Polar decompositions of normal operators in indefinite inner product spaces

Christian Mehl* André C. M. Ran† Leiba Rodman‡

Abstract

Polar decompositions of normal matrices in indefinite inner product spaces are studied. The main result of this paper provides sufficient conditions for a normal operator in a Krein space to admit a polar decomposition. As an application of this result, we show that any normal matrix in a finite dimensional indefinite inner product space admits a polar decomposition which answers affirmatively an open question formulated in [2]. Furthermore, necessary and sufficient conditions are given for normal matrices to admit a polar decomposition or to admit a polar decomposition with commuting factors.

1 Introduction

Let $\mathcal{H}$ be a (complex) Hilbert space, and let H be a (bounded) selfadjoint operator on $\mathcal{H}$, which is boundedly invertible. The operator H defines a Krein space structure on $\mathcal{H}$, via the indefinite inner product

$$[x, y] = \langle Hx, y \rangle, \qquad x, y \in \mathcal{H},$$

where $\langle \cdot, \cdot \rangle$ is the Hilbert inner product in $\mathcal{H}$. All operators in the paper are assumed to be linear and bounded. We denote by $\mathcal{L}(\mathcal{H})$ the Banach algebra of bounded linear operators on $\mathcal{H}$. The adjoint of an operator $X \in \mathcal{L}(\mathcal{H})$ with respect to $\langle \cdot, \cdot \rangle$ will be denoted by X^*.

*Fakultät II; Institut für Mathematik, Technische Universität Berlin, D-10623 Berlin, Germany. *A large part of this work was performed while this author was visiting the Vrije Universiteit Amsterdam and while he was supported by the Research Training Network HPRN-CT-2000-00116 of the European Union, "Classical Analysis, Operator Theory, Geometry of Banach Spaces, their interplay and their applications"*

†Afdeling Wiskunde, Faculteit der Exacte Wetenschappen, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands

‡College of William and Mary, Department of Mathematics, P.O.Box 8795, Williamsburg, VA 23187-8795. *The research of this author was supported in part by NSF grant DMS-9988579 and by a Faculty Research Assignment grant from the College of William and Mary.*

An operator $X \in \mathcal{L}(\mathcal{H})$ is said to be an H-*isometry* if $[Xx, Xy] = [x, y]$ for all $x, y \in \mathcal{H}$, and is called H-*selfadjoint* if $[Xx, y] = [x, Xy]$ for all $x, y \in \mathcal{H}$. An operator $X \in \mathcal{L}(\mathcal{H})$ is called H-*normal* if

$$XX^{[*]} = X^{[*]}X,$$

where $X^{[*]}$ is the adjoint of X with respect to the indefinite inner product $[\cdot, \cdot]$.

Given a (linear bounded) operator X on $\mathcal{H}$, a decomposition of the form

$$X = UA,$$

where U is an invertible H-isometry (in other words, U is H-*unitary*) and A is H-selfadjoint, is called an H-*polar decomposition* of X. An analogous decomposition of the form $X = AU$ will be called a *right* H-polar decomposition for X.

In the context of positive definite inner products, polar decompositions (which are usually taken with the additional requirement that A be positive semidefinite and the relaxation that U need be a partial isometry only instead of an invertible one) are a basic tool of operator theory. In context of indefinite inner products, they have been studied extensively in recent years (see, e.g., [4, 2, 3, 16, 13]), in particular, in connection with matrix computations [7, 8].

Remark 1 An operator $X \in \mathcal{L}(\mathcal{H})$ admits an H-polar decomposition if and only if it admits a right H-polar decomposition. This follows easily from the fact that $X = UA = (UAU^{-1})U$.

Our main result, Theorem 4, is stated and proved in the next section. In particular, it follows from Theorem 4 that for a finite dimensional $\mathcal{H}$ every H-normal operator admits an H-polar decomposition, thereby settling in the affirmative an open question formulated in [2]. In Sections 3 and 4 we apply the main result to other properties that H-normal operators may have in connection with H-polar decompositions, assuming that $\mathcal{H}$ is finite dimensional. In particular, we provide necessary and sufficient conditions for normal matrices to admit a polar decomposition or to admit a polar decomposition with commuting factors.

2 The main result

In this section, we will provide sufficient conditions for an H-normal operator to admit an H-polar decomposition. The proof of the main result will be based on the following decomposition that is of interest in itself.

Lemma 2 *Let $X \in \mathcal{L}(\mathcal{H})$, and let $Q_{\operatorname{Ker} X}$ be the orthogonal (in the Hilbert space sense) projection onto* $\operatorname{Ker} X$. *Assume that the operator*

$$Q_{\operatorname{Ker} X} H Q_{\operatorname{Ker} X}|_{\operatorname{Ker} X} \; : \; \operatorname{Ker} X \;\longrightarrow\; \operatorname{Ker} X \tag{1}$$

has closed range. Then there exists an invertible operator $P \in \mathcal{L}(\mathcal{H})$, a Hilbert space orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \widetilde{\mathcal{H}}_0 \tag{2}$$

and a Hilbert space isomorphism $H_{14} : \mathcal{H}_0 \to \widetilde{\mathcal{H}}_0$, such that

$$\mathrm{Ker}\,(P^{-1}XP) = \mathcal{H}_0 \oplus \mathcal{H}_1, \tag{3}$$

and with respect to decomposition (2)*, $P^{-1}XP$, P^*HP, and $P^{-1}X^{[*]}P$ have the following block operator matrix forms:*

$$P^{-1}XP = \begin{bmatrix} 0 & 0 & X_{13} & X_{14} \\ 0 & 0 & X_{23} & X_{24} \\ 0 & 0 & X_{33} & X_{34} \\ 0 & 0 & X_{43} & X_{44} \end{bmatrix}, \quad P^*HP = \begin{bmatrix} 0 & 0 & 0 & H_{14} \\ 0 & H_{22} & 0 & 0 \\ 0 & 0 & H_{33} & 0 \\ H_{14}^* & 0 & 0 & 0 \end{bmatrix}, \tag{4}$$

and

$$P^{-1}X^{[*]}P = \begin{bmatrix} H_{14}^{-*}X_{44}^*H_{14}^* & H_{14}^{-*}X_{24}^*H_{22} & H_{14}^{-*}X_{34}^*H_{33} & H_{14}^{-*}X_{14}^*H_{14} \\ 0 & 0 & 0 & 0 \\ H_{33}^{-1}X_{43}^*H_{14}^* & H_{33}^{-1}X_{23}^*H_{22} & H_{33}^{-1}X_{33}^*H_{33} & H_{33}^{-1}X_{13}^*H_{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

where $H_{14}^{-*} := (H_{14}^*)^{-1}$. Moreover, if X is H-normal, then $X_{23} = 0$, $X_{43} = 0$, and X_{33} is H_{33}-normal.

Proof. Let $\mathcal{H} = \mathcal{G}_0 \oplus \mathcal{G}_1$ where $\mathcal{G}_0 = \mathrm{Ker}\,X$ and $\mathcal{G}_1 = (\mathrm{Ker}\,X)^\perp$. Then with respect to this decomposition, X and H have the forms

$$X = \begin{bmatrix} 0 & \widehat{X}_{12} \\ 0 & \widehat{X}_{22} \end{bmatrix}, \quad H = \begin{bmatrix} \widehat{H}_{11} & \widehat{H}_{12} \\ \widehat{H}_{12}^* & \widehat{H}_{22} \end{bmatrix}.$$

By the hypothesis, $\widehat{H}_{11}$ has closed range, so we may further orthogonally decompose $\mathcal{G}_0 = \mathcal{H}_0 \oplus \mathcal{H}_1$ such that with respect to the decomposition $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{G}_1$ the operators X and H have the forms

$$X = \begin{bmatrix} 0 & 0 & \widehat{X}_{13} \\ 0 & 0 & \widehat{X}_{23} \\ 0 & 0 & \widehat{X}_{33} \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & \widehat{H}_{33} \end{bmatrix},$$

where $H_{22} : \mathcal{H}_1 \to \mathcal{H}_1$ is invertible. Then setting

$$P_1 := \begin{bmatrix} I & 0 & 0 \\ 0 & I & -H_{22}^{-1}H_{23} \\ 0 & 0 & I \end{bmatrix}$$

implies

$$P_1^{-1}XP_1 = \begin{bmatrix} 0 & 0 & \widehat{X}_{13} \\ 0 & 0 & \widehat{X}_{23} + H_{22}^{-1}H_{23}\widehat{X}_{33} \\ 0 & 0 & \widehat{X}_{33} \end{bmatrix}, \quad P_1^*HP_1 = \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & 0 \\ H_{13}^* & 0 & \widetilde{H}_{33} \end{bmatrix}.$$

Since H is invertible, we obtain that H_{13} is right invertible. Let $\mathcal{H}_2 = \operatorname{Ker} H_{13}$, $\widetilde{\mathcal{H}}_0 = (\operatorname{Ker} H_{13})^{\perp}$, and decompose $\mathcal{G}_1 = \mathcal{H}_2 \oplus \widetilde{\mathcal{H}}_0$. Then there exist invertible operators $S : \mathcal{H}_0 \to \mathcal{H}_0$ and $T : \mathcal{G}_1 \to \mathcal{G}_1$ such that $S^*H_{13}T = \begin{bmatrix} 0 & H_{14} \end{bmatrix}$, where $H_{14} : \mathcal{H}_0 \to \widetilde{\mathcal{H}}_0$ is a Hilbert space isomorphism. Setting $P_2 = P_1 \cdot (S \oplus I_{\mathcal{H}_1} \oplus T)$, we get

$$P_2^{-1}XP_2 = \begin{bmatrix} 0 & 0 & \widetilde{X}_{13} & \widetilde{X}_{14} \\ 0 & 0 & \widetilde{X}_{23} & \widetilde{X}_{24} \\ 0 & 0 & \widetilde{X}_{33} & \widetilde{X}_{34} \\ 0 & 0 & \widetilde{X}_{43} & \widetilde{X}_{44} \end{bmatrix}, \quad P_2^*HP_2 = \begin{bmatrix} 0 & 0 & 0 & H_{14} \\ 0 & H_{22} & 0 & 0 \\ 0 & 0 & H_{33} & H_{34} \\ H_{14}^* & 0 & H_{34}^* & H_{44} \end{bmatrix}.$$

Finally, setting

$$P := P_1P_2 \begin{bmatrix} I & 0 & -(H_{14}^*)^{-1}H_{34}^* & -\frac{1}{2}(H_{14}^*)^{-1}H_{44} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

we obtain that $P^{-1}XP$ and P^*HP have the form as in (4). A straightforward computation shows that $P^{-1}X^{[*]}P$ has the form (5). Furthermore,

$$P^{-1}X^{[*]}XP = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, let X be H-normal, i.e., $P^{-1}XX^{[*]}P = P^{-1}X^{[*]}XP$. This implies that the first two operator columns of $P^{-1}XX^{[*]}P$ are zero, i.e.,

$$\begin{bmatrix} X_{13} \\ X_{23} \\ X_{33} \\ X_{43} \end{bmatrix} \begin{bmatrix} H_{33}^{-1}X_{43}^*H_{14}^* & H_{33}^{-1}X_{23}^*H_{22} \end{bmatrix} = 0. \tag{6}$$

Observe that the first operator matrix in (6) has zero kernel, because of (3). This implies $X_{43} = 0$ and $X_{23} = 0$. Then comparing the blocks in the $(3, 3)$-positions of $P^{-1}XX^{[*]}P$ and $P^{-1}X^{[*]}XP$, we obtain $X_{33}H_{33}^{-1}X_{33}^{*}H_{33} = H_{33}^{-1}X_{33}^{*}H_{33}X_{33}$, i.e., X_{33} is H_{33}-normal. $\square$

Next, we state a lemma that is of a general nature. We say that a point $\lambda \in \sigma(X)$, $X \in \mathcal{L}(\mathcal{H})$, is an *eigenvalue of finite type* if λ is an isolated point of the spectrum $\sigma(X)$ and the spectral projection $(2\pi i)^{-1} \int_{|\xi|=\epsilon} (\xi I - X)^{-1} d\xi$, where $\epsilon > 0$ is sufficiently small, has finite rank. It is easy to see (by using the decomposition of $\mathcal{H}$ as a direct sum of two X-invariant subspaces so that $X - \lambda I$ is invertible on one of them, and $X - \lambda I$ is nilpotent on the other) that if λ is an eigenvalue of finite type of X, and if $\mathcal{M}$ is an X-invariant subspace such that $\lambda \in \sigma(X|_{\mathcal{M}})$, then λ is an eigenvalue of finite type of the restriction $X|_{\mathcal{M}}$.

Lemma 3 *Let $X \in \mathcal{L}(\mathcal{H})$ be such that* 0 *is an eigenvalue of finite type of X. Then we have that* $\dim \operatorname{Ker} X = \dim \operatorname{Ker} X^{[*]}$.

Proof. By the assumption the spectral subspace $\mathcal{H}_0$ of X corresponding to the zero eigenvalue is finite dimensional. Write $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$, and with respect to this decomposition write

$$X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}.$$

Then $\sigma(X_{11}) = \{0\}$ and X_{22} is invertible. Now $\dim \operatorname{Ker} X^{[*]} = \dim \operatorname{Ker} X^*$. We have

$$\operatorname{Ker} X^* = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; x_1 \in \operatorname{Ker} X_{11}^*,\ x_2 = -(X_{22}^*)^{-1}X_{12}^*x_1 \right\}.$$

Also $\dim \operatorname{Ker} X_{11}^* = \dim \operatorname{Ker} X_{11}$ as $\mathcal{H}_0$ is finite dimensional. So

$$\dim \operatorname{Ker} X^* = \dim \operatorname{Ker} X_{11}^* = \dim \operatorname{Ker} X_{11} = \dim \operatorname{Ker} X,$$

as required. $\square$

We are now ready to state our main result.

Theorem 4 *Assume that $X \in \mathcal{L}(\mathcal{H})$ satisfies the following properties:*

(a) *X is H-normal;*

(b) *either X is invertible, or* 0 *is an eigenvalue of X of finite type;*

(c) *$\sigma(X)$ does not surround zero, i.e., there exists a continuous path in the complex plane that connects a sufficiently small neighborhood of zero with infinity and lies entirely in the resolvent set $\mathbb{C} \setminus \sigma(X)$.*

Assume in addition that one of the following conditions hold:

(i) $\mathrm{Ker}\, X = \mathrm{Ker}\, X^{[*]}$*;*

(ii) $\mathcal{H}$ *with the indefinite inner product generated by* H *is a Pontryagin space, i.e., at least one of the two spectral subspaces of* H *corresponding to the positive part of* $\sigma(H)$ *and to the negative part of* $\sigma(H)$ *is finite dimensional.*

Then X *admits an* H*-polar decomposition.*

Proof. The proof starts with a general construction that is independent of whether we assume the additional conditions (i) or (ii) or not.

By Lemma 2 we may assume that

$$X = \begin{bmatrix} 0 & 0 & X_{13} & X_{14} \\ 0 & 0 & 0 & X_{24} \\ 0 & 0 & X_{33} & X_{34} \\ 0 & 0 & 0 & X_{44} \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 0 & 0 & H_{14} \\ 0 & H_{22} & 0 & 0 \\ 0 & 0 & H_{33} & 0 \\ H_{14}^* & 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

with respect to an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \widetilde{\mathcal{H}}_0,$$

where $\mathrm{Ker}\, X = \mathcal{H}_0 \oplus \mathcal{H}_1$, where X_{33} is H_{33}-normal, and where $H_{14} : \mathcal{H}_0 \to \widetilde{\mathcal{H}}_0$ is a Hilbert space isomorphism. (Note that by the hypotheses of the theorem, clearly the operator (1) has closed range). In the following, we will identify $\mathcal{H}_0$ and $\widetilde{\mathcal{H}}_0$ via the isomorphism H_{14}, i.e., we assume without loss of generality that $\mathcal{H}_0 = \widetilde{\mathcal{H}}_0$ and $H_{14} = I_{\mathcal{H}_0}$.

We use induction on the dimension of the spectral subspace of X corresponding to the eigenvalue 0. The base of induction, i.e., the case when X is invertible, was proved in [13] (note that the finite dimensional proof given in [13] carries over to the infinite dimensional case using the property (c) of X).

We have

$$\sigma(X_{33}) \cup \{0\} = \sigma(\widetilde{X}), \quad \text{where} \quad \widetilde{X} := \left(\begin{bmatrix} 0 & 0 & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & X_{33} \end{bmatrix} \right).$$

Moreover, the unbounded component of $\mathbb{C} \setminus \sigma(\widetilde{X})$ contains the unbounded component of $\mathbb{C} \setminus \sigma(X)$ (this is a general property of the spectrum of a restriction of an operator to its invariant subspace). Thus, the property (c) holds true for X_{33}.

To see that X_{33} satisfies property (b), we have to show that either X_{33} is invertible, or 0 is an eigenvalue of finite type of X_{33}. Assume then that X_{33} is not invertible. Since 0 is an eigenvalue of finite type of X, it is also an eigenvalue of finite type for X restricted to its invariant subspace $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. In order to show that 0 is an eigenvalue of finite type of X_{33} all we need to show is that

$\dim \mathrm{Ker}\ X_{33}^n$ is uniformly bounded. We have that $\dim \mathrm{Ker}\ \widetilde{X}^n \leq \dim \mathrm{Ker}\ X^n$, and so $\dim \mathrm{Ker}\ \widetilde{X}^n$ is uniformly bounded. Now

$$\widetilde{X}^n = \begin{bmatrix} 0 & 0 & X_{13}X_{33}^{n-1} \\ 0 & 0 & 0 \\ 0 & 0 & X_{33}^n \end{bmatrix},$$

and so

$$\mathrm{Ker}\ \widetilde{X}^n = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathrm{Ker}\ X_{33}^{n-1},$$

where the latter equality follows from

$$\mathrm{Ker} \begin{bmatrix} X_{13} \\ X_{33} \end{bmatrix} = \{0\} \tag{8}$$

by construction of the form (7). Hence we have that $\dim \mathrm{Ker}\ X_{33}^{n-1}$ is uniformly bounded, and so 0 is an eigenvalue of finite type of X_{33} whenever X_{33} is not invertible.

If (ii) is satisfied, i.e., if $\mathcal{H}$ with the indefinite inner product generated by H is a Pontryagin space, then also $\mathcal{H}_2$ with the indefinite inner product generated by H_{33} is a Pontryagin space. On the other hand, if (i) is satisfied, i.e., $\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]}$, then we obtain $X_{24} = 0$ and $X_{44} = 0$, and

$$\begin{aligned} X^{[*]}X &= \begin{bmatrix} 0 & 0 & X_{34}^* H_{33} X_{33} & X_{34}^* H_{33} X_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X_{33}^{[*]} X_{33} & X_{33}^{[*]} X_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ XX^{[*]} &= \begin{bmatrix} 0 & 0 & X_{13} X_{33}^{[*]} & X_{13} H_{33}^{-1} X_{13}^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X_{33} X_{33}^{[*]} & X_{33} H_{33}^{-1} X_{13}^* \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Assume that $x \in \mathrm{Ker}\ X_{33}$. Then

$$XX^{[*]} \begin{bmatrix} 0 & 0 & x & 0 \end{bmatrix}^T = X^{[*]}X \begin{bmatrix} 0 & 0 & x & 0 \end{bmatrix}^T = 0$$

which implies

$$\begin{bmatrix} X_{13} \\ X_{33} \end{bmatrix} X_{33}^{[*]} x = 0.$$

Because of (8), we obtain $X_{33}^{[*]}x = 0$ and $\mathrm{Ker}\ X_{33} \subseteq \mathrm{Ker}\ X_{33}^{[*]}$. The other inclusion follows analogously. So, $\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]}$ implies that $\mathrm{Ker}\ X_{33} = \mathrm{Ker}\ X_{33}^{[*]}$.

Hence, X_{33} satisfies all assumptions of the theorem. By the induction hypothesis, X_{33} admits an H_{33}-polar decomposition and by Remark 1 also a right H-polar decomposition $X_{33} = A_{33}U_{33}$, where U_{33} is an invertible H_{33}-isometry, and A_{33} is H_{33}-selfadjoint. In the following, we construct an H-polar decomposition for X. This will be done in five steps.

1. First, we show that there exists α real such that the operator $L - \alpha M$ is invertible, where

$$L = H_{33} A_{33} \qquad \text{and} \qquad M = (U_{33}^{-1})^* X_{13}^* X_{13} U_{33}^{-1}$$

are selfadjoint operators. For this purpose, observe that $H_{33}^{-1} L U_{33} = X_{33}$ is Fredholm, and therefore so is L. Denote by $Q_{\operatorname{Ker} L}$ the orthogonal projection onto the finite dimensional subspace $\operatorname{Ker} L$. We claim that

$$\operatorname{Ker} \, (Q_{\operatorname{Ker} L} M|_{\operatorname{Ker} L}) = \{0\}. \tag{9}$$

To this end note that $\operatorname{Ker} X_{13} \cap \operatorname{Ker} X_{33} = \{0\}$ by (8), and hence

$$\operatorname{Ker} M \cap \operatorname{Ker} L = \{0\}. \tag{10}$$

Let x be such that $Lx = 0$, $Q_{\operatorname{Ker} L} Mx = 0$. Then

$$\langle Mx, x \rangle = \langle Mx, Q_{\operatorname{Ker} L} x \rangle = \langle Q_{\operatorname{Ker} L} Mx, x \rangle = 0,$$

thus $Mx = 0$ (because M is positive semidefinite), and $x = 0$ in view of (10). This proves the claim (9). Now, with respect to the orthogonal decomposition $\mathcal{H}_2 = \operatorname{Ker} L \oplus (\operatorname{Ker} L)^{\perp}$, we have

$$L - \alpha M = \begin{bmatrix} -\alpha M_1 & -\alpha M_2 \\ -\alpha M_2^* & L_1 - \alpha M_3 \end{bmatrix}, \qquad \alpha \in \mathbb{R},$$

where L_1 and M_1 (because of (9) and the Fredholmness of L) are invertible. Using Schur complements we obtain that $L - \alpha M$ is invertible if and only if $\alpha \neq 0$ and the operator

$$L_1 + \alpha(-M_3 + M_2^* M_1^{-1} M_2)$$

is invertible. Clearly, such α's exist.

2. We construct an H-selfadjoint polar factor for X. For this, let $\alpha \neq 0$, $\alpha \in \mathbb{R}$, be such that $L - \alpha M$ is invertible. Then set

$$A_{13} := X_{13} U_{33}^{-1}, \quad A_{14} := \alpha^{-1} I_q, \quad A_{34} := H_{33}^{-1} A_{13}^* = H_{33}^{-1} \left(U_{33}^{-1}\right)^* X_{13}^*,$$

and

$$A := \begin{bmatrix} 0 & 0 & A_{13} & A_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation shows that A is H-selfadjoint.

3. Next, we show $A^2 = X^{[*]}X$. Indeed, we obtain from the identities

$$\begin{aligned}
A_{13}A_{33} &= X_{13}U_{33}^{-1}A_{33} = X_{13}H_{33}^{-1}H_{33}U_{33}^{-1}A_{33} = X_{13}H_{33}^{-1}U_{33}^{*}H_{33}A_{33} \\
&= X_{13}H_{33}^{-1}U_{33}^{*}A_{33}^{*}H_{33} = X_{13}H_{33}^{-1}X_{33}^{*}H_{33}, \\
A_{13}A_{34} &= X_{13}U_{33}^{-1}H_{33}^{-1}(U_{33}^{*})^{-1}X_{13}^{*} = X_{13}H_{33}^{-1}X_{13}^{*}, \\
A_{33}^2 &= A_{33}H_{33}^{-1}A_{33}^{*}H_{33} = X_{33}U_{33}^{-1}H_{33}^{-1}(U_{33}^{*})^{-1}X_{33}^{*}H_{33} = X_{33}H_{33}^{-1}X_{33}^{*}H_{33}, \\
A_{33}A_{34} &= X_{33}U_{33}^{-1}H_{33}^{-1}(U_{33}^{*})^{-1}X_{13}^{*} = X_{33}H_{33}^{-1}X_{13}^{*},
\end{aligned}$$

that

$$\begin{aligned}
A^2 &= \begin{bmatrix} 0 & 0 & A_{13}A_{33} & A_{13}A_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33}^2 & A_{33}A_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & X_{13}H_{33}^{-1}X_{33}^{*}H_{33} & X_{13}H_{33}^{-1}X_{13}^{*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X_{33}H_{33}^{-1}X_{33}^{*}H_{33} & X_{33}H_{33}^{-1}X_{13}^{*} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= XX^{[*]} = X^{[*]}X.
\end{aligned}$$

4. Finally, we show $\mathrm{Ker}\ X = \mathrm{Ker}\ A$. From the construction, it is clear that $\mathrm{Ker}\ X \subseteq \mathrm{Ker}\ A$. For the other implication, let $v = \begin{bmatrix} a & b & c & d \end{bmatrix}^T \in \mathrm{Ker}\ A$. Then

$$0 = A_{13}c + A_{14}d = X_{13}U_{33}^{-1}c + \alpha^{-1}d \implies d = -\alpha X_{13}U_{33}^{-1}c.$$

Moreover,

$$0 = A_{33}c + A_{34}d = A_{33}c - \alpha H_{33}^{-1}\left(U_{33}^{-1}\right)^{*} X_{13}^{*}X_{13}U_{33}^{-1}c.$$

The choice of α implies $c = 0$ and thus, we also obtain $d = 0$. Hence, $v \in \mathrm{Ker}\ X$.

Thus, we constructed an H-selfadjoint operator A that satisfies $A^2 = X^{[*]}X$ and $\mathrm{Ker}\ X = \mathrm{Ker}\ A$. Since X is Fredholm of index zero, it is easy to see that $X^{[*]}X$ and therefore also A are Fredholm operators of index zero. Define the operator U_0 on the range of A by $U_0x = Xy$, where y is such that $x = Ay$. It is a standard exercise to check that U_0 is a well-defined H-isometry on the range of A, and the range of U_0 coincides with the range of X. Moreover, since A and X have generalized inverses and $\mathrm{Ker}\, A = \mathrm{Ker}\, X$, it follows that U_0 is bounded and $\|U_0x\| \geq \varepsilon\|x\|$, $x \in \mathrm{Range}\, A$, where the positive constant ε is independent of x.

5. Extension of U_0 to an invertible H-isometry. This is where the assumptions (i) or (ii) come in that have not been used so far. First we consider the case where $\mathcal{H}$ is a Pontryagin space. By Lemma 3 we have $\dim \mathrm{Ker}\ X = \dim \mathrm{Ker}\ X^{[*]}$, so

$$\begin{aligned} \text{codim Range}\, A &= \dim(\text{Range}\, A)^{[\perp]} = \dim \mathrm{Ker}\ A^{[*]} = \dim \mathrm{Ker}\ A = \dim \mathrm{Ker}\ X \\ &= \dim \mathrm{Ker}\ X^{[*]} = \dim(\text{Range}\, X)^{[\perp]} = \text{codim Range}\, X. \end{aligned}$$

Then we can use [16, Theorem 2.5] to show that in case $\mathcal{H}$ is a Pontryagin space with respect to the indefinite scalar product generated by H, U_0 can be extended to an invertible H-isometry. This proves the theorem in case (ii) holds true.

Next, we consider the case that $\mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]}$. Then we have the equalities

$$(\text{Range}\, A)^{[\perp]} = \mathrm{Ker}\ A^{[*]} = \mathrm{Ker}\ A = \mathrm{Ker}\ X = \mathrm{Ker}\ X^{[*]} = (\text{Range}\, X)^{[\perp]}, \quad (11)$$

and so we have that $\text{Range}\, A = \text{Range}\, X$. In particular we have

$$\mathcal{H}_0 \oplus \mathcal{H}_1 = \mathrm{Ker}\ X = (\text{Range}\, A)^{[\perp]} = H^{-1}(\text{Range}\, A)^{\perp}$$

which implies $(\text{Range}\, A)^{\perp} = \widetilde{\mathcal{H}}_0 \oplus \mathcal{H}_1$ and $\text{Range}\, A = \mathcal{H}_0 \oplus \mathcal{H}_2$. Because of (11), the isotropic part of $\text{Range}\, A$ (which is the finite dimensional space $\mathcal{H}_0$) is the same as the isotropic part of $\text{Range}\, X$. Choose a $\langle\cdot,\cdot\rangle$-orthonormal set of vectors $\{e_1, \cdots, e_n\}$ that form a basis for $\mathcal{H}_0$. Moreover, the $\langle\cdot,\cdot\rangle$-orthogonal complement of $\mathcal{H}_0$ in $\text{Range}\, A$ (which is $\mathcal{H}_2$) is an H-nondegenerate subspace. Choose a basis $\{f_1, \cdots, f_n\}$ of $\widetilde{\mathcal{H}}_0$ that is skewly linked to $\{e_1, \cdots, e_n\}$, that is, $[e_i, f_j] = \delta_{ij}$ and $[f_i, f_j] = 0$. (For details on construction of skewly linked bases see, e.g., [10, 16, 3]; although it is assumed there that the indefinite inner product space is a Pontryagin space, the construction goes through without change for finite dimensional subspaces of Krein spaces.) Then $\text{Range}\, A \oplus \widetilde{\mathcal{H}}_0 = \text{Range}\, X \oplus \widetilde{\mathcal{H}}_0$ is H-nondegenerate.

We start by showing that U_0 maps $\mathcal{H}_0$ into itself. Indeed, for $x_0 \in \mathcal{H}_0$ we have that $U_0 x_0$ is H-orthogonal to the whole of $\text{Range}\, X$, and hence is in $\mathcal{H}_0$. So, if we write U_0 with respect to the decomposition $\mathcal{H}_0 \oplus \mathcal{H}_2$ of $\text{Range}\, A = \text{Range}\, X$ as a two by two block operator matrix, we have

$$U_0 = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix},$$

Clearly, since U_0 is one-to-one and maps onto $\text{Range}\, X$, it follows that U_0 and therefore also U_{11} and U_{22} are invertible maps.

With respect to the decomposition $\mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \widetilde{\mathcal{H}}_0$ we have for H the following form (where we choose the basis in $\mathcal{H}_0$ and in $\widetilde{\mathcal{H}}_0$ as above)

$$H = \begin{bmatrix} 0 & 0 & I \\ 0 & H_{33} & 0 \\ I & 0 & 0 \end{bmatrix}.$$

We shall define $\widetilde{U}_0 : \operatorname{Range} A \oplus \tilde{\mathcal{H}}_0 \to \operatorname{Range} X \oplus \tilde{\mathcal{H}}_0$ as the following 3×3 block operator matrix

$$\widetilde{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix},$$

where $U_{13} := -\frac{1}{2} U_{12} H_{22}^{-1} U_{12}^* (U_{11}^*)^{-1}$, and $U_{23} := -U_{22} H_{22}^{-1} U_{12}^* (U_{11}^*)^{-1}$, and finally $U_{33} := (U_{11}^*)^{-1}$. Computing $\widetilde{U}_0^* H \widetilde{U}_0$ on $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \widetilde{\mathcal{H}}_0$ we have that it equals to

$$\begin{bmatrix} 0 & 0 & I \\ 0 & H_{33} & U_{12}^*(U_{11}^*)^{-1} + U_{22}^* H_{33} U_{23} \\ I & U_{23}^* H_{33} U_{22} + U_{11}^{-1} U_{12} & U_{13}^* (U_{11}^*)^{-1} + U_{11}^{-1} U_{13} + U_{23}^* H_{33} U_{23} \end{bmatrix}. \qquad (12)$$

We see from the definition of U_{23} that the $(2,3)$-entry of the operator matrix (12) is zero. Next,

$$U_{23}^* H_{33} U_{23} = U_{11}^{-1} U_{12} H_{33}^{-1} U_{12}^* (U_{11}^*)^{-1}.$$

Thus, from the definition of U_{13} we see that also the $(3,3)$-entry of (12) is zero. Hence $\widetilde{U}_0$ is indeed an H-isometry. The fact that $\widetilde{U}_0$ is one-to-one and maps onto $\operatorname{Range} X \oplus \tilde{\mathcal{H}}_0$ follows easily from the invertibility of U_{11}, U_{22}, and U_{33}.

Now using [1, Theorem VI.4.4] we see that $\widetilde{U}_0$ can be extended to an H-unitary operator on the whole space $\mathcal{H}$. This concludes the proof. □

3 Applications of the main result

For the remainder of the paper, we assume that $\mathcal{H}$ is finite dimensional, and identify $\mathcal{L}(\mathcal{H})$ with $\mathbb{C}^{n \times n}$, the algebra of $n \times n$ complex matrices. Then Theorem 4 has some important corollaries. First of all, it answers affirmatively the question posed in [2] whether each H-normal matrix allows an H-polar decomposition.

Corollary 5 *Let $X \in \mathbb{C}^{n \times n}$ be H-normal. Then X admits an H-polar decomposition.*

Corollary 5 was known to be correct for invertible H-normal matrices and for some special cases of singular H-normal matrices (see [2, 12, 11, 13]). The result for the general case is new. The next corollary gives a criterion for the existence of H-polar decompositions in terms of well-known canonical forms of pairs (A, H), where A is H-selfadjoint, under transformations of the form $(A, H) \mapsto (P^{-1}AP, P^*HP)$, where P is invertible, see, for example, [6].

Corollary 6 *Let $X \in \mathbb{C}^{n \times n}$. Then X admits an H-polar decomposition if and only if $(X^{[*]}X, H)$ and $(XX^{[*]}, H)$ have the same canonical form.*

Proof. If $X = UA$ is a polar decomposition, then

$$XX^{[*]} = UAA^{[*]}U^{[*]} = UA^2U^{-1} \quad \text{and} \quad X^{[*]}X = A^{[*]}U^{[*]}UA = A^2,$$

i.e., $(XX^{[*]}, H)$ and $(X^{[*]}X, H)$ have the same canonical forms, because U is H-unitary. On the other hand, if $(XX^{[*]}, H)$ and $(X^{[*]}X, H)$ have the same canonical forms, then there exists an H-unitary matrix U such that $UXX^{[*]}U^{-1} = X^{[*]}X$. Then $\widetilde{X} = UX$ is H-normal, since

$$\widetilde{X}^{[*]}\widetilde{X} = X^{[*]}X = UXX^{[*]}U^{-1} = \widetilde{X}\widetilde{X}^{[*]}.$$

By Corollary 5 $\widetilde{X}$ admits an H-polar decomposition $\widetilde{X} = VA$, where V is H-unitary and A is H-selfadjoint. Then $X = (U^{-1}V)A$ is an H-polar decomposition for X. □

Thus, up to multiplication by an H-unitary matrix from the left, H-normal matrices are the only matrices that admit H-polar decompositions. Corollary 6 has been conjectured in [12, 11], where also a proof has been given for the case that X is invertible or that the eigenvalue zero of $X^{[*]}X$ has equal algebraic and geometric multiplicities.

Theorem 4 also answers a question on sums of squares of H-selfadjoint matrices that has been posed in [14]. In general, the set $\{A^2 : A \text{ is } H\text{-selfadjoint}\}$ (where H is fixed) is not convex, in contrast to the convexity of the cone of positive semidefinite matrices with respect to the Euclidean inner product, as the following example shows: Let

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_1^2 + A_2^2 = \begin{bmatrix} -3 & 2 \\ 0 & -3 \end{bmatrix}.$$

Then $A_1^2 + A_2^2$ is not a square of any H-selfadjoint matrix, since $A_1^2 + A_2^2$ has only one Jordan block associated with the eigenvalue -3. This contradicts the conditions for the existence of an H-selfadjoint square root, see Theorem 3.1 in [15]. Instead, we have the following result.

Corollary 7 *If A_1 and A_2 are two commuting H-selfadjoint matrices, then there exist an H-selfadjoint matrix A such that $A_1^2 + A_2^2 = A^2$.*

Proof. Let $X = A_1 + iA_2$. Then X is H-normal, because X and $X^{[*]} = A_1 - iA_2$ commute. By Corollary 5, X admits an H-polar decomposition $X = UA$, where U is H-unitary and A is H-selfadjoint. This implies $A_1^2 + A_2^2 = X^{[*]}X = A^2$. □

4 Polar decompositions with commuting factors

Again, we assume that $\mathcal{H}$ is finite dimensional, and identify $\mathcal{L}(\mathcal{H})$ with $\mathbb{C}^{n\times n}$, the algebra of $n \times n$ complex matrices. It is well known that a normal matrix X

(normal with respect to the standard inner product) allows a polar decomposition $X = UA$ with commuting factors, see [5], for example. The question arises whether this is still true for indefinite inner products. In [13], it has been shown by a Lie group theoretical argument that nonsingular H-normal matrices allow an H-polar decomposition with commuting factors. (For a different proof of this fact, see [12].) On the other hand, there exist singular H-normal matrices that do not allow such H-polar decompositions. The following example is borrowed from [13].

Example 8 Let

$$X = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then X is H-normal. In fact, $X^{[*]}X = XX^{[*]} = 0$. It is straightforward to check that all H-polar decompositions $X = UA$ of X are described by the formulas

$$U = \begin{bmatrix} 0 & ix \\ ix^{-1} & y \end{bmatrix}, \quad A = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix},$$

where $x \neq 0$ and y are arbitrary real numbers. Clearly, U and A do not commute for any values of the parameters x and y.

In the following, we will give necessary and sufficient conditions for the existence of H-polar decompositions with commuting factors. The proof will be based on the following result on particular square roots of H-unitary matrices.

Theorem 9 *Let $V \in \mathbb{C}^{n\times n}$ be H-unitary and let $M \in \mathbb{C}^{n\times n}$ be such that M and V commute. Then there exists an H-unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^2 = V$ and $MU = UM$.*

Proof. First, assume that there are no eigenvalues of V on the negative real line (including zero). Let Γ be a simple (i.e., without self-intersections) closed rectifiable contour in the complex plane such that Γ is symmetric with respect to the real axis, the eigenvalues of V are inside Γ, and the negative real axis $(-\infty, 0]$ is outside Γ. Let $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be the branch of the square root that assigns to $z \in \mathbb{C} \setminus (-\infty, 0]$ the solution c of $c^2 = z$ that has positive real part. Then f is analytic on Γ and analytic in the interior of Γ and hence, the matrix $f(V)$ given by the functional calculus

$$f(V) = \frac{1}{2\pi i} \int_\Gamma f(z)(zI - V)^{-1} dz \tag{13}$$

is well defined. From the fact that V is H-unitary, we obtain the formula

$$H(zI-V)^{-1} = \Big((zI-V)H^{-1}\Big)^{-1} = \Big(H^{-1}(zI-(V^*)^{-1})\Big)^{-1} = (zI-(V^*)^{-1})^{-1}H.$$

This implies $Hf(V) = f((V^*)^{-1})H$. Since $f(z^{-1}) = f(z)^{-1}$, we obtain that $f(V^{-1}) = f(V)^{-1}$, see [9, Corollary 6.2.10].

We then obtain from $f(\overline{z}) = \overline{f(z)}$, the symmetry of Γ with respect to the real axis, and the general fact that $f(M^T) = f(M)^T$, that

$$f\Big((V^*)^{-1}\Big) = \Big(f(V)^*\Big)^{-1}.$$

This implies that $U := f(V)$ is H-unitary. Clearly, $U^2 = V$ and $UM = MU$. For the case that there are negative eigenvalues of V, there exists $0 \leq \theta < 2\pi$ such that the ray $re^{i\theta}$ $(r > 0)$ does not contain an eigenvalue of V. Then $\widetilde{V} = e^{i(\pi-\theta)}V$ is still H-unitary, satisfies $M\widetilde{V} = \widetilde{V}M$, and does not have negative eigenvalues. Hence, there exists an H-unitary matrix $\widetilde{U}$ such that $\widetilde{U}^2 = \widetilde{V}$ and $M\widetilde{U} = \widetilde{U}M$. Then $U = e^{i(\theta-\pi)/2}\widetilde{U}$ is an H-unitary square root of V satisfying $MU = UM$. □

The following result provides necessary and sufficient conditions for the existence of polar decompositions with commuting factors.

Theorem 10 *Let $X \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent.*

i) *X admits an H-polar decomposition with commuting factors.*

ii) *X is H-normal and* Ker $(X) =$ Ker $(X^{[*]})$.

iii) *There exists an H-unitary matrix V such that $X = VX^{[*]}$.*

Proof. *i) ⇒ ii)*: If X allows an H-polar decomposition $X = UA$ with commuting factors, then $X^{[*]} = (UA)^{[*]} = AU^{-1} = U^{-1}A$. But then X is H-normal, because

$$XX^{[*]} = UAAU^{-1} = AU^{-1}UA = X^{[*]}X.$$

In addition, we have Ker $(X) =$ Ker $(A) =$ Ker $(X^{[*]})$.

ii) ⇒ iii): This is a special case of Witt's Theorem and coincides with [4, Lemma 4.1].

iii) ⇒ i): Let V be an H-unitary matrix such that $X = VX^{[*]}$. Note that X and V commute.

$$XV = VX^{[*]}V = V(VX^{[*]})^{[*]}V = VXV^{[*]}V = VX.$$

Then Theorem 9 implies that V has an H-unitary square root U that commutes with X. Now consider $X = UA$, where $A := U^{-1}X$. Clearly, U and A commute. Furthermore, A is H-selfadjoint, because

$$(U^{-1}X)^{[*]} = X^{[*]}U = V^{-1}XU = V^{-1}UX = U^{-2}UX = U^{-1}X.$$

Thus $X = UA$ is an H-polar decomposition for X with commuting factors. □

Note that if $X = UA$ is an H-polar decomposition of X, i.e., U is H-unitary and A is H-selfadjoint, then

$$UA = AU \iff UX = XU \implies XA = AX.$$

If A is invertible, then $XA = AX \implies UA = AU$, but in general $XA = AX \not\Longrightarrow UA = AU$ as the next two examples show. Thus, the equality $XA = AX$ gives a commutativity property of H-polar decomposition which is strictly weaker than commuting factors. Example 8 shows that not every H-normal matrix admits an H-polar decomposition with this weaker commutativity property.

We conclude the paper with two examples; the second example is borrowed from [14].

Example 11 Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then X is H-normal, but $\mathrm{Ker}\,(X) \neq \mathrm{Ker}\,(X^{[*]})$. Thus, X cannot have a polar decomposition with commuting factors by Theorem 10. On the other hand, consider the matrices

$$U = \begin{bmatrix} 1 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then U is H-unitary, A is H-selfadjoint and $X = UA$. Moreover, A and X commute, but A and U do not.

Example 12 Let

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r & z \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad r > 0,\ z = \pm 1, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

A possible H-polar decomposition $X = UA$, where U is H-unitary and A is

H-selfadjoint, is the following:

$$U = \begin{bmatrix} 1 & -\frac{r}{2z} & \frac{r^2}{4z} & -\frac{r^4}{32} & 0 \\ 0 & 1 & \frac{r}{2} & -\frac{3r^3}{16z} & -\frac{r^2}{8} \\ 0 & 0 & 1 & -\frac{r^2}{2z} & -\frac{r}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{r}{2z} & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 & \frac{r}{2} \\ 0 & 0 & 0 & \frac{r}{2} & z \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

Note that A and X commute; but A and U do not commute. A MAPLE computation even shows that there does not exist an H-unitary $\widetilde{U}$ such that $X = \widetilde{U}A = A\widetilde{U}$ for the special choice of A in as an H-selfadjoint polar factor of X.

However, note that Ker X = Ker $X^{[*]}$, i.e., by Theorem 10 there exists an H-polar decomposition $X = \widehat{U}\widehat{A}$ with commuting factors. Indeed, let

$$\widehat{U} = \begin{bmatrix} 1 & -\frac{rz}{2} & \frac{r^2z}{8} & -\frac{9r^4z^2}{128} & -\frac{r^3z}{8} \\ 0 & 1 & \frac{r}{2} & 0 & -\frac{r^2}{8} \\ 0 & 0 & 1 & -\frac{3r^2z}{8} & -\frac{r}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{rz}{2} & 1 \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} 0 & 0 & 1 & \frac{r^2z}{8} & \frac{r}{2} \\ 0 & 0 & 0 & \frac{r}{2} & z \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

Then $\widehat{U}$ is H-unitary, $\widehat{A}$ is H-selfadjoint, and $X = \widehat{U}\widehat{A} = \widehat{A}\widehat{U}$. It is interesting to note that a straightforward but tedious MAPLE computation reveals that the polar factor A is unique up to a sign, i.e., all H-polar decompositions for X with commuting factors necessarily have the H-selfadjoint polar factor A (or $-A$) as in (15).

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