair* $(\lambda_j, \overline{\lambda_j})$ *of nonreal unimodular eigenvalues:*

$$(5.22)\ U_j = \begin{bmatrix} \Lambda_j & \Theta_j & -r_2\Lambda_j & \dots & -r_{n-1}\Lambda_j \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -r_2\Lambda_j \\ \vdots & & \ddots & \ddots & \Theta_j \\ 0 & \cdots & \cdots & 0 & \Lambda_j \end{bmatrix}, H_j = \begin{cases} \varepsilon_j R_j \otimes I_2 & \text{if } \delta = 1 \\ \varepsilon_j R_j \otimes \Sigma_2 & \text{if } \delta = -1 \end{cases}$$

*where* $|\lambda_j| = 1$*,* $\mathrm{Im}(\lambda_j) > 0$*,* $m_j \in \mathbb{N}$*,* $\varepsilon_j = \pm 1$*, and*

$$(5.23)\quad \Lambda_j = \begin{bmatrix} \mathrm{Re}(\lambda_j) & \mathrm{Im}(\lambda_j) \\ -\mathrm{Im}(\lambda_j) & \mathrm{Re}(\lambda_j) \end{bmatrix}, \quad \Theta_j = \begin{bmatrix} \mathrm{Im}(\lambda_j) & -\mathrm{Re}(\lambda_j) \\ \mathrm{Re}(\lambda_j) & \mathrm{Im}(\lambda_j) \end{bmatrix}.$$

*Moreover,* $r_k = 0$ *for odd* $k$ *and the parameters* $r_k$ *for even* $k$ *are real and uniquely determined by the recursive formula*

$$(5.24)\qquad r_2 = \frac{1}{2}, \quad r_k = \frac{1}{2}\left(\sum_{\nu=1}^{\frac{k}{2}-1} r_{2\cdot\nu} r_{2\cdot(\frac{k}{2}-\nu)}\right), \quad 4 \le k \le m_j;$$

*v) blocks associated with a quadruplet* $(\alpha_j \pm i\beta_j, \frac{1}{\alpha_j \pm i\beta_j})$ *of nonreal, nonunimodular eigenvalues:*

$$(5.25)\quad U_j = \begin{bmatrix} \mathcal{J}_{m_j}(\alpha_j, \beta_j) & 0 \\ 0 & \mathcal{J}_{m_j}(\alpha_j, \beta_j)^{-T} \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & I_{2m_j} \\ \delta I_{2m_j} & 0 \end{bmatrix},$$

*where* $\alpha_j \in \mathbb{R}$*,* $\beta_j > 0$*,* $\alpha_j^2 + \beta_j^2 > 0$*, and* $m_j \in \mathbb{N}$*.*

*Moreover, the form (5.17) is unique up to the permutation of blocks.*

*Proof.* Since $U$ is $H$-unitary, we have that $U$ is polynomially $H$-normal with $H$-normality polynomial $p$ satisfying $p(U) = U^{-1}$. In particular, this implies $p(\lambda) = \lambda^{-1}$ for all eigenvalues $\lambda \in \mathbb{C}$ of $U$. Without loss of generality, we may assume that $U$ and $H$ are already essentially decomposed in the sense of Theorem 4.3. Observe that the real numbers $\lambda = 1$ and $\lambda = -1$ are the only numbers satisfying $p(\lambda) = \lambda$. Consequently, $U$ has no complex bilinear part. Thus, we have

$$U = \widetilde{U}_1 \oplus \widetilde{U}_2, \quad H = \widetilde{H}_1 \oplus \widetilde{H}_2,$$

where $\sigma(\widetilde{U}_1) \subseteq \mathbb{R}$ and $\widetilde{U}_2 = \phi(\mathcal{U}_2)$, $\widetilde{H}_2 = \phi(\mathcal{H}_2)$, where $\mathcal{U}_2$ is $\mathcal{H}_2$-skewadjoint with respect to the complex sesquilinear form induced by the complex (skew-)Hermitian matrix $\mathcal{H}_2$. Here, we choose the eigenvalues of $\mathcal{U}_2$ such that

$$\lambda \in \sigma(U_2) \text{ and } |\lambda| \ge 1 \;\Rightarrow\; \mathrm{Im}(\lambda) > 0.$$

Since the eigenvalues of $\mathcal{U}_2$ are either unimodular or occur in pairs $(\lambda, \overline{\lambda}^{-1})$, see, e.g., [15, Theorem 6.5], this implies in particular that $\mathrm{Im}(\lambda) < 0$ for all eigenvalues $\lambda$ of $\mathcal{U}_2$

with modulus smaller than one. In view of Theorem 4.5, we may furthermore assume that the pairs $(\widetilde{U}_1, \widetilde{H}_1)$ and $(\mathcal{U}_2, \mathcal{H}_2)$ are in canonical form.

The canonical form of the pair $(\widetilde{U}_1, \widetilde{H}_1)$ can be obtained from Theorem 2.4 giving the blocks of the forms (5.18)–(5.21). The detailed argument follows exactly the lines of the proof of Theorem 7.5 in [15] and will therefore not be repeated here. The canonical form of the pair $(\mathcal{U}_2, \mathcal{H}_2)$ in the case $\delta = \pm 1$ can be read off [15, Theorem 6.5]. (The canonical form is explicitly given for Hermitian $\mathcal{H}_2$ only, but the result for $\mathcal{H}_2$ skew-Hermitian can be obtained by considering the Hermitian matrix $i\mathcal{H}_2 = \sqrt{-1}\mathcal{H}_2$ instead.) Having in mind the assumption on the spectrum of $\mathcal{U}_2$, this form consists either of blocks of the forms

$$\mathcal{U}_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & \mathcal{J}_{m_j}(\lambda_j)^{-*} \end{bmatrix}, \quad \mathcal{H}_j = \begin{bmatrix} 0 & \sqrt{\delta} I_{m_j} \\ \sqrt{\delta} I_{m_j} & 0 \end{bmatrix}, \tag{5.26}$$

or of blocks of the forms

$$\mathcal{U}_j = \omega_j I_{m_j} + i\omega_j T(0, 1, ir_2, \ldots, ir_{m_j-1}), \quad \mathcal{H}_j = \varepsilon_j \sqrt{\delta} R_{m_j}, \tag{5.27}$$

where $|\omega_j| = 1$, $\mathrm{Im}(\omega_j) > 0$, $m_j \in \mathbb{N}$, and $\varepsilon_j = \pm 1$. Moreover, $r_k = 0$ for odd $k$ and the parameters $r_k$ for even $k$ are real and uniquely determined by the recursive formula

$$r_2 = \frac{1}{2}, \quad r_k = \frac{1}{2} \left( \sum_{\nu=1}^{\frac{k}{2}-1} r_{2\cdot\nu} r_{2\cdot(\frac{k}{2}-\nu)} \right), \quad 4 \le k \le m_j;$$

The blocks (5.27) results in blocks of the form (5.22). Concerning the blocks (5.26), we immediately obtain blocks of the form (5.25) in the case $\delta = 1$. For the case $\delta = -1$, observe that a transformation with the matrix $\mathcal{Q} := I_{m_j} \oplus (-iI_{m_j})$ yields

$$\mathcal{Q}^{-1}\mathcal{U}_j\mathcal{Q} = \mathcal{U}_j = \begin{bmatrix} \mathcal{J}_{m_j}(\lambda_j) & 0 \\ 0 & \mathcal{J}_{m_j}(\lambda_j)^{-*} \end{bmatrix}, \quad \mathcal{Q}^*\mathcal{H}_j\mathcal{Q} = \begin{bmatrix} 0 & I_{m_j} \\ -I_{m_j} & 0 \end{bmatrix},$$

which results again in blocks of the form (5.25). □

REMARK 5.6. A result in the direction of Theorem 5.5 for the case $\delta = -1$ has been obtained in [21] concerning existence of a decomposition of $H$-unitary matrices into indecomposable blocks. Also, the possible Jordan canonical forms of the indecomposable blocks have been fully described. The part in what Theorem 5.5 differs from the result in [21] is that not only the $H$-unitary matrix $U$, but also $H$ has been reduced to a canonical form and that the form of Theorem 5.5 is unique up to permutation of blocks. Canonical forms for $H$-unitary matrices for the case $\delta = 1$ have been developed in [20] and for some special cases in [2].

REMARK 5.7. We note that it follows from the proofs of the Theorems 5.1–5.5 that $H$-selfadjoint matrices have no complex sesquilinear part and that $H$-skew-adjoint and $H$-unitary matrices have no complex bilinear part.

**6. Semidefinite invariant subspaces.** In this section, we apply the essential decomposition to prove existence of semidefinite invariant subspaces for polynomially $H$-normal matrices. Let $\mathcal{H} \in \mathbb{F}^{n\times n}$ be Hermitian. Then a subspace $\mathcal{S} \subseteq \mathbb{F}^n$ is called $\mathcal{H}$-*nonnegative* if $[x, x] \ge 0$ for all $x \in \mathcal{S}$. An $\mathcal{H}$-nonnegative subspace $\mathcal{M} \subseteq \mathbb{F}^n$ is called *maximal $\mathcal{H}$-nonnegative* if $\mathcal{H}$ is not contained in a larger $\mathcal{H}$-nonnegative subspace. It is well known and easy to verify that an $\mathcal{H}$-nonnegative subspace $\mathcal{M} \subseteq \mathbb{F}^n$ is maximal $\mathcal{H}$-nonnegative if and only if $\dim \mathcal{M} = \nu_+(\mathcal{H})$, where $\nu_+(\mathcal{H})$ is the number of positive eigenvalues of $\mathcal{H}$, see, e.g., [16]. It is also well known that if $\mathcal{X} \in \mathbb{C}^{n\times n}$ is $\mathcal{H}$-normal, then $\mathcal{X}$ has an invariant subspace $\mathcal{M}$ that is also maximal $\mathcal{H}$-nonnegative, see [3, Corollary 3.4.12] for a more general result in Krein spaces or [16] for a proof depending on finite dimensionality. However, the corresponding statement for the case $\mathbb{F} = \mathbb{R}$ is not true as the following example shows.

Example 6.1. Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$, and consider the matrices

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $A$ is $H$-selfadjoint and $H$ has one positive eigenvalue. However, $A$ has no real nontrivial invariant subspaces and thus, no invariant subspace that is also $H$-nonnegative.

In the following, we give a sufficient condition for a polynomially $H$-normal matrix to have an invariant subspace $\mathcal{M} \subseteq \mathbb{R}^n$ that is also maximal $H$-nonnegative.

Theorem 6.2. *Let $H \in \mathbb{R}^{n\times n}$ be symmetric and let $X \in \mathbb{R}^{n\times n}$ be polynomially $H$-normal such that $X$ has no complex bilinear part in its essential decomposition. Then $X$ has an invariant subspace $\mathcal{M} \subseteq \mathbb{R}^n$ that is also maximal $H$-nonnegative.*

*Proof.* Without loss of generality, we may assume that $X$ and $H$ are in the form (4.1). Thus, since $X$ has no complex bilinear part, we find that

$$X = \widetilde{X}_1 \oplus \widetilde{X}_2, \quad H = \widetilde{H}_1 \oplus \widetilde{H}_2,$$

where $\sigma(\widetilde{X}_1) \subseteq \mathbb{R}$ and $\widetilde{X}_2 = \phi(\mathcal{X}_2)$, $\widetilde{H}_2 = \phi(\mathcal{H}_2)$, where $\mathcal{X}_2$ is polynomially normal with respect to the sesquilinear form induced by $\mathcal{H}_2$. In view of Theorem 4.5 we may furthermore assume that $\widetilde{X}_1$ and $\widetilde{H}_1$ are in the canonical form (2.2). It is sufficient to consider the case that $X$ equals either $\widetilde{X}_2$ or one of the indecomposable blocks in (2.2), because if each such block has an invariant subspace that is maximal $H$-nonnegative then an invariant maximal $H$-nonnegative subspace for $X$ can be obtained as the direct sum of all those subspaces.

If $X$ and $H$ are in the form (2.3) or (2.5), then we choose

$$\mathcal{M} = \begin{cases} \mathrm{Span}(e_1, \ldots, e_{\frac{n}{2}}) & \text{if } n \text{ is even} \\ \mathrm{Span}(e_1, \ldots, e_{\frac{n+1}{2}}) & \text{if } n \text{ is odd and } \varepsilon = 1 \\ \mathrm{Span}(e_1, \ldots, e_{\frac{n-1}{2}}) & \text{if } n \text{ is odd and } \varepsilon = -1 \end{cases}.$$

Indeed, it is easily seen that $\mathcal{M}$ is $X$-invariant and maximal $H$-nonnegative. If $X$ and $H$ are in the form (2.6) or (2.7), then $\mathcal{M} = \mathrm{Span}(e_1, \ldots, e_{\frac{n}{2}})$ is the desired invariant subspace that is also maximal $H$-nonnegative.

Next consider the case that $X = \widetilde{X}_2 = \phi(\mathcal{X}_2)$ and $H = \widetilde{H}_2 = \phi(\mathcal{H}_2)$. Since $\mathcal{X}_2$ is polynomially $\mathcal{H}_2$-normal (and thus, $\mathcal{X}_2$ is in particular $\mathcal{H}_2$-normal), $\mathcal{X}_2$ has an invariant subspace $\mathcal{M}_{\mathbb{C}}$ that is maximal $\mathcal{H}_2$-nonnegative. Let $k$ be the number of positive eigenvalues of $\mathcal{H}_2$ and let $\mathcal{M}_{\mathbb{C}}$ be spanned by the columns of the matrix $\mathcal{T} \in \mathbb{C}^{n\times k}$. Then there exists a $k\times k$ matrix $\mathcal{B} \in \mathbb{C}^{k\times k}$ such that $\mathcal{X}_2\mathcal{T} = \mathcal{T}\mathcal{B}$. Applying $\phi$ to this identity, we obtain

$$\widetilde{X}_2\, \phi(\mathcal{T}) = \phi(\mathcal{X}_2\mathcal{T}) = \phi(\mathcal{T}\mathcal{B}) = \phi(\mathcal{T})\phi(\mathcal{B}),$$

that is, $\mathcal{M} := \mathrm{Range}\, \phi(\mathcal{T})$ is an invariant subspace for $\widetilde{X}_2$. The dimension of $\mathcal{M}$ is $2k$, because $\phi$ doubles the rank of matrices. (This follows easily by considering the singular value decomposition and using that $\phi$ is an isomorphism.) Finally, $\mathcal{M}$ is $\widetilde{H}_2$-nonnegative (and, consequently, maximal $\widetilde{H}_2$-nonnegative, because $\widetilde{H}_2 = \phi(\mathcal{H}_2)$ necessarily has $2k$ positive eigenvalues which follows by diagonalizing $\mathcal{H}_2$ and then using that $\phi$ is an isomorphism), because

$$\phi(\mathcal{T})^T \widetilde{H}_2 \, \phi(\mathcal{T}) = \phi(\mathcal{T}^* \mathcal{H}_2 \mathcal{T})$$

is positive semidefinite if and only if $\mathcal{T}^*\mathcal{H}_2\mathcal{T}$ is. (Again, diagonalize $\mathcal{T}^*\mathcal{H}_2\mathcal{T}$ and use that $\phi$ is an isomorphism.) Thus, the $\mathcal{H}_2$-nonnegativity of $\operatorname{Range}\mathcal{T}$ implies the $\widetilde{H}_2$-nonnegativity of $\operatorname{Range}\phi(\mathcal{T})$. $\square$

In view of Remark 5.7, we immediately obtain the following corollary.

COROLLARY 6.3. *Let $X \in \mathbb{R}^{n\times n}$ be $H$-skew-adjoint or $H$-unitary. Then $X$ has an invariant subspace $\mathcal{M} \subseteq \mathbb{R}^n$ that is also maximal $H$-nonnegative.*

REMARK 6.4. The hypothesis of $X$ having no complex bilinear part is essential in Theorem 6.2. Indeed, this fact is illustrated by Example 6.1. Thus, in contrast to $H$-skewadjoint and $H$-unitary matrices, $H$-selfadjoint matrices need not have an invariant maximal $H$-nonnegative subspace, because $H$-selfadjoint matrices may have a non-vanishing complex bilinear part.

**7. Conclusions.** We have introduced the essential decomposition of polynomially $H$-normal matrices. With the help of this decomposition, real polynomially $H$-normal matrices can be described by a polynomially $H$-normal matrix having real spectrum only and two complex matrices that are polynomially normal with respect to a sesquilinear form and bilinear form, respectively. The main motivation for the development of the essential decomposition is the construction of canonical forms for real polynomially $H$-normal matrices, but it is expected to be useful for the investigation of polynomially $H$-normal matrices in general, in particular, for the investigation of $H$-selfadjoint, $H$-skewadjoint, and $H$-unitary matrices, because the real part, the complex sesquilinear part, and the complex bilinear part of a polynomially $H$-normal matrix can be considered separately.

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