Extension to maximal semidefinite invariant subspaces for hyponormal matrices in indefinite inner products

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Abstract

It is proved that under certain essential additional hypotheses, a nonpositive invariant subspace of a hyponormal matrix admits an extension to a maximal nonpositive subspace which is invariant for both the matrix and its adjoint. Nonpositivity of subspaces and the hyponormal property of the matrix are understood in the sense of a nondegenerate inner product in a finite dimensional complex vector space. The obtained theorem combines and extends several previously known results. A Pontryagin space formulation, with essentially the same proof, is offered as well.

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1 Introduction

On the vector space $\mathbb{C}^n$, equipped with the standard inner product, we fix an indefinite inner product $[\cdot,\cdot]$ determined by an invertible Hermitian $n \times n$ matrix H via the formula

$$[x, y] = \langle Hx, y\rangle, \qquad x, y \in \mathbb{C}^n.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the standard inner product.

A subspace $\mathcal{M} \subseteq \mathbb{C}^n$ is said to be *H-nonnegative* if $[x, x] \geq 0$ for every $x \in \mathcal{M}$, *H-positive* if $[x, x] > 0$ for every nonzero $x \in \mathcal{M}$, *H-nonpositive* if $[x, x] \leq 0$ for every $x \in \mathcal{M}$, *H-negative* if $[x, x] < 0$ for every nonzero $x \in \mathcal{M}$, and *H-neutral* if $[x, x] = 0$ for every

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$x \in \mathcal{M}$. Note that by default the zero subspace is H-positive as well as H-negative. An H-nonnegative subspace is said to be *maximal H-nonnegative* if it is not properly contained in any larger H-nonnegative subspace. It is easy to see that an H-nonnegative subspace is maximal if and only if its dimension is equal to the number $i_+(H)$ of positive eigenvalues of H (counted with multiplicities). Analogously, an H-nonpositive subspace is maximal if and only its dimension is equal to the number of negative eigenvalues of H.

Let $X^{[*]}$ denote the adjoint of a matrix $X \in \mathbb{C}^{n\times n}$ with respect to the indefinite inner product, i.e., $X^{[*]}$ is the unique matrix satisfying $[x, Xy] = [X^{[*]}x, y]$ for all $x, y \in \mathbb{C}^n$. One easily sees that $X^{[*]} = H^{-1}X^*H$. We recall that a matrix $X \in \mathbb{C}^{n\times n}$ is called *H-normal* if $X^{[*]}X = XX^{[*]}$, and *H-hyponormal* if $H(X^{[*]}X - XX^{[*]}) \geq 0$ (positive semidefinite). We note that it is easy to check that if X is H-normal, resp., H-hyponormal, then $P^{-1}XP$ is P^*HP-normal, resp., P^*HP-hyponormal, provided that $P \in \mathbb{C}^{n\times n}$ is nonsingular.

It is well known that several classes of matrices in indefinite inner product spaces allow extensions of invariant H-nonnegative subspaces to invariant maximal H-nonnegative subspaces. Those classes are for example the ones of H-expansive matrices (including H-unitary matrices), H-dissipative matrices (including H-selfadjoints), and H-skew-adjoint matrices, see, e.g. [5] for a proof. The natural question arises if this extension problem still has a solution for arbitrary H-normal matrices. A partial answer to this question is contained in the following result.

Theorem 1 *Let $X \in \mathbb{C}^{n\times n}$ be H-normal, and let $\mathcal{M}_0$ be an H-neutral X-invariant subspace. Then there exists an X-invariant subspace $\mathcal{M}$ which is also maximal H-nonnegative, i.e., H-nonnegative of dimension $i_+(H)$, and such that $\mathcal{M}_0 \subseteq \mathcal{M}$. Also, there exists an X-invariant maximal H-nonpositive subspace containing $\mathcal{M}_0$.*

Theorem 1 can be obtained from results of [2], [3], and it holds also for Pontryagin spaces; see [6] for details. A more general theorem is proved in [5]. The proof of Theorem 1 given in [5] depends essentially on the H-neutrality of the given invariant subspace $\mathcal{M}_0$.

Moreover, it was proven in [6] that if $\mathcal{M}$ is a maximal H-nonnegative subspace invariant under an H-normal X, then it is also invariant under $X^{[*]}$. Also, the authors proved an extension result in the framework of H-hyponormal matrices. For sake of convenience, we recall the two main results from that paper.

Theorem 2 *Let $X \in \mathbb{C}^{n\times n}$ be H-hyponormal. If the spectrum of $X + X^{[*]}$ is real or if the spectrum of $X - X^{[*]}$ is purely imaginary (including zero), then there exists an X-invariant maximal H-nonnegative subspace that is also invariant for $X^{[*]}$. Also, there exists an X-invariant maximal H-nonpositive subspace that is also invariant for $X^{[*]}$.*

The assumption that either the spectrum of $X+X^{[*]}$ is real or the spectrum of $X-X^{[*]}$ is purely imaginary in Theorem 2 was shown in [6] to be essential even for the case of H-normal matrices.

For a subspace $\mathcal{M}_0 \subseteq \mathbb{C}^n$, we denote by

$$\mathcal{M}_0^{[\perp]} = \{x \in \mathbb{C}^n \mid [x, y] = 0 \text{ for every } y \in \mathcal{M}_0\}$$

the H-orthogonal companion of $\mathcal{M}_0$.

Theorem 3 *Let $X \in \mathbb{C}^{n\times n}$ be H-hyponormal and let $\mathcal{M}_0$ be an X-invariant H-negative subspace. Define $X_{22} = X^{[*]}|_{\mathcal{M}_0^{[\perp]}} : \mathcal{M}_0^{[\perp]} \to \mathcal{M}_0^{[\perp]}$. Equip $\mathcal{M}_0^{[\perp]}$ with the indefinite inner product induced by H. Assume that at least one of the two inclusions $\sigma(X_{22}^{[*]}+X_{22}) \subset \mathbb{R}$ and $\sigma(X_{22}^{[*]} - X_{22}) \subset i\mathbb{R}$ holds true. Then there exists an X-invariant maximal H-nonpositive subspace that contains $\mathcal{M}_0$.*

The aim of this note is to unify and complete the theory of extensions of semidefinite subspaces for H-normal and H-hyponormal subspaces. In particular, we prove a generalization of Theorem 3, where we start with an H-nonpositive X-invariant subspace $\mathcal{M}_0$ instead of an H-negative one. The extension result is then not true without further conditions, as it was already shown in [6].

2 Extension of nonpositive invariant subspaces.

We start by generalizing the fact that, for H-normal matrices X, invariant maximal H-semidefinite subspaces are also invariant under the adjoint $X^{[*]}$. Indeed, it turns out that this results holds true even for H-hyponormal matrices if the subspace under consideration is assumed to be H-nonpositive.

Proposition 4 *Let $X \in \mathbb{C}^{n\times n}$ be H-hyponormal and let $\mathcal{M}$ be an X-invariant maximal H-nonpositive subspace. Then $\mathcal{M}$ is invariant also for $X^{[*]}$.*

Proof. The proof is essentially the same as the corresponding proof for the case that X is H-normal (see [6]). Nevertheless we provide the proof here to keep the paper self-contained. Applying otherwise a suitable transformation $X \mapsto P^{-1}XP$, $H \mapsto P^*HP$, where P is invertible, we may assume that $\mathcal{M}$ is spanned by the first (say) m unit vectors and that X and H have the forms

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ 0 & 0 & X_{33} & X_{34} \\ 0 & 0 & X_{43} & X_{44} \end{bmatrix}, \quad H = \begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{1}$$

Indeed, this follows easily by decomposing $\mathcal{M} = \mathcal{M}_p \oplus \mathcal{M}_0$ into an H-neutral subspace $\mathcal{M}_0$ and its orthogonal complement $\mathcal{M}_p$ (in $\mathcal{M}$), and choosing an H-neutral subspace $\mathcal{M}_{sl}$ that is skewly linked to $\mathcal{M}_0$ (see [1], [4] for the definition and properties of skewly linked subspaces). Note that the H-orthogonal complement to $\mathcal{M}\dot{+}\mathcal{M}_{sl}$ is necessarily an H-positive subspace due to the maximality of $\mathcal{M}$. Then, selecting appropriate bases in all subspaces constructed above, and putting the bases as the consecutive columns of a matrix P, we get a transformation that yields the desired result. From (1), we then obtain that

$$X^{[*]} = \begin{bmatrix} X_{11}^* & 0 & -X_{21}^* & 0 \\ -X_{13}^* & X_{33}^* & X_{23}^* & X_{43}^* \\ -X_{12}^* & 0 & X_{22}^* & 0 \\ -X_{14}^* & X_{34}^* & X_{24}^* & X_{44}^* \end{bmatrix} \tag{2}$$

and

$$
\begin{aligned}
&H(X^{[*]}X - XX^{[*]}) \\
&= \begin{bmatrix} * & * & * & * \\ * & -X_{12}^*X_{12} - X_{34}X_{34}^* & * & * \\ * & * & * & * \\ * & * & * & X_{44}^*X_{44} - X_{14}^*X_{14} + X_{24}^*X_{34} + X_{34}^*X_{24} - X_{44}X_{44}^* \end{bmatrix} \quad (3)
\end{aligned}
$$

Since X is H-hyponormal, i.e., $H(X^{[*]}X - XX^{[*]}) \geq 0$, we obtain from the block $(2,2)$-entry in (3) that $X_{12} = 0$ and $X_{34} = 0$. But then the inequality for the block $(4,4)$-entry of (3) becomes

$$
X_{44}^*X_{44} - X_{44}X_{44}^* \geq X_{14}^*X_{14} \geq 0, \tag{4}
$$

which is easily seen to imply (by taking traces of both sides in (4)) that X_{44} is normal and that $X_{14} = 0$. Thus, we obtain from (2) that $\mathcal{M}$ is also invariant for $X^{[*]}$. □

The following example illustrates Proposition 4 and shows that we cannot replace H-nonpositivity in the hypothesis of the proposition by H-nonnegativity.

Example 5 Let

$$
X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{5}
$$

Then one easily computes

$$
X^{[*]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad H(X^{[*]}X - XX^{[*]}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X + X^{[*]} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix}
$$

that is, X is H-hyponormal and the spectrum of $\sigma(X + X^{[*]}) = \{2\}$ is real. Then the only X-invariant subspace that is maximal H-nonpositive is given by $\mathcal{M}_- = \mathrm{Span}(e_2, e_3)$. Obviously, $\mathcal{M}_-$ is also invariant under $X^{[*]}$. On the other hand, $\mathcal{M}_+ = \mathrm{Span}(e_1)$ is a maximal H-nonnegative subspace that is invariant under X, but $\mathcal{M}_+$ is not invariant under $X^{[*]}$. However, Theorem 2 implies that X has a maximal H-nonnegative subspace that is also invariant under $X^{[*]}$. Such a subspace is given by $\widetilde{\mathcal{M}}_+ = \mathrm{Span}(e_2)$. □

The main results of this note is the following. It combines elements of Theorems 1, 2, and 3.

Theorem 6 *Let X be H-hyponormal, and let $\mathcal{M}$ be an H-nonpositive subspace that is invariant under X. Let $\mathcal{M}_0$ be the isotropic part of $\mathcal{M}$ and decompose $\mathcal{M}^{[\perp]}$ as*

$$
\mathcal{M}^{[\perp]} = \mathcal{M}_0 \dot{+} \mathcal{M}_{nd}, \tag{6}
$$

for an H-nondegenerate subspace $\mathcal{M}_{nd}$. Denote by X_{44} and H_4 the compressions of X and H to $\mathcal{M}_{nd}$, respectively. Assume that $\mathcal{M}_0$ is invariant under $X^{[*]}$ and that, in addition, one of the three following conditions holds:

(a) $\sigma(X_{44}+X_{44}^{[*]})\subset\mathbb{R}$,

(b) $\sigma(X_{44}-X_{44}^{[*]})\subset i\mathbb{R}$,

(c) X_{44} *is* H_4*-normal.*

Then $\mathcal{M}$ *can be extended to a maximal* H*-nonpositive subspace* $\mathcal{M}_-$ *that is invariant under both* X *and* $X^{[*]}$.

The conditions (a)–(c) *are independent of the particular choice of a nondegenerate subspace* $\mathcal{M}_{nd}$ *subject to* (6).

Proof. A decomposition similar to (1) will be used. Since $\mathcal{M}_0$ is the isotropic part of $\mathcal{M}$ we have that $\mathcal{M}_0=\mathcal{M}\cap\mathcal{M}^{[\perp]}$. Let $\mathcal{M}_{sl}$ be a subspace skewly linked to $\mathcal{M}_0$, let $\mathcal{M}_2$ be a nondegenerate subspace of $\mathcal{M}$ which is H-orthogonal to both $\mathcal{M}_0$ and $\mathcal{M}_{sl}$, and finally, let $\mathcal{M}_4$ be the H-orthogonal complement of $\mathcal{M}_0\dot{+}\mathcal{M}_2\dot{+}\mathcal{M}_{sl}$. Observe that $\mathcal{M}_2$ is an H-negative subspace in $\mathcal{M}$ while $\mathcal{M}_4$ is a nondegenerate subspace in $\mathcal{M}^{[\perp]}$. With respect to the decomposition

$$\mathbb{C}^n=\left(\mathcal{M}_0\dot{+}\mathcal{M}_2\dot{+}\mathcal{M}_{sl}\right)[\dot{+}]\mathcal{M}_4, \tag{7}$$

where $[\dot{+}]$ stands for an H-orthogonal sum, and with respect to an appropriate choice of basis in each of the components we write

$$H=\begin{bmatrix}0&0&I&0\\0&-I&0&0\\I&0&0&0\\0&0&0&H_4\end{bmatrix},\qquad X=\begin{bmatrix}X_{11}&X_{12}&X_{13}&X_{14}\\X_{21}&X_{22}&X_{23}&X_{24}\\0&0&X_{33}&X_{34}\\0&0&X_{43}&X_{44}\end{bmatrix}.$$

Using this we easily see that $X^{[*]}$ is given by

$$X^{[*]}=\begin{bmatrix}X_{33}^*&-X_{23}^*&X_{13}^*&X_{43}^*H_4\\0&X_{22}^*&-X_{12}^*&0\\0&-X_{21}^*&X_{11}^*&0\\H_4^{-1}X_{34}^*&-H_4^{-1}X_{24}^*&H_4^{-1}X_{14}^*&H_4^{-1}X_{44}^*H_4\end{bmatrix}$$

Partitioning $Y:=H(X^{[*]}X-XX^{[*]})$ conformably with respect to the decomposition (7), we obtain that the $(4,4)$-block Y_{44} takes the form

$$Y_{44}=X_{34}^*X_{14}-X_{24}^*X_{24}+X_{14}^*X_{34}+H_4(X_{44}^{[*]}X_{44}-X_{44}X_{44}^{[*]}), \tag{8}$$

where $X_{44}^{[*]}$ denotes the H_{44}-adjoint $H_{44}^{-1}X_{44}^*H_{44}$ of X_{44}. By assumption, the isotropic part $\mathcal{M}_0$ of $\mathcal{M}$ is invariant under $X^{[*]}$ which implies $X_{34}=0$. But then, we obtain that X_{44} is H_4-hyponormal, because we get from (8) that

$$H_4(X_{44}^{[*]}X_{44}-X_{44}X_{44}^{[*]})=Y_{44}+X_{24}^*X_{24}\geq Y_{44}\geq 0,$$

since X is H-hyponormal and, therefore, Y and Y_{44} are positive semidefinite.

Next, we show that the conditions (a)– (c) are independent of the particular choice of a nondegenerate subspace $\mathcal{M}_{nd}$ subject to (6), i.e., we may assume without loss of generality that $\mathcal{M}_{nd} = \mathcal{M}_4$. Indeed, choosing another nondegenerate subspace $\mathcal{M}_{nd}$ in $\mathcal{M}^{[\perp]}$ in place of $\mathcal{M}_4$ amounts to a change of basis in $\mathcal{M}^{[\perp]}$ given by a matrix of the form

$$S = \begin{bmatrix} I & 0 & 0 & S_{14} \\ 0 & I & 0 & S_{24} \\ 0 & 0 & I & S_{34} \\ 0 & 0 & 0 & S_{44} \end{bmatrix},$$

with S_{44} invertible. Thus, we obtain that with respect to the new decomposition

$$\mathbb{C}^n = \left(\mathcal{M}_0 \dot{+} \mathcal{M}_2 \dot{+} \mathcal{M}_{sl}\right) \dot{+} \mathcal{M}_{nd},$$

and the new basis, X and H take the forms

$$\widetilde{X} = S^{-1}XS = \begin{bmatrix} X_{11} & X_{12} & * & * \\ X_{21} & X_{22} & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & S_{44}^{-1}X_{44}S_{44} + S_{44}^{-1}X_{43}S_{34} \end{bmatrix}$$

$$\widetilde{H} = S^{*}HS = \begin{bmatrix} 0 & 0 & I & S_{34} \\ 0 & -I & 0 & -S_{24} \\ I & 0 & 0 & S_{14} \\ S_{34}^{*} & -S_{24}^{*} & S_{14}^{*} & S_{44}H_4S_{44} + S_{34}^{*}S_{14} + S_{14}^{*}S_{34} - S_{24}^{*}S_{24} \end{bmatrix}$$

Since $\mathcal{M}_{nd}$ is assumed to be a subspace in $\mathcal{M}^{[\perp]}$, we must have

$$0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} (S^{*})^{-1}(S^{*}HS) \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} H \begin{bmatrix} S_{14} \\ S_{24} \\ S_{34} \\ S_{44} \end{bmatrix} = \begin{bmatrix} S_{34} \\ -S_{24} \end{bmatrix}$$

which implies $S_{24} = 0$ and $S_{34} = 0$. Thus, the compressions $\widetilde{X}_{44}$ and $\widetilde{H}_{44}$ of $\widetilde{X}$ resp. $\widetilde{H}$ to $\mathcal{M}_{nd}$ are

$$\widetilde{X}_{44} = S_{44}^{-1}X_{44}S_{44}, \quad \widetilde{H}_4 = S_{44}^{*}H_4S_{44}.$$

Clearly it follows from this that if each of the three conditions (a)– (c) holds for $\widetilde{X}_{44}$ and $\widetilde{H}_4$, then it holds also for X_{44} and H_4. In particular, the conditions (a)– (c) are independent of the choice of $\mathcal{M}_{nd}$.

Consequently, assuming $\mathcal{M}_{nd} = \mathcal{M}_4$ and that we have either $\sigma(X_{44} + X_{44}^{[*]}) \subset \mathbb{R}$ or $\sigma(X_{44} - X_{44}^{[*]}) \subset i\mathbb{R}$ or that X_{44} is H_4-normal, we obtain from Theorems 1 and 2 and Proposition 4 that there exists an X_{44}-invariant maximal H_4-nonpositive subspace $\mathcal{N}_4$ that is also invariant under $X_{44}^{[*]}$. In that case $\mathcal{M}_{-} := \mathcal{M} \dot{+} \mathcal{N}_4$ is maximal H-nonpositive, X-invariant, and thus, by Proposition 4 also $X^{[*]}$-invariant. □

The following example, adapted from [6], shows that the conditions (a)–(c) are essential in Theorem 6.

Example 7 Let

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}.$$

Then one easily calculates

$$X^{[*]} = \begin{bmatrix} i & i \\ -i & -i \end{bmatrix}, \quad A := \frac{1}{2}(X + X^{[*]}) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad S := \frac{1}{2}(X - X^{[*]}) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

and $H(X^{[*]}X - XX^{[*]}) = 4 \cdot I$. Hence X is H-hyponormal but not normal. Moreover, the spectrum of A is not real, and neither is the spectrum of S purely imaginary. Clearly, the zero space $\{0\}$ is H-neutral, invariant both under X and $X^{[*]}$, and coincides with its isotropic subspace. Now the only nontrivial invariant subspace for X is

$$\mathcal{M}_+ = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

which is easily seen to be maximal H-nonnegative, but it is not invariant under $X^{[*]}$, because otherwise it would also be invariant for A and S which is obviously not the case. Thus, $\{0\}$ cannot be extended neither to a maximal H-nonnegative nor to a maximal H-nonpositive subspace that is invariant for both X and $X^{[*]}$.

On the other hand, Example 5 shows that also the hypothesis in Theorem 6 that the isotropic subspace $\mathcal{M}_0$ of $\mathcal{M}$ is $X^{[*]}$-invariant is essential. Thus, the question arises under which conditions the isotropic subspace $\mathcal{M}_0$ of an X-invariant H-nonpositive subspace $\mathcal{M}$ (where X is an H-hyponormal matrix) is $X^{[*]}$-invariant. One immediate answer is given in the following remark that can be verified in a straightforward manner.

Remark 8 If X is H-hyponormal and $\mathcal{M}$ is a maximal H-nonpositive subspace that is invariant under both X and $X^{[*]}$, then its isotropic part $\mathcal{M}_0 = \mathcal{M} \cap \mathcal{M}^{[\perp]}$ is also invariant under both X and $X^{[*]}$.

Remark 9 Theorem 6 contains Theorem 3 as a special case, because clearly, the isotropic part of an H-negative subspace is the zero space which is always invariant under $X^{[*]}$.

We conclude the note with an observation that Proposition 4 and Theorem 6 are valid also for Pontryagin space operators, where H is an invertible selfadjoint operator on a Hilbert space with only finite dimensional invariant subspace corresponding to the positive part of the spectrum of H. In the case of Theorem 6 an additional hypothesis that the codimension of $\mathcal{M}$ is finite has to be imposed; this hypothesis would guarantee that $\mathcal{M}_{nd}$ is finite dimensional. The proofs remain essentially the same.

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