Invariant maximal positive subspaces and polar decompositions

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Abstract

It is proved that invertible operators on a Krein space which have an invariant maximal uniformly positive subspace and map its orthogonal complement into a nonnegative subspace allow polar decompositions with additional spectral properties. As a corollary, several classes of Krein space operators are shown to allow polar decompositions. An example in a finite dimensional Krein space shows that there exist dissipative operators that do not allow polar decompositions.

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1 Introduction and main result

Let $\mathcal{H}$ be a (complex) Hilbert space with the inner product $\langle \cdot\, , \, \cdot \rangle$, and let J be an invertible (bounded) selfadjoint operator on $\mathcal{H}$. The operator J induces a Krein space structure on $\mathcal{H}$ in a standard way: The generally indefinite inner product on $\mathcal{H}$ is defined by $[x, y] = \langle Jx, y\rangle$, $x, y \in \mathcal{H}$. A closed (in the topology induced by $\langle \cdot\, , \, \cdot \rangle$) subspace $\mathcal{M}$ of $\mathcal{H}$ is called *uniformly J-positive* if $[x, x] \geq \epsilon\langle x, x\rangle$ for every $x \in \mathcal{M}$, where $\epsilon > 0$ is independent of x. A uniformly J-positive subspace is called *maximal uniformly J-positive* if no strictly larger subspace of $\mathcal{H}$ is uniformly J-positive. For example, the spectral subspace of J corresponding to the positive part of the spectrum of J is maximal uniformly J-positive. The reader is referred to the books [1], [3], [2], [10] (finite dimensional Krein spaces only), [11] for information on geometry and classes of linear operators in Krein spaces.

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All operators on $\mathcal{H}$ are assumed to be linear and bounded (with respect to the Hilbert norm $\|x\| = \sqrt{\langle x, x\rangle}$). The *adjoint* operator $Y^{[*]}$ of an operator Y with respect to J is defined by $[Yx, y] = [x, Y^{[*]}y]$, $x, y \in \mathcal{H}$; the Hilbert space adjoint will be denoted Y^*. An operator Y on $\mathcal{H}$ is called *J-selfadjoint* if $Y = Y^{[*]}$, and *J-unitary* if Y is invertible and $Y^{-1} = Y^{[*]}$. If $\mathcal{M} \subseteq \mathcal{H}$ is a subspace (all subspaces are assumed to be closed), then we denote by $\mathcal{M}^{[\perp]}$ the orthogonal companion of $\mathcal{M}$, i.e., the subspace formed by the vectors J-orthogonal to $\mathcal{M}$.

A *J-polar decomposition* of an operator X is a decomposition of the form $X = UA$, where U is J-unitary and A is J-selfadjoint. A particular kind of J-polar decompositions, involving the notion of J-modulus, was introduced in [14], [15]. Recently, polar decompositions in finite dimensional Krein spaces were studied in [7], [4], [5], [6], [12], and in Π_κ spaces in [13]. In contrast with the Hilbert space case, there exist operators already on a 2-dimensional Krein space that do not admit a J-polar decomposition.

Of particular interest are J-polar decompositions in which the operator A has additional spectral properties. For example, the spectrum of J-modulus is assumed to be positive. In the finite dimensional case, if a J-polar decomposition exists, one can always choose A to have its spectrum in the closed right halfplane (this follows easily from the results in [5]).

In this paper we prove the following result. It asserts existence and uniqueness of a J-polar decomposition of X with the spectrum of A located in a quarterplane centered about the positive half-axis, provided X has an invariant subspace that satisfies certain geometric conditions.

Theorem 1.1 *Let X be an invertible operator on $\mathcal{H}$, and suppose that X has an invariant maximal uniformly J-positive subspace $\mathcal{M}$ such that $X(\mathcal{M}^{[\perp]})$ is J-nonpositive. Then X allows a J-polar decomposition $X = UA$ such that*

$$\sigma(A) \subseteq \{z \in \mathbb{C} : \mathrm{Re}\,(z) \geq |\mathrm{Im}\,(z)|\} \setminus \{0\}. \tag{1.1}$$

Moreover, the J-polar decomposition $X = UA$ with the property (1.1) *is unique.*

If in addition, the restriction of X to $\mathcal{M}$ is invertible, and the subspace $X(\mathcal{M}^{[\perp]})$ is uniformly J-negative, then for the unique J-polar decomposition with (1.1) *we actually have*

$$\sigma(A) \subseteq \{z \in \mathbb{C} : \mathrm{Re}\,(z) > |\mathrm{Im}\,(z)|\}. \tag{1.2}$$

Note that invertibility of $X|_{\mathcal{M}}$ follows automatically from that of X if at least one of the two spectral subspaces of J corresponding to the positive part and to the negative part of $\sigma(J)$ is finite dimensional.

The proof is based on a lemma which is independently interesting.

Lemma 1.2 *If an invertible operator X is such that $X^{[*]}X$ has no spectrum in the open, resp. closed, left halfplane, then X allows a J-polar decomposition $X = UA$ such that* (1.1), *resp.,* (1.2), *holds true. Moreover, the J-polar decomposition $X = UA$ with the property* (1.1), *resp.,* (1.2), *is unique.*

Proof Using the functional calculus, define

$$A = \frac{1}{2\pi i} \int_\Gamma z^{1/2} (zI - X^{[*]}X)^{-1} dz,$$

where Γ is a closed simple rectifiable contour that does not intersect the negative semiaxis, contains the spectrum of $X^{[*]}X$ in its interior, and is symmetric with respect to the real axis ($z \in \Gamma$ implies $\overline{z} \in \Gamma$), and where $z^{1/2}$ is the analytic branch of the square root function defined on Γ and its interior and such that $z^{1/2} > 0$ if $z > 0$. Then $A^2 = X^{[*]}X$, and one easily checks that A is J-selfadjoint. Moreover, by the spectral mapping theorem (1.1) or (1.2), as the case may be, holds true. Next, we show that $U := XA^{-1}$ is J-unitary. Clearly, U is invertible, and $UU^{[*]} = XA^{-2}X^{[*]} = X(X^{[*]}X)^{-1}X^{[*]} = I$.

It remains to prove the uniqueness. Let $X = UA$ be a polar decomposition, where A satisfies (1.1). (In particular, this case contains polar decompositions, where A satisfies (1.2).) Then $A^2 = X^{[*]}X$. Again, let Γ be a closed simple rectifiable contour that does not intersect the negative semiaxis, contains the spectrum of $X^{[*]}X$ in its interior, and is symmetric with respect to the real axis and let $z^{1/2}$ be the analytic branch of the square root function defined on Γ and its interior and such that $z^{1/2} > 0$ if $z > 0$. Define

$$A_1 = \frac{1}{2\pi i} \int_\Gamma z^{1/2} (zI - X^{[*]}X)^{-1} dz = \frac{1}{2\pi i} \int_\Gamma z^{1/2} (zI - A^2)^{-1} dz.$$

Now

$$(z - A^2)^{-1} = \frac{1}{2} A^{-1} \left((z^{1/2} - A)^{-1} - (z^{1/2} + A)^{-1} \right).$$

So,

$$AA_1 = \frac{1}{4\pi i} \left(\int_\Gamma z^{1/2} (z^{1/2} - A)^{-1} dz - \int_\Gamma z^{1/2} (z^{1/2} + A)^{-1} dz \right).$$

We substitute $z^{\frac{1}{2}} = t$, and define $\Gamma' = \{ z^{\frac{1}{2}} \mid z \in \Gamma \}$. Then $z = t^2$ on Γ with $t \in \Gamma'$, and substitution gives

$$AA_1 = \frac{1}{2\pi i} \left(\int_{\Gamma'} t^2 (t - A)^{-1} dt - \int_{\Gamma'} t^2 (t + A)^{-1} dt \right).$$

Since the real part of t is nonnegative on Γ', we have that $\sigma(-A)$ is in the exterior of Γ'. So the second integral above is zero, as the integrand is analytic inside Γ'. Hence

$$AA_1 = \frac{1}{2\pi i} \int_{\Gamma'} t^2 (t - A)^{-1} dt.$$

Now since $\sigma(A^2)$ is contained in the interior of Γ and since A satisfies (1.1), we have that $\sigma(A)$ is contained in the interior of Γ'. Therefore, by the functional calculus of A, we have that

$$AA_1 = A^2,$$

and as A is invertible, it follows that $A = A_1$. Thus A is unique, and hence also $U = XA^{-1}$. $\square$

We mention in passing that the uniqueness of A follows also from the following general result concerning a monic operator polynomial $L(\lambda)$ and its monic operator polynomial right divisor $L_1(\lambda)$ of degree k (we apply the result with $L(\lambda) = z^2 I - X^{[*]}X$ and $L_1(\lambda) = zI - A$): If γ is a closed rectifiable contour such that the spectrum of $L_1(\lambda)$ is inside γ and the spectrum of the operator polynomial $L(\lambda)(L_1(\lambda))^{-1}$ is outside γ, then there exists only one operator polynomial right divisor of $L(\lambda)$ with spectrum inside γ and the same degree k, namely $L_1(\lambda)$. This follows easily from the spectral theory of operator polynomials [9], also [17]. For further details we refer the reader to these sources.

Proof (of the theorem). By the lemma we need to show that

$$\sigma(X^{[*]}X) \cap \{z \in \mathbb{C} \,:\, \mathrm{Re}\,(z) < 0\} = \emptyset. \tag{1.3}$$

Write X and J as 2×2 block operator matrices with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus (\mathcal{M})^{\perp}$:

$$X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}, \quad J = \begin{pmatrix} J_{11} & J_{12} \\ J_{12}^* & J_{22} \end{pmatrix}.$$

Here, J_{11} is positive definite and invertible. Applying a transformation

$$X \;\mapsto\; P^{-1}XP, \quad J \;\mapsto\; P^*JP, \qquad \text{where } P = \begin{pmatrix} J_{11}^{-1/2} & -J_{11}^{-1}J_{12} \\ 0 & I \end{pmatrix},$$

we can (and will) assume without loss of generality that $J_{11} = I$ and $J_{12} = 0$. Since $\mathcal{M}$ is maximal uniformly J-positive, the $(2,2)$-block J_{22} is necessarily congruent to $-I$. Thus, we may assume that X and J have the forms

$$X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}, \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{1.4}$$

Then one easily computes that

$$X^{[*]}X = \begin{pmatrix} X_{11}^*X_{11} & X_{11}^*X_{12} \\ -X_{12}^*X_{11} & X_{22}^*X_{22} - X_{12}^*X_{12} \end{pmatrix}. \tag{1.5}$$

As X is invertible, so is $X^{[*]}X$.

Arguing by contradiction, suppose that $X^{[*]}X$ has spectrum in the open left half plane, and let $\lambda \in \mathbb{C}$, $\mathrm{Re}\,(\lambda) < 0$ be a boundary point of $\sigma(X^{[*]}X)$. Then λ belongs to the approximate point spectrum (see, e.g., [8]), i.e., there is a sequence $\{z_n = (x_n, y_n)\}_{n=1}^{\infty}$, $x_n \in \mathcal{M}$, $y_n \in (\mathcal{M})^{\perp}$ such that $\|z_n\| = 1$ and $(X^{[*]}X - \lambda I)z_n \longrightarrow 0$ as $n \longrightarrow \infty$:

$$X_{11}^*X_{11}x_n + X_{11}^*X_{12}y_n - \lambda x_n \;\longrightarrow\; 0, \tag{1.6}$$

$$-X_{12}^*X_{11}x_n + (X_{22}^*X_{22} - X_{12}^*X_{12})y_n - \lambda y_n \;\longrightarrow\; 0. \tag{1.7}$$

From the fact that $\mathrm{Re}\,(\lambda)$ is negative, we obtain that $\lambda I - X_{11}^*X_{11}$ is invertible and the inverse $(\lambda I - X_{11}^*X_{11})^{-1}$ has a negative definite and invertible selfadjoint part. Recall that for any operator X on $\mathcal{H}$, the operator $\frac{1}{2}(X + X^*)$ is called the *selfadjoint part* of X.

We get from (1.6):

$$x_n - (\lambda I - X_{11}^* X_{11})^{-1} X_{11}^* X_{12} y_n \;\longrightarrow\; 0. \tag{1.8}$$

Inserting this in (1.7) we obtain

$$\Big(\lambda I - (X_{22}^* X_{22} - X_{12}^* X_{12}) + X_{12}^* X_{11}(\lambda I - X_{11}^* X_{11})^{-1} X_{11}^* X_{12}\Big) y_n \;\longrightarrow\; 0. \tag{1.9}$$

We set

$$F(\lambda) = \lambda I - (X_{22}^* X_{22} - X_{12}^* X_{12}) + X_{12}^* X_{11}(\lambda I - X_{11}^* X_{11})^{-1} X_{11}^* X_{12}. \tag{1.10}$$

The condition that $X(\mathcal{M}^{[\perp]})$ is J-nonpositive translates into $X_{22}^* X_{22} - X_{12} X_{12}^*$ being positive semidefinite. It then follows from (1.10) that $F(\lambda)$ has a negative definite and invertible selfadjoint part. In particular, $F(\lambda)$ is invertible.

Hence from (1.9) we see that $y_n \longrightarrow 0$. Then (1.8) implies that also $x_n \longrightarrow 0$, a contradiction with $\|z_n\| = 1$.

The proof of the additional part of Theorem 1.1 follows the same lines. We now have $\mathrm{Re}\,(\lambda) \leq 0$. The invertibility of $X|_{\mathcal{M}}$ implies that $X_{11}^* X_{11}$ is invertible, hence again $\lambda I - X_{11}^* X_{11}$ is invertible and the inverse $(\lambda I - X_{11}^* X_{11})^{-1}$ has a negative definite and invertible selfadjoint part. The condition that $X(\mathcal{M}^{[\perp]})$ is uniformly J-negative means that $X_{22}^* X_{22} - X_{12} X_{12}^*$ is positive definite invertible. So we conclude again from (1.10) that $F(\lambda)$ is invertible, and obtain a contradiction. $\square$

Remark 1.3 The theorem obviously remains true if $\mathcal{M}$ is assumed to be an invariant maximal uniformly J-negative subspace of X such that $X(\mathcal{M}^{[\perp]}$ is J-nonnegative. (Replace J with $-J$ in the theorem.)

2 Polar decompositions for various classes of operators and examples

Several consequences of Theorem 1.1 and illustrative examples are presented in this section.

Corollary 2.1 *Let X be an invertible operator such that the spectrum of X does not intersect the unit circle, and assume that one of the following two conditions holds:*

(a.) *the spectrum of X does not intersect the unit circle, and X is strictly monotone; that is, either $[Xx, Xx] > [x, x]$ for every nonzero $x \in \mathcal{H}$, or $[Xx, Xx] < [x, x]$ for every nonzero $x \in \mathcal{H}$.*

(b.) *the spectral subspace of J corresponding to the positive part of $\sigma(J)$ is finite dimensional, and $[Xx, Xx] > [x, x]$ for every nonzero $x \in \mathcal{H}$ with $[x, x] \geq 0$.*

Then X admits a J-polar decomposition with the property (1.1).

Proof First consider case (a.) Assume that $[Xx, Xx] > [x, x]$ for every nonzero $x \in \mathcal{H}$, and that the spectrum of X does not intersect the unit circle. The proof of the case $[Xx, Xx] < [x, x]$ is similar.

According to [11, Theorem 11.1] there exist two subspaces $\mathcal{H}_-$ and $\mathcal{H}_+$ which are X-invariant and maximal J-negative, respectively, maximal J-positive, and for which we have the direct sum decomposition $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$. Observe that this direct sum decomposition is not necessarily J-orthogonal. Note that the statements cited from [11] are made for the case of Π_κ spaces, that is, for spaces for which the spectral subspace of J corresponding to the positive part of $\sigma(J)$ is finite dimensional. However, the proof given there carries over directly to the general case, as is already remarked in [11] (Note 2 on page 80).

According to [1, Theorem 5.2] the spaces $\mathcal{H}_-$ and $\mathcal{H}_+$ are uniformly J-negative, respectively, uniformly J-positive. In order to be able to apply Theorem 1.1, we will establish that $X(\mathcal{H}_-^{[\perp]})$ is J-nonnegative. Then in view of Remark 1.3, we can apply Theorem 1.1 with "positive" replaced by "negative" everywhere in the statement. So, let $x \in \mathcal{H}_-^{[\perp]} \setminus \{0\}$. According to [3, Lemma I.6.3] the space $\mathcal{H}_-^{[\perp]}$ is J-nonnegative. So, $[x, x] \geq 0$. Since $[Xx, Xx] > [x, x]$ it follows that Xx is a J-positive vector. Hence $X(\mathcal{H}_-^{[\perp]})$ is J-nonnegative.

In case (b.), the result follows in the same way, but using [11, Theorem 11.4] instead of [11, Theorem 11.1]. $\square$

It is known that in finite dimensional Krein spaces strictly monotone operators always allow J-polar decompositions, see [14], [4, Theorem 2.4].

Example 2.2 Let $\lambda > 0$, $\varepsilon = \pm 1$ and consider

$$J = \varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \varepsilon \begin{pmatrix} -i\lambda & \frac{i}{2\lambda} \\ 0 & i\lambda \end{pmatrix}.$$

Then

$$i(X^*J - JX) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix},$$

so that X is J-dissipative. Recall that X is J-dissipative if $\frac{1}{2i}(JX - X^*J)$ is a positive semidefinite matrix. If X were to admit a J-polar decomposition, then $X^{[*]}X$ would be the square of the J-selfadjoint factor. However,

$$X^{[*]}X = \begin{pmatrix} -\lambda^2 & 1 \\ 0 & -\lambda^2 \end{pmatrix}$$

and this does not have a J-selfadjoint square root (see also [5, Theorem 4.4]). We conclude that not every J-dissipative operator admits a J-polar decomposition.

Recall that a J-dissipative operator in a finite dimensional Krein space always has an invariant maximal J-nonnegative subspace (see, e.g., [16]). In Example 2.2, the X-invariant maximal J-nonnegative subspaces are

$$\mathcal{M}_1 := \text{Span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varepsilon = \pm 1$$

and

$$\mathcal{M}_2 := \text{Span}\begin{pmatrix} 1 \\ 4\lambda^2 \end{pmatrix}, \quad \varepsilon = 1.$$

Clearly, $\mathcal{M}_1^{[\perp]} = \mathcal{M}_1$, hence $X(\mathcal{M}_1^{[\perp]}) = \mathcal{M}_1$ is J-nonpositive.

Thus, we cannot replace the condition that X has an invariant maximal uniformly J-positive subspace in Theorem 1.1 by the condition that X has an invariant maximal J-nonnegative subspace, not even in the finite dimensional case.

Example 2.3 Let

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = i\begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix}, \quad \alpha \in \mathbb{C}, \quad |\alpha| \leq 2.$$

Then B is strictly J-dissipative, i.e., $i(B^*J - JB)$ is positive definite, for $|\alpha| < 2$ and J-dissipative, i.e., $i(B^*J - JB)$ is positive semidefinite, for $|\alpha| \leq 2$. Moreover,

$$B^{[*]}B = \begin{pmatrix} 1 & \alpha \\ -\overline{\alpha} & 1 - |\alpha|^2 \end{pmatrix}.$$

One easily checks that this matrix has the eigenvalues

$$1 - \frac{1}{2}|\alpha|^2 \pm \frac{1}{2}\sqrt{|\alpha|^4 - 4|\alpha|^2}.$$

Thus, $B^{[*]}B$ has no eigenvalues on the negative half axis for $|\alpha| < 2$ and hence, B does admit J-polar decomposition by the results in [5].

Take $\mathcal{M}_1 = \text{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $\mathcal{M}_1$ is a B-invariant maximal uniformly J-positive subspace. Then

$$B(\mathcal{M}_1^{[\perp]}) = B\left(\text{Span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \text{Span}\begin{pmatrix} \alpha \\ -1 \end{pmatrix}.$$

Clearly this is J-nonpositive only if $|\alpha| \leq 1$. So, for the case $|\alpha| \leq 1$ Theorem 1.1 applies and asserts unique existence of a J-polar decomposition $B = UA$, where A satisfies (1.1) or (1.2).

However, for $1 < |\alpha| < 2$ Theorem 1.1 does not apply, not even in the version with "positive" replaced by "negative" everywhere in the statement. Indeed, consider $\mathcal{M}_2 = \text{Span}\begin{pmatrix} \alpha \\ -2 \end{pmatrix}$. Then $\mathcal{M}_2$ is a B-invariant maximal uniformly J-negative subspace, and

$$B(\mathcal{M}_2^{[\perp]}) = B\left(\text{Span}\begin{pmatrix} 2 \\ -\overline{\alpha} \end{pmatrix}\right) = \text{Span}\begin{pmatrix} |\alpha|^2 - 2 \\ -\overline{\alpha} \end{pmatrix}.$$

This space is J-negative for $1 < |\alpha| < 2$, because

$$\left(|\alpha|^2 - 2\,,\, -\alpha \right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} |\alpha|^2 - 2 \\ -\overline{\alpha} \end{pmatrix} = |\alpha|^4 - 5|\alpha|^2 + 4,$$

which is negative for the indicated values of α. Observe that for $\sqrt{2} \leq |\alpha| < 2$ the eigenvalues of $B^{[*]}B$ are located in the open left half plane, so B cannot have a J-polar decomposition $B = UA$ such that A satisfies (1.1) or (1.2). However, B still admits a J-polar decomposition. When $|\alpha| = 2$, B is only J-dissipative, but not strictly J-dissipative. In this case $B^{[*]}B$ is similar to a Jordan block of size 2 associated with the eigenvalue -1. Hence B does not allow a J-polar decomposition because $B^{[*]}B$ does not have a J-selfadjoint square root. Again see also [5, Theorem 4.4].

The last observation in Example 2.3 gives rise to the following open question.

Problem 1 *Does any strictly J-dissipative operator allow a J-polar decomposition?*

The following result can be seen quite quickly as a corollary from our main theorem (although a more direct approach is possible as well, which in the finite dimensional case is probably more straightforward).

Corollary 2.4 *Assume that X is invertible and commutes with a uniformly positive operator, that is $XY = YX$ for some J-selfadjoint Y satisfying $JY \geq \varepsilon I > 0$, where $\varepsilon > 0$. Then X admits a J-polar decomposition with the property* (1.1).

Proof From [3, Theorem VIII.1.2] it follows that X is fundamentally reducible. Let $\mathcal{M}_+$ and $\mathcal{M}_-$ be a fundamentally reducing pair of subspaces, i.e., they are both X-invariant, they are uniformly J-positive and uniformly J-negative respectively, and $\mathcal{H} = \mathcal{M}_+ [\dot{+}] \mathcal{M}_-$, where this is a J-orthogonal direct sum decomposition. Hence, we can apply Theorem 1.1 to get the desired result. $\square$

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