STABILITY RADII FOR LINEAR HAMILTONIAN SYSTEMS WITH DISSIPATION UNDER STRUCTURE-PRESERVING PERTURBATIONS

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Abstract. Dissipative Hamiltonian (DH) systems are an important concept in energy based modeling of dynamical systems. One of the major advantages of the DH formulation is that system properties are encoded in an algebraic way. For instance, the algebraic structure of DH systems guarantees that the system is automatically stable. In this paper the question is discussed when a linear constant coefficient DH system is on the boundary of the region of asymptotic stability, i.e., when it has purely imaginary eigenvalues, or how much it has to be perturbed to be on this boundary. For unstructured systems this *distance to instability (stability radius)* is well-understood. In this paper, explicit formulas for this distance under structure-preserving perturbations are determined. It is also shown (via numerical examples) that under structure-preserving perturbations the asymptotical stability of a DH system is much more robust than under general perturbations, since the distance to instability can be much larger when structure-preserving perturbations are considered.

Keywords dissipative Hamiltonian system, port-Hamiltonian system, distance to instability, structure-preserving distance to instability, restricted distance to instability
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1. Introduction. In recent years energy based modeling approaches have gained great attention. When a model arises from variational principles, then it is often characterized by a *port-Hamiltonian (PH) system*, see [5, 9, 26, 28, 29, 32, 31, 34, 35, 36] for some major references.

Linear constant coefficient input-state-output PH systems have the form

$$\begin{aligned} \dot{x} &= (J-R)\,Qx + (B-P)u, \\ y &= \ (B+P)^H Qx + (S+N)u, \end{aligned} \tag{1.1}$$

where x is the *state*, u the *input*, and y the *output*. The *Hamiltonian*, i.e., the function $x \mapsto x^H Qx$ with $Q = Q^H \in \mathbb{C}^{n,n}$ being positive definite, describes the energy of the system; $J = -J^H \in \mathbb{C}^{n,n}$ is the *structure matrix* describing the energy flux among energy storage elements within the system; $R = R^H \in \mathbb{C}^{n,n}$ is the *dissipation matrix* describing energy dissipation/loss in the system; $B \pm P \in \mathbb{C}^{n,m}$ are the *port* matrices, describing the manner in which energy enters and exits the system, and the matrix $S+N$, with $S = S^H \in \mathbb{C}^{m,m}$ and $N = -N^H \in \mathbb{C}^{m,m}$, describes direct feed-through from input to output. In a PH system the matrices R, P, and S must satisfy

$$K = \begin{bmatrix} R & P \\ P^H & S \end{bmatrix} \geq 0; \tag{1.2}$$

i.e., K is symmetric positive semidefinite. In particular, R must also be positive semidefinite.

PH systems have many important geometric and algebraic properties that are nicely encoded in the way the system is represented, see [5, 18, 29]. In this paper, we focus on the property that PH systems are *stable*, i.e., all eigenvalues of the system matrix $A = (J-R)Q$ are contained in the closed left half complex plane and all eigenvalues on the imaginary axis are semisimple. To study stability, the port matrices can be ignored, and so one is left with a *dissipative Hamiltonian* (DH) system of the form

$$\dot{x} = (J-R)\,Qx. \tag{1.3}$$

The stability of the system is then due to the fact that Q is Hermitian positive definite. Indeed, for any nonzero vector z one has

$$\mathrm{Re}\left(z^H(Q^{1/2}AQ^{-1/2})z\right) = \mathrm{Re}\left(z^H(Q^{1/2}JQ^{1/2} - Q^{1/2}RQ^{1/2})z\right) = -z^H Q^{1/2}RQ^{1/2}z \leq 0$$

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since R is positive semidefinite. (Concerning semisimplicity of the eigenvalues on the imaginary axis, we refer to Lemma 3.1.)

If one would multiply out the product to form the matrix A and forget about the DH-structure of the system, then stability would not be obvious anymore. To check whether the system is stable, one can compute the eigenvalues or use Lyapunov's theorem [21]. If A has purely imaginary eigenvalues then arbitrarily small perturbations (such as data or roundoff errors) may move eigenvalues into the right half plane. This is particularly the case for linear systems which arise as linearization of nonlinear systems around stationary reference solutions [4], from data driven realizations, see, e.g., [1, 27], or from classical finite element modeling [11]. In all these and many other cases the system model is subject to perturbations and the stability of the systems can only be guaranteed when the system has a reasonable distance to instability, see [14, 16]. Computing the distance to instability [3, 7, 13, 40] is an optimization problem and again subject to perturbations.

The situation is different for DH systems which are automatically stable, whatever the perturbations are, as long as they preserve the DH structure. However, DH systems are not necessarily *asymptotically stable*, i.e., they may have purely imaginary eigenvalues. So for a DH system it is important to know whether the system is just stable or even asymptotically stable, and even more whether it is *robustly asymptotically stable*, i.e., small (structured) perturbations keep it asymptotically stable. The latter requires that the system has a reasonable distance to a DH system with purely imaginary eigenvalues. To study this question is an important topic in many applications, in particular, in power system and circuit simulation, see, e.g. [24, 25, 23, 30], and multi-body systems, see, e.g. [11, 37, 41].

Example 1.1. In the finite element analysis of disk brake squeal [11], large scale second order differential equations arise that have the form

$$M\ddot{q} + (D+G)\dot{q} + (K+N)q = f,$$

where $M = M^H > 0$ is the mass matrix, $D = D^H \geq 0$ models material and friction induced damping, $G = -G^H$ models gyroscopic effects, $K = K^H > 0$ models the stiffness and N, is a nonsymmetric matrix modeling circulatory effects. An appropriate first order formulation is associated with the linear pencil $\lambda I + (J-R)Q$, where

$$J := \begin{bmatrix} G & K+\frac{1}{2}N \\ -(K+\frac{1}{2}N^H) & 0 \end{bmatrix},\ R := \begin{bmatrix} D & \frac{1}{2}N \\ \frac{1}{2}N^H & 0 \end{bmatrix},\ Q := \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}^{-1}, \tag{1.4}$$

where I denotes the identity matrix.

Break squeal is associated with eigenvalues in the right half plane. If the matrix N vanishes, then the system is automatically stable, since it is a DH system. One can view the matrix N as a (small-rank) perturbation of a DH system since in the industrial examples considered in [11], the matrix N has a rank of order 2000 and the size of the system is of order 1 million. It is obvious that for $N \neq 0$ the pencil $\lambda I + (J-R)Q$ is missing one of the essential properties of a DH system, because the matrix R is then indefinite and thus the system may be unstable which is the reason for squeal. To analyze properties of the system (1.1) when this happens is one of the motivations for our work.

Example 1.2. A different and more general class of DH descriptor systems of the form

$$M\dot{x} = (J-R)\,Qx \tag{1.5}$$

arises in circuit simulation as well as power system modeling. Consider e.g. a simple example of an RLC network, see [8], given by a differential-algebraic equation

$$\underbrace{\begin{bmatrix} G_c\mathcal{C}G_c^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=M} \begin{bmatrix} \dot{v}(t) \\ \dot{i_l}(t) \\ \dot{i_v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -G_r\mathcal{R}^{-1}G_r^T & -G_l & -G_v \\ G_l^T & 0 & 0 \\ G_v^T & 0 & 0 \end{bmatrix}}_{:=J-R} \begin{bmatrix} v(t) \\ i_l(t) \\ i_v(t) \end{bmatrix}, \tag{1.6}$$

with real symmetric matrices $\mathcal{L} > 0$, $\mathcal{C} > 0$, $\mathcal{R} > 0$ incorporating the resistances of the resistors, capacitances of the capacitors, and inductances between the inductors, respectively.

Here, $(J - R)$ is the graph incidence matrix, G_v is of full rank, and the subscripts r, c, l, v and i refer to edge quantities corresponding to the resistors, capacitors, inductors, voltage sources and current sources, respectively, of the given RLC network. In this case we have

$$J = \begin{bmatrix} 0 & -G_l & -G_v \\ G_l^T & 0 & 0 \\ G_v^T & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} G_r \mathcal{R}^{-1} G_r^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q := I.$$

Since M is singular, this system has algebraic constraints (arising from Kirchhoff's laws), i.e., eigenvalues at ∞ and since G_v has full row rank, it is is of index two, i.e., the system has Jordan blocks at ∞ of size two, [6]. Applying an index reduction procedure [20, 33] and solving the algebraic constraint equations (which one would not do in practice) leads to a DH system for the dynamic variables

$$\widetilde{M}\dot{z} = (\widetilde{J} - \widetilde{R})z, \tag{1.7}$$

where $\widetilde{M}$ is invertible. Setting $\widetilde{Q} = \widetilde{M}^{-1}$ and $\widetilde{x} = \widetilde{M}z$ then gives a DH system as in (1.3).

In this paper, we focus on perturbations of DH systems that affect only one of the coefficient matrices R, J, or Q. We also allow perturbations of the form $B\Delta C$, where $B \in \mathbb{C}^{n,r}$ and $C \in \mathbb{C}^{\ell,n}$ are of full column rank or full row rank, respectively. This allows the consideration of perturbations that only affect restricted parts of matrices. For example, if

$$D \in \mathbb{C}^{r,r}, \quad R = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{n,n}, \quad B = C^H = \begin{bmatrix} I_r \\ 0 \end{bmatrix} \in \mathbb{C}^{n,r},$$

then perturbations of the form $B\Delta C$ will only affect the block D, but will leave the zero blocks in R unchanged. While perturbations of the form $B\Delta C$ were called *structured perturbations* in [15], we will call them *restricted perturbations* instead, because "*structured*" could be misinterpreted as referring to the additional port-Hamiltonian structure of the system.

The paper is organized as follows. In Section 2 we study some mapping theorems that will be needed to characterize the stability distances under consideration. In Section 3 we define the various stability distances that we will discuss in this paper and give explicit formulas when only one of the matrices R, J, or Q is perturbed and structure is ignored. Then we develop explicit formulas or bounds for stability distances while focussing on structure-preserving perturbations that individually perturb only R, J, or Q in Sections 4, 5, and 6, respectively. In Section 7 we provide some numerical experiments to illustrate our results and, in particular, to show that the stability distances under structure-reserving perturbations may differ significantly from the corresponding ones under general perturbations.

In the following $\|\cdot\|$ denotes the spectral norm of a vector or a matrix while $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. By $\Lambda(A)$ we denote the spectrum of a matrix $A \in \mathbb{C}^{n,n}$, where $\mathbb{C}^{n,r}$ is the set of complex $n \times r$ matrices, with the special case $\mathbb{C}^n = \mathbb{C}^{n,1}$. The sets $\mathrm{Herm}(n)$ and $\mathrm{SHerm}(n)$, respectively, denote the set of complex Hermitian and skew-Hermitian matrices in $\mathbb{C}^{n,n}$. We use the notation $A \geq 0$ and $A \leq 0$ if $A \in \mathbb{C}^{n,n}$ is positive or negative semidefinite, respectively, and $A > 0$ if A is positive definite.

For a matrix $A \in \mathbb{C}^{n,r}$ we denote by $A^\dagger \in \mathbb{C}^{r,n}$ the *Moore-Penrose inverse* of A, see e.g., [10]. We denote the identity matrix of size n by I_n. Finally, $\sigma_{min}(A)$ denotes the smallest singular value of A, and if A is Hermitian, then $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its largest or smallest eigenvalue, respectively.

2. Mapping theorems. An important tool in the theory of distance problems are so-called *structured mapping problems*, i.e., finding necessary and sufficient conditions on vectors $x, y \in \mathbb{C}^n$ for the existence of matrices Δ with a given symmetry structure that map x to y, and characterizing all such matrices that are of minimal norm. In this section we discuss some mapping results that will be necessary to compute the stability distances.

The minimal norm solutions for the *Hermitian mapping problem* with respect to both the spectral norm and the Frobenius norm are well known, see [22]. In order to allow a direct application in later sections of this paper, we restate the result concerning the spectral norm in the form given in [2] which is a slightly different form than the one in [22].

THEOREM 2.1. *Let $x, y \in \mathbb{C}^n \setminus \{0\}$. Then there exists a matrix $H \in \mathrm{Herm}(n)$ such that $Hx = y$ if and only if $x^H y \in \mathbb{R}$. If the latter condition is satisfied then we have*

$$\min\left\{ \|H\| \;\middle|\; H \in \mathbb{C}^{n,n},\, H^H = H,\, Hx = y \right\} = \frac{\|y\|}{\|x\|}$$

and the minimum is attained for the matrix

$$\widetilde{H} := \frac{\|y\|}{\|x\|} \begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|} \end{bmatrix} \begin{bmatrix} \frac{y^H x}{\|x\|\,\|y\|} & 1 \\ 1 & \frac{x^H y}{\|x\|\,\|y\|} \end{bmatrix}^{-1} \begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|} \end{bmatrix}^H \tag{2.1}$$

if x and y are linearly independent and for $\widetilde{H} := \frac{yx^H}{x^H x}$ otherwise.

As the minimal norm matrices presented in Theorem 2.1 typically have rank 2 unless x and y are linearly dependent, one may ask whether there exists also matrices solving the Hermitian mapping problem that have rank one, and indeed it is well known that such matrices exists, see, e.g., [22, Theorem 5.1]. Interestingly, as we will show below, these matrices are not only minimal in rank, but they are also minimal norm solutions to the slightly different mapping problem, where the matrices are not only required to be Hermitian, but also to be semidefinite. For the proof of the following theorem, where we will characterize all Hermitian semidefinite solutions to our mapping problem, we will need the following lemma.

LEMMA 2.2 ([38, Lemma 1.3]). *Let $A \in \mathbb{C}^{p,m}$, $B \in \mathbb{C}^{n,q}$, $C \in \mathbb{C}^{p,q}$, and*

$$\Upsilon = \left\{ E \in \mathbb{C}^{m,n} \;\middle|\; AEB = C \right\}.$$

Then $\Upsilon \neq \emptyset$ if and only if A, B, C satisfy $AA^\dagger CB^\dagger B = C$. If the latter condition is satisfied then

$$\Upsilon = \left\{ A^\dagger CB^\dagger + Z - A^\dagger AZBB^\dagger \;\middle|\; Z \in \mathbb{C}^{n,n} \right\}.$$

Using this Lemma, we have the following mapping theorem with Hermitian positive semidefinite solutions.

THEOREM 2.3. *Let $x, y \in \mathbb{C}^n \setminus \{0\}$ and let*

$$\mathcal{S} := \left\{ H \in \mathbb{C}^{n,n} \;\middle|\; H^H = H,\, H \geq 0,\, Hx = y \right\}. \tag{2.2}$$

Then there exists a positive semidefinite Hermitian matrix $H \in \mathrm{Herm}(n)$ such that $Hx = y$ (i.e., we have $\mathcal{S} \neq \emptyset$) if and only if $x^H y > 0$. If the latter condition is satisfied then

$$\min\left\{ \|H\| \;\middle|\; H \in \mathcal{S} \right\} = \frac{\|y\|^2}{x^H y} \tag{2.3}$$

and the minimum is attained for the rank one matrix

$$\widetilde{H} = \frac{1}{x^H y} yy^H. \tag{2.4}$$

Furthermore, we have

$$\mathcal{S} = \left\{ \widetilde{H} + \left(I_n - \frac{xx^H}{\|x\|^2} \right) K^H K \left(I_n - \frac{xx^H}{\|x\|^2} \right) \;\middle|\; K \in \mathbb{C}^{n,n} \right\}, \tag{2.5}$$

where $\widetilde{H}$ is as in (2.4).

Proof. If $H \in \mathcal{S}$, then $H^H = H \geq 0$ and $Hx = y$. This implies that $x^H y = x^H H x \geq 0$. If $x^H y = 0$ then $x^H H x = 0$ and hence $y = Hx = 0$ (as $H \geq 0$) in contradiction to the assumption that y is nonzero. Thus, we have $x^H y > 0$.

Conversely, let $x^H y > 0$. Then $\widetilde{H}$ as in (2.4) is well defined. Furthermore, it is easy to see that $\widetilde{H}^H = \widetilde{H}$ and $\widetilde{H} x = y$. Also, $\widetilde{H}$ is of rank one with a positive eigenvalue $\|y\|^2/(x^H y)$ and hence $\widetilde{H}$ is positive semidefinite.

To show (2.5), note that any matrix H of the form as in the right hand side of (2.5) satisfies $H^H = H$ and $Hx = y$, and also $H \geq 0$, because it is the sum of two positive semidefinite matrices. This proves the inclusion "$\supseteq$". For the other inclusion, let $H \in \mathcal{S}$. Then we have $H^H = H \geq 0$ and $Hx = y$. Since H is positive semidefinite, we can write $H = A^H A$ for some $A \in \mathbb{C}^{n,n}$. Setting $z := Ax$, we have $Ax = z$, $A^H z = y$, and $\|z\|^2 = x^H y$, and by Lemma 2.2, the matrix A has the form

$$A = \frac{zx^H}{\|x\|^2} + Z\left(I_n - \frac{xx^H}{\|x\|^2}\right)$$

for some $Z \in \mathbb{C}^{n,n}$. Let $U := \begin{bmatrix} \frac{x}{\|x\|} & U_2 \end{bmatrix} \in \mathbb{C}^{n,n}$ be a unitary matrix, then $U_2 U_2^H = I_n - \frac{xx^H}{\|x\|^2}$, and we can write A as

$$A = \frac{zx^H}{\|x\|^2} + ZU_2U_2^H. \tag{2.6}$$

Multiplying A^H by z from the right, we obtain

$$y = A^H z = \frac{xz^H z}{\|x\|^2} + U_2 U_2^H Z^H z,$$

which implies $y^H U_2 = z^H Z U_2$, since the columns of U_2 are orthogonal to x and $U_2^H U_2 = I_{n-1}$. By applying Lemma 2.2 to $z^H Z U_2$, we obtain that Z has the form

$$Z = \frac{zy^H U_2 U_2^H}{\|z\|^2} + L - \frac{zz^H L U_2 U_2^H}{\|z\|^2}$$

for some $L \in \mathbb{C}^{n,n}$. Inserting this Z into (2.6) and that U_2 has orthonormal columns, we get

$$\begin{aligned} A &= \frac{zx^H}{\|x\|^2} + \frac{zy^H U_2 U_2^H}{\|z\|^2} + LU_2U_2^H - \frac{zz^H L U_2 U_2^H}{\|z\|^2} \\ &= \frac{zx^H}{\|x\|^2} + \frac{zy^H U_2 U_2^H}{\|z\|^2} + \left(I_n - \frac{zz^H}{\|z\|^2}\right) L U_2 U_2^H. \end{aligned}$$

Then, using $A^H z = y$ and $z^H(I_n - \frac{zz^H}{\|z\|^2}) = 0$ as well as $\|z\|^2 = x^H y$ and the orthonormality of U, we obtain that

$$\begin{aligned} H = A^H A &= \frac{yx^H}{\|x\|^2} + \frac{yy^H U_2 U_2^H}{\|z\|^2} + A^H \left(I_n - \frac{zz^H}{\|z\|^2}\right) L U_2 U_2^H \\ &= \frac{yy^H xx^H}{(x^H y)\|x\|^2} + \frac{yy^H U_2 U_2^H}{x^H y} + U_2 U_2^H L^H \left(I_n - \frac{zz^H}{\|z\|^2}\right)\left(I_n - \frac{zz^H}{\|z\|^2}\right) L U_2 U_2^H \\ &= \frac{yy^H}{x^H y} + \left(I_n - \frac{xx^H}{\|x\|^2}\right) K^H K \left(I_n - \frac{xx^H}{\|x\|^2}\right), \end{aligned} \tag{2.7}$$

where $K = \left(I_n - \frac{zz^H}{\|z\|^2}\right) L$, thus (2.5) holds.

To show (2.3), let $H \in \mathcal{S}$ be in the form (2.7) for some $K \in \mathbb{C}^{n,n}$. Since the matrices $\frac{yy^H}{x^H y}$ and $U_2 U_2^H K^H K U_2 U_2^H$ are Hermitian positive semidefinite, we have that

$$\left\|\frac{yy^H}{x^H y}\right\| \leq \left\|\frac{yy^H}{x^H y} + \left(I_n - \frac{xx^H}{\|x\|^2}\right) K^H K \left(I_n - \frac{xx^H}{\|x\|^2}\right)\right\|$$

for all $K \in \mathbb{C}^{n,n}$, which implies that

$$\left\| \frac{yy^H}{x^H y} \right\| \leq \inf_{K \in \mathbb{C}^{n,n}} \left\| \frac{yy^H}{(x^H y)} + \left(I_n - \frac{xx^H}{\|x\|^2} \right) K^H K \left(I_n - \frac{xx^H}{\|x\|^2} \right) \right\| = \inf_{H \in \mathcal{S}} \|H\|. \tag{2.8}$$

A possible choice for obtaining equality in (2.8) is $K = 0$. This gives $\widetilde{H} = \frac{yy^H}{x^H y}$ such that $\widetilde{H} \in \mathcal{S}$ and $\|\widetilde{H}\| = \frac{\|y\|^2}{x^H y} = \min_{H \in \mathcal{S}} \|H\|$. $\square$

Remark 2.4. Although we concentrate on the spectral norm in this paper, we note that the matrix $\widetilde{H}$ from Theorem 2.3 is not only the solution of the semidefinite mapping problem that has minimal spectral norm, but it is also minimal in Frobenius norm. Indeed, let $H \in \mathcal{S}$ be in the form (2.7) for some $K \in \mathbb{C}^{n,n}$. Then for $B = \frac{y}{\sqrt{x^H y}}$ and $C = KU_2U_2^H$, using that U_2 has orthonormal columns, it follows that

$$\begin{aligned} \|BB^H + C^H C\|_F^2 &= \|BB^H\|_F^2 + 2\,\mathrm{Re}\left(\mathrm{trace}(BB^H C^H C)\right) + \|C^H C\|_F^2 \\ &= \|BB^H\|_F^2 + 2\|CB\|_F^2 + \|C^H C\|_F^2, \end{aligned}$$

where we have used that $\mathrm{trace}(BB^H C^H C) = \mathrm{trace}(B^H C^H CB) = \|CB\|_F^2$. Thus, we obtain

$$\|H\|_F^2 = \frac{\|y\|^4}{(x^H y)^2} + \frac{2}{x^H y} \left\| KU_2U_2^H y \right\|_F^2 + \left\| U_2U_2^H K^H KU_2U_2^H \right\|_F^2,$$

since $\|yy^H\|_F = \|y\|_F^2$. Hence setting $K = 0$, we obtain

$$\widetilde{H} = \frac{yy^H}{x^H y}$$

as the unique matrix in $\mathcal{S}$ of minimal Frobenius norm, i.e.,

$$\|\widetilde{H}\|_F = \frac{\|y\|^2}{x^H y} = \min_{H \in \mathcal{S}} \|H\|_F.$$

Remark 2.5. Although Theorem 2.3 has been stated for Hermitian positive semidefinite mappings only, there is a corresponding result for the Hermitian negative semidefinite case. Indeed, for $x, y \in \mathbb{C} \setminus \{0\}$ there exists a negative semidefinite matrix $H \in \mathbb{C}^{n,n}$ such that $Hx = y$ if and only if $x^H y < 0$. Furthermore, it follows immediately from Theorem 2.3 by replacing y with $-y$ and H with $-H$ that a minimal solution in spectral norm is given by

$$\widetilde{H} = \frac{1}{x^H y} yy^H, \quad \|\widetilde{H}\| = \frac{\|y\|^2}{|x^H y|}.$$

Therefore, we will refer to Theorem 2.3 also in the case that we are seeking solutions for the negative semidefinite mapping problem.

When considering perturbations, it is often useful to consider perturbations that only perturb a particular part of a matrix. As mentioned in the introduction, we will describe such perturbations with the help of a so-called *restriction matrix* $B \in \mathbb{C}^{n,r}$. The following simple lemmas will be useful when applying the mapping results in the case of restricted perturbations.

Lemma 2.6. *Let $B \in \mathbb{C}^{n,r}$ with $\mathrm{rank}(B) = r$, let $y \in \mathbb{C}^r \setminus \{0\}$, and let $z \in \mathbb{C}^n \setminus \{0\}$. Then there exists a positive semidefinite $\Delta = \Delta^H \in \mathbb{C}^{r,r}$ satisfying $B\Delta y = z$ if and only if $y^H B^\dagger z > 0$ and $BB^\dagger z = z$.*

Proof. If $\Delta = \Delta^H \in \mathbb{C}^{r,r}$ is positive semidefinite and satisfies $B\Delta y = z$, then, since $B^\dagger B = I_r$, we have

$$y^H B^\dagger z = y^H B^\dagger B \Delta y = y^H \Delta y \geq 0,$$

because Δ is positive semidefinite. If $y^H \Delta y = 0$, then we would have $\Delta y = 0$ in contradiction to $0 \neq z = BB^\dagger z = B\Delta y$. Thus, we have that $y^H B^\dagger z > 0$. For the converse, consider

$$\Delta := \frac{(B^\dagger z)(B^\dagger z)^H}{y^H B^\dagger z}$$

which is Hermitian and, since $y^H B^\dagger z > 0$, also positive semidefinite. Furthermore, we have that

$$B\Delta y = \frac{B(B^\dagger z)(B^\dagger z)^H y}{y^H B^\dagger z} = BB^\dagger z = z. \quad \square$$

We also have a version of the lemma without semidefiniteness.

LEMMA 2.7. *Let $B \in \mathbb{C}^{n,r}$ with $\operatorname{rank}(B) = r$, let $y \in \mathbb{C}^r \setminus \{0\}$, and let $z \in \mathbb{C}^n \setminus \{0\}$. Then there exist $\Delta \in \mathbb{C}^{r,r}$ such that $\Delta^H = \Delta$ and $B\Delta y = z$ if and only if $y^H B^\dagger z \in \mathbb{R}$ and $BB^\dagger z = z$.*

Proof. The proof is similar to the one of Lemma 2.6. $\square$

Finally, the next lemma reveals under which conditions the "B" in the identity $B\Delta y = z$ can be "moved" to the other side of the identity.

LEMMA 2.8. *Let $B \in \mathbb{C}^{n,r}$ with $\operatorname{rank}(B) = r$, let $y \in \mathbb{C}^r \setminus \{0\}$, and let $z \in \mathbb{C}^n \setminus \{0\}$. Then for all $\Delta \in \mathbb{C}^{r,r}$ we have that $B\Delta y = z$ if and only if $\Delta y = B^\dagger z$ and $BB^\dagger z = z$.*

Proof. If $B\Delta y = z$, then $BB^\dagger z = BB^\dagger B\Delta y = B\Delta y = z$. Since B has full column rank, we have $B^\dagger B = I_r$, and thus $B\Delta y = z$ implies that $\Delta y = B^\dagger z$. Conversely, if $\Delta y = B^\dagger z$ and $BB^\dagger z = z$, then $B\Delta y = BB^\dagger z = z$. $\square$

The theorem on Hermitian semidefinite mappings that we have proved in this section will now be employed in several ways to compute the smallest distance of an asymptotically stable DH system to one which is only stable.

3. Stability radii for DH systems. In this section we discuss smallest perturbations to the individual factors J, R, Q that make a DH system of the form (1.3) lose its asymptotic stability. Our first step in this direction is the following characterization when a DH system has purely imaginary eigenvalues.

LEMMA 3.1. *Let J, Q, $R \in \mathbb{C}^{n,n}$ be such that $J^H = -J$, $Q^H = Q > 0$, and $R^H = R \geq 0$. Furthermore, let $V \in \mathbb{C}^{n,k}$ be such that $V^H V = I_k$ and $\omega \in \mathbb{R}$. Then the following statements are equivalent.*

1) *The columns of V form an orthonormal basis for an invariant subspace for $(J-R)Q$ associated with the eigenvalue $i\omega$.*
2) *The columns of V form an orthonormal basis for an invariant subspace for JQ associated with the eigenvalue $i\omega$ and $RQV = 0$.*

In particular $(J-R)Q$ has an eigenvalue on the imaginary axis if and only if $RQx = 0$ for some eigenvector x of JQ, and all purely imaginary eigenvalues of $(J-R)Q$ are semisimple.

Proof. The proof of "2) $\Rightarrow$ 1)" is obvious. For the converse, let $W \in \mathbb{C}^{k,k}$ be such that $\Lambda(W) = \{i\omega\}$ and $(J-R)QV = VW$. Furthermore, let L be the Cholesky factor of the positive definite matrix $V^H QV$. Then

$$L^{-1}V^H Q(J-R)QVL^{-H} = L^{-1}V^H QVWL^{-H} = L^H WL^{-H}.$$

Let $U \in \mathbb{C}^{k,k}$ be unitary such that $U^H L^H WL^{-H}U = i\omega I_k + N$ is in Schur form, where $N \in \mathbb{C}^{k,k}$ is strictly upper triangular and set $S = QVL^{-H}U$. Then

$$S^H(J-R)S = i\omega I_k + N. \tag{3.1}$$

Comparing the Hermitian and skew-Hermitian parts on both sides of (3.1) yields

$$-S^H RS = \frac{1}{2}(N + N^H)$$

and thus $N = 0$, because on the left hand side of this identity we have a negative semidefinite matrix and the diagonal of the matrix on the right hand side is zero. But then it follows that $0 = RS = RQVL^{-H}U$, and hence $RQV = 0$.

Finally observe that $JQ = Q^{-1/2}Q^{1/2}JQ^{1/2}Q^{1/2}$, i.e., JQ is similar to a skew-Hermitian matrix. Therefore, all eigenvalues of JQ and thus also of $(J-R)Q$ are purely imaginary and semisimple. $\square$

In the following we consider perturbations in the individual matrices $F \in \{J, R, Q\}$ of a DH system, and we also consider restrictions to the perturbations of the form $F + B\Delta_F C$, where B, C are given restriction matrices. Thus, we consider the three individual types of perturbed systems $\tilde{A}_F$, given by

$$\tilde{A}_J = ((J + B\Delta_J C) - R)Q, \ \tilde{A}_R = (J - (R + B\Delta_R C))Q, \ \text{ and } \ \tilde{A}_Q = (J-R)(Q + B\Delta_Q C). \quad (3.2)$$

For complex unstructured linear systems that are asymptotically stable, the smallest norm of a perturbation that moves an eigenvalue to the imaginary axis is called the *(complex) stability radius*, since arbitrary small perturbations can then move an eigenvalue to the right half plane and thus make the system unstable. For real systems, there is also the *real stability radius* which refers to perturbations that are constrained to be real. This is subject to future research.

In the case of DH systems, if we use perturbations that preserve the DH structure, then we may lose asymptotic stability, but the systems stays stable. Despite this property we keep the terminology *stability radius* as in the following definition.

DEFINITION 3.2. *Consider a DH system of the form* (1.3) *and let* $B \in \mathbb{C}^{n,r}$ *and* $C \in \mathbb{C}^{q,n}$ *be given restriction matrices.*

For $F \in \{J, R, Q\}$ *the* stability radius $r(F; B, C)$ *of the matrix triple* (J, R, Q) *with respect to individual perturbations to* F *under the restriction* (B, C) *is defined by*

$$r(F; B, C) := \inf \left\{ \|\Delta\| \,\middle|\, \Delta \in \mathbb{C}^{r,q}, \ \Lambda(\tilde{A}_F) \cap i\mathbb{R} \neq \emptyset \right\},$$

where $\tilde{A}_F$ *is as in* (3.2), *and the* distance to singularity *with respect to perturbations to* Q *is defined by*

$$d(Q; B, C) = \inf \left\{ \|\Delta\| \ \middle| \ \Delta \in \mathbb{C}^{r,q}, \ \det(Q + B\Delta C) = 0 \right\}.$$

For structure-preserving, restricted perturbations of the individual $F \in \{J, R, Q\}$ *we consider the following cases.*

1) *The* stability radius $r^{\mathcal{S}_d}(R; B)$ *with respect to Hermitian negative semidefinite perturbations to* R *from the perturbation set*

$$\mathcal{S}_d(R, B) := \left\{ \Delta \in \mathbb{C}^{r,r} \,\middle|\, \Delta^H = \Delta \leq 0 \ \text{ and } \ (R + B\Delta B^H) \geq 0 \right\} \quad (3.3)$$

is defined by

$$r^{\mathcal{S}_d}(R; B) := \inf \left\{ \|\Delta\| \,\middle|\, \Delta \in \mathcal{S}_d(R, B), \, \Lambda\big((J-R)Q - (B\Delta B^H)Q\big) \cap i\mathbb{R} \neq \emptyset \right\}.$$

2) *The* stability radius $r^{\mathcal{S}_i}(R; B)$ *with respect to Hermitian, but possibly indefinite, perturbations to* R *from the perturbation set*

$$\mathcal{S}_i(R, B) := \left\{ \Delta \in \mathbb{C}^{r,r} \,\middle|\, \Delta^H = \Delta \ \text{ and } \ (R + B\Delta B^H) \geq 0 \right\} \quad (3.4)$$

is defined by

$$r^{\mathcal{S}_i}(R; B) := \inf \left\{ \|\Delta\| \,\middle|\, \Delta \in \mathcal{S}_i(R, B), \, \Lambda\big((J-R)Q - (B\Delta B^H)Q\big) \cap i\mathbb{R} \neq \emptyset \right\},$$

3) *The* eigenvalue backward error $\eta^{\mathrm{Herm}}(R; B, \lambda)$, $\lambda \in \mathbb{C}$ *and the* stability radius $r^{\mathrm{Herm}}(R; B)$ *with respect to Hermitian indefinite perturbations to* R *are, respectively, defined as*

$$\eta^{\mathrm{Herm}}(R; B, \lambda) := \inf \left\{ \|\Delta\| \,\middle|\, \Delta \in \mathrm{Herm}(r), \ \lambda \in \Lambda\big(JQ - (R + B\Delta B^H)Q\big) \right\},$$

and

$$\begin{aligned} r^{\mathrm{Herm}}(R; B) &:= \inf_{\omega \in \mathbb{R}} \eta^{\mathrm{Herm}}(R; B, i\omega) \\ &= \inf \left\{ \|\Delta\| \,\middle|\, \Delta \in \mathrm{Herm}(r), \ \Lambda\big(JQ - (R + B\Delta B^H)Q\big) \cap i\mathbb{R} \neq \emptyset \right\}. \end{aligned}$$

4) *The* stability radius $r^{\mathcal{S}}(J;B)$ *with respect to structure-preserving perturbations to* J *is defined by*

$$r^{\mathcal{S}}(J;B) := \inf\left\{\|\Delta\| \,\middle|\, \Delta \in \mathrm{SHerm}(r),\ \Lambda\big((J+B\Delta B^H)Q - RQ\big)\cap i\mathbb{R} \neq \emptyset\right\}.$$

5) *The* stability radius $r^{\mathcal{S}_d}(Q;B)$ *with respect to Hermitian negative semidefinite perturbations to* Q *from the perturbation set*

$$\mathcal{S}_d(Q,B) := \left\{\Delta \in \mathbb{C}^{r,r} \,\middle|\, \Delta^H = \Delta \leq 0 \ \text{ and } \ (Q+B\Delta B^H) \geq 0\right\} \tag{3.5}$$

is defined by

$$r^{\mathcal{S}_d}(Q;B) := \inf\left\{\|\Delta\| \,\middle|\, \Delta \in \mathcal{S}_d(Q,B),\ \Lambda\big((J-R)(Q+B\Delta B^H)\big)\cap i\mathbb{R} \neq \emptyset\right\}.$$

6) *The* stability radius $r^{\mathcal{S}_i}(Q;B)$ *with respect to Hermitian, but possibly indefinite, structured perturbations to* Q *from the perturbation set*

$$\mathcal{S}_i(Q,B) := \left\{\Delta \in \mathbb{C}^{r,r} \,\middle|\, \Delta^H = \Delta \ \text{ and } \ (Q+B\Delta B^H) \geq 0\right\} \tag{3.6}$$

is defined by

$$r^{\mathcal{S}_i}(Q;B) := \inf\left\{\|\Delta\| \,\middle|\, \Delta \in \mathcal{S}_i(Q,B),\ \Lambda\big((J-R)(Q+B\Delta B^H)\big)\cap i\mathbb{R} \neq \emptyset\right\}$$

7) *Finally we introduce the* distances to singularity *with respect to structure-preserving perturbations to* Q *by*

$$d^{\mathcal{S}_d}(Q;B) := \inf\left\{\|\Delta\| \,\middle|\, \Delta \in \mathcal{S}_d(Q,B),\ \det(Q+B\Delta B^H) = 0\right\}$$

and

$$d^{\mathcal{S}_i}(Q;B) := \inf\left\{\|\Delta\| \,\middle|\, \Delta \in \mathcal{S}_i(Q,B),\ \det(Q+B\Delta B^H) = 0\right\},$$

respectively.

If the perturbation is restricted to be of rank one, then we denote this by adding an index 1*, i.e., we write* r_1 *for the corresponding radius.*

The characterization of the stability radii $r(F,B,C)$, $F \in \{J,R,Q\}$ can be easily obtained by slightly modifying the general approach of [14, Proposition 2.1].

THEOREM 3.3. *Consider an asymptotically stable DH system of the form* (1.3)*. Furthermore, let* $B \in \mathbb{C}^{n,r}$ *and* $C \in \mathbb{C}^{q,n}$ *be given restriction matrices. Then:*

1) $r(Q;B,C)$ *is finite if and only if* $G_Q(\omega) := C\big(i\omega I_n - (J-R)Q\big)^{-1}(J-R)B$ *is not identically zero for* $\omega \in \mathbb{R}$*. In the latter case, we have*

$$r(Q;B,C) = \inf_{\omega\in\mathbb{R}} \frac{1}{\|G_Q(\omega)\|}. \tag{3.7}$$

2) $r(R;B,C)$ *is finite if and only if* $G_R(\omega) := CQ\big(i\omega I_n - (J-R)Q\big)^{-1}B$ *is not identically zero if and only if* $r(J;B,C)$ *is finite. In that case, we have*

$$r(R;B,C) = r(J;B,C) = \inf_{\omega\in\mathbb{R}} \frac{1}{\|G_R(\omega)\|}. \tag{3.8}$$

In the following sections we discuss formulae for stability radii when we consider structure-preserving, restricted perturbations for the three different cases of individually perturbing the matrices $F \in \{J,R,Q\}$. It is clear that the stability radius $r(F;B,C)$ gives a lower bound for the radii obtained under structure-preserving perturbations, but as we will show, the latter stability radii may be much larger than this lower bound.

4. Stability radii under structure-preserving perturbations of the dissipation matrix R. In this section we discuss stability radii for perturbations to the dissipation matrix R. We consider three cases of perturbation matrices Δ_R: negative semidefinite perturbations that keep $R+\Delta_R \geq 0$, indefinite perturbations that keep $R+\Delta_R \geq 0$ and perturbations that possibly make $R+\Delta_R$ indefinite.

4.1. The structured restricted stability radius $r^{\mathcal{S}_d}(R;B)$. We first give explicit formulas for the stability radius in the case that R is perturbed by a restricted perturbation from $\mathcal{S}_d(R;B)$, i.e., the perturbation matrix Δ_R is negative semidefinite. In this case we also show that the perturbation matrix Δ_R of minimal norm that perturbs the triple (J,R,Q) in such a way that the matrix $(J-(R+\Delta_R))Q$ has an eigenvalue on the imaginary axis can be chosen to have rank one, so that we actually have $r^{\mathcal{S}_d}(R;B) = r_1^{\mathcal{S}_d}(R;B) = r_1^{\mathcal{S}_i}(R;B)$. We need the following Lemma.

LEMMA 4.1. *Let R, $W \in \mathbb{C}^{n,n}$ be such that $R^H = R \geq 0$ and W is nonsingular. Suppose that $x \in \mathbb{C}^n \setminus \{0\}$ is such that $RWx \neq 0$ and set*

$$\Delta_R := -\frac{(RWx)(RWx)^H}{x^H W^H RWx},$$

then $R+\Delta_R$ is Hermitian positive semidefinite.

Proof. Obviously, Δ_R is a Hermitian matrix of rank one and negative semidefinite. Thus, $R+\Delta_R$ is clearly Hermitian. We will now show that $R+\Delta_R$ is positive semidefinite by showing that all its eigenvalues are nonnegative. Since W is nonsingular, we have $Wx \neq 0$ and

$$(R+\Delta_R)Wx = RWx - \frac{(RWx)(x^H W^H RWx)}{x^H W^H RWx} = 0, \tag{4.1}$$

and hence Wx is an eigenvector of $R+\Delta_R$ corresponding to the eigenvalue zero.

Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of R and let $\mu_1,\ldots,\mu_n$ be the eigenvalues of $R+\Delta_R$, where both lists are arranged in nondecreasing order, i.e.,

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \quad \text{and} \quad \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n.$$

Since Δ_R is of rank one, by the Cauchy interlacing theorem [17, Theorem 4.3.4], we have that

$$\lambda_k \leq \mu_{k+1} \quad \text{and} \quad \mu_k \leq \lambda_{k+1} \tag{4.2}$$

for $k = 1,\ldots,n-1$. This implies that $0 \leq \mu_2 \leq \mu_3 \leq \cdots \leq \mu_n$, and thus the proof is finished once we show that $\mu_1 = 0$.

If R is positive definite, then $\lambda_1,\ldots,\lambda_n$ satisfy $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and therefore $0 < \mu_2 \leq \cdots \leq \mu_n$. Therefore, we have $\mu_1 = 0$ by (4.1).

If R is positive semidefinite but singular, then let k be the dimension of the kernel of R. We then have $k < n$, because $R \neq 0$. Letting ℓ be the dimension of the kernel of $R+\Delta_R$, together with (4.2), this implies that

$$k-1 \leq \ell \leq k+1.$$

Note that we have $\mu_1 = 0$ if we can show that $\ell = k+1$. Since W is nonsingular, the kernels of R and RW have the same dimension k. Let $x_1, x_2, \ldots, x_k$ be linearly independent eigenvectors of RW associated with the eigenvalue zero, i.e., we have $RWx_i = 0$ for $i = 1,\ldots,k$. Then

$$\Delta_R W x_i = \frac{(RWx)(x^H W^H R)}{x^H W^H RWx} W x_i = 0 \quad \text{for} \quad i = 1,\ldots,k$$

and hence, $(R+\Delta_R)Wx_i = 0$ for $i = 1,\ldots,k$. The linear independence of $x_1,\ldots,x_k$ together with the nonsingularity of W implies that $Wx_1,\ldots,Wx_k$ are linearly independent. By (4.1) we have that $(R+\Delta_R)Wx = 0$, and moreover, the vectors $Wx, Wx_1,\ldots,Wx_k$ are linearly independent, because $RWx_i = 0$ for $i = 1,\ldots,k$, but $RWx \neq 0$. Thus, the dimension of the kernel of $R+\Delta R$ is at least $k+1$ and hence we must have $\mu_1 = 0$. $\square$

As a consequence of Lemma 4.1 we obtain the formula for the stability radius $r^{\mathcal{S}_d}(R,B)$ for perturbations in R that preserve positive semidefiniteness.

Theorem 4.2. *Consider an asymptotically stable DH system of the form* (1.3) *and let the matrix* $B \in \mathbb{C}^{n,r}$ *have full rank* r*. Then* $r^{\mathcal{S}_d}(R,B)$ *is finite if and only if* $BB^\dagger RQx = RQx$ *for some eigenvector* x *of* JQ*. If this is case, then we have*

$$r^{\mathcal{S}_d}(R;B) = \min_{x\in\Omega} \left\| \frac{(B^\dagger RQx)(B^\dagger RQx)^H}{x^H QRQx} \right\|, \tag{4.3}$$

where Ω *is the set of eigenvectors of* JQ *with the property* $BB^\dagger RQx = RQx$.

Proof. By definition we have

$$r^{\mathcal{S}_d}(R;B) := \inf\Big\{ \|\Delta_R\| \;\Big|\; \Delta_R \in \mathcal{S}_d(R,B),\; \Lambda\big((J-R)Q - (B\Delta_R B^H)Q\big) \cap i\mathbb{R} \neq \emptyset \Big\}.$$

Since for $\Delta_R \in \mathcal{S}_d(R,B)$ the perturbed matrix $R + B\Delta B^H$ is, by definition of $\mathcal{S}_d(R,B)$, Hermitian positive semidefinite, we obtain by using Lemma 3.1 that

$$\begin{aligned}
&r^{\mathcal{S}_d}(R;B)\\
&= \inf\Big\{ \|\Delta_R\| \;\Big|\; \Delta_R \in \mathcal{S}_d(R,B),\; (R + B\Delta_R B^H)Qx = 0 \text{ for some eigenvector } x \text{ of } JQ \Big\}\\
&= \inf\Big\{ \|\Delta_R\| \;\Big|\; \Delta_R \in \mathcal{S}_d(R,B),\; B\Delta_R B^H Qx = -RQx \text{ for some eigenvector } x \text{ of } JQ \Big\}\\
&= \inf\Big\{ \|\Delta_R\| \;\Big|\; \Delta_R \in \mathcal{S}_d(R,B),\; \Delta_R B^H Qx = -B^\dagger RQx \text{ for some } x \in \Omega \Big\},
\end{aligned} \tag{4.4}$$

because by Lemma 2.8 we have $B\Delta_R B^H Qx = -RQx$ if and only if $\Delta_R B^H Qx = -B^\dagger RQx$ and $BB^\dagger RQx = RQx$. From (4.4) and $\mathcal{S}_d(R,B) \subseteq \big\{\Delta \in \mathbb{C}^{r,r} \;\big|\; \Delta^H = \Delta \leq 0\big\}$ we obtain

$$r^{\mathcal{S}_d}(R;B) \geq \inf\Big\{ \|\Delta\| \;\Big|\; \Delta \in \mathbb{C}^{r,r},\; \Delta^H = \Delta \leq 0,\; \Delta(B^H Qx) = -B^\dagger RQx \text{ for some } x \in \Omega \Big\}. \tag{4.5}$$

The infimum on the right hand side of (4.5) is finite. Indeed, in view of Remark 2.5 this follows from Theorem 2.3, because for x satisfying $BB^\dagger(RQx) = RQx$ there exist $\Delta \leq 0$ such that $\Delta(B^H Qx) = -B^\dagger RQx$ if and only if $x^H Q^H BB^\dagger RQx > 0$. Clearly, we have

$$x^H Q^H BB^\dagger RQx = x^H Q^H RQx \geq 0,$$

because R is positive semidefinite. Now if $0 = x^H Q^H RQx$ for some $x \in \Omega$ then the positive semidefiniteness of R implies $RQx = 0$ and thus we have $(J-R)Qx = JQx$. This implies that x is an eigenvector of $(J-R)Q$ associated with an eigenvalue on the imaginary axis which is a contradiction to the assumption that (1.3) is asymptotically stable.

Our next step is to show that we have equality in (4.5). Using mappings of minimal norm from Theorem 2.3 in (4.5) and the fact that $x \in \Omega$ implies $BB^\dagger(RQx) = RQx$, we obtain

$$\begin{aligned}
r^{\mathcal{S}_d}(R;B) &\geq \inf\big\{ \|\Delta\| \;\big|\; \Delta \in \mathbb{C}^{r,r},\; \Delta^H = \Delta \leq 0,\; \Delta(B^H Qx) = -B^\dagger RQx \text{ for some } x \in \Omega \big\}\\
&= \inf\left\{ \left\| \frac{(B^\dagger RQx)(B^\dagger RQx)^H}{x^H Q^H BB^\dagger RQx} \right\| \;\middle|\; x \in \Omega \right\}\\
&= \inf_{x\in\Omega} \left\| \frac{(B^\dagger RQx)(B^\dagger RQx)^H}{x^H QRQx} \right\|.
\end{aligned} \tag{4.6}$$

Since we can scale vectors $x \in \Omega$ to norm one without changing the quotient of norms in (4.6), a compactness argument shows that the infimum is actually a minimum and attained for some $x = \hat{x}$. Then setting

$$\widehat{\Delta}_R := -\frac{(B^\dagger RQ\hat{x})(B^\dagger RQ\hat{x})^H}{\hat{x}^H QRQ\hat{x}},$$

we can show that equality holds in (4.5) if we prove that $(R + B\widehat{\Delta}_R B^H)$ is positive semidefinite, because in that case we have $\widehat{\Delta}_R \in \mathcal{S}_d(R, B)$. But this follows from Lemma 4.1 by noting that $BB^\dagger RQ\hat{x} = RQ\hat{x}$ and

$$
\begin{aligned}
R + B\widehat{\Delta}_R B^H &= R - B\left(\frac{(B^\dagger RQ\hat{x})(B^\dagger RQ\hat{x})^H}{\hat{x}^H QRQ\hat{x}^H}\right)B^H \\
&= R - \frac{(BB^\dagger RQ\hat{x})(BB^\dagger RQ\hat{x})^H}{\hat{x}^H QRQ\hat{x}^H} \\
&= R - \frac{(RQ\hat{x})(RQ\hat{x})^H}{\hat{x}^H QRQ\hat{x}^H}. \qquad \square
\end{aligned}
$$

REMARK 4.3. It follows from the proof of Theorem 4.2 that the desired perturbation $\widehat{\Delta}_R$ of minimal norm can be chosen to be of rank one. Since on the other hand any Hermitian matrix of rank one is necessarily semi-definite and as discussed in the introduction only a negative semidefinite Hermitian matrix Δ_R in $\big(J-(R+B\Delta_R B^H\big)Q$ can move eigenvalues of $(J-R)Q$ to the right, we see that any Hermitian rank one perturbation Δ_R of $(J-R)Q$ such that $\big(J-(R+\Delta_R)\big)Q$ has an eigenvalue on the imaginary axis is necessarily negative semidefinite and thus has a norm of at least $r^{\mathcal{S}_d}(R; B)$. Consequently, we have

$$r^{\mathcal{S}_d}(R; B) = r_1^{\mathcal{S}_d}(R; B) = r_1^{\mathcal{S}_i}(R; B).$$

In this subsection we have discussed negative semidefinite perturbation matrices Δ_R and shown that the minimal perturbation that moves an eigenvalue to the imaginary axis is achieved by a rank one perturbation. In the next section we discuss indefinite perturbation matrices Δ_R.

4.2. The stability radius $r^{\mathcal{S}_i}(R; B)$. This subsection is devoted to the computation of the stability radius $r^{\mathcal{S}_i}(R; B)$, where the perturbation matrix Δ_R is now assumed to be only Hermitian, but not necessarily negative semidefinite. We still require that the system stays DH though, i.e., that $R + B\Delta_R B^H \geq 0$. To derive the formula for $r^{\mathcal{S}_i}(R; B)$, we employ the Hermitian mapping problem from Theorem 2.1 and we will use the following lemma.

LEMMA 4.4. *Let R, $\Delta_R \in \mathrm{Herm}(n)$ be such that $R > 0$ and such that Δ_R has at most one negative eigenvalue. If $R + \Delta_R$ is singular then $R + \Delta_R \geq 0$.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Δ_R. As Δ_R has at most one negative eigenvalue, we may assume that $\lambda_2, \ldots, \lambda_n \geq 0$, and we have the spectral decomposition

$$\Delta_R = \sum_{j=1}^{n} \lambda_i u_i u_i^H$$

with unit norm vectors $u_1, \ldots, u_n$. Clearly, then

$$\widetilde{R} := R + \sum_{j=2}^{n} \lambda_i u_i u_i^H > 0.$$

Since $\lambda_1 u_1 u_1^H$ is of rank one, we can apply the Cauchy interlacing theorem [17, Theorem 4.3.4], and obtain that $R + \Delta_R = \widetilde{R} + \lambda_1 u_1 u_1^H$ has at least $n-1$ positive eigenvalues, and hence the singularity of $R + \Delta_R$ implies that $R + \Delta_R \geq 0$. $\square$

THEOREM 4.5. *Consider an asymptotically stable DH system of the form* (1.3)*, let $B \in \mathbb{C}^{n,r}$ have full rank r, and let Ω be the set of eigenvectors x of JQ such that $BB^\dagger RQx = RQx$.*

1) *If $R > 0$, then $r^{\mathcal{S}_i}(R; B)$ is finite if and only if $\Omega \neq \emptyset$. In that case we have*

$$r^{\mathcal{S}_i}(R; B) = \min_{x \in \Omega} \frac{\|(B^\dagger RQx)\|}{\|B^H Qx\|}. \tag{4.7}$$

2) *If $R \geq 0$ is singular and if $r^{\mathcal{S}_i}(R; B)$ is finite, then we have $\Omega \neq \emptyset$ and*

$$r^{\mathcal{S}_i}(R; B) \geq \min_{x \in \Omega} \frac{\|(B^\dagger RQx)\|}{\|B^H Qx\|}. \tag{4.8}$$

Proof. By definition, we have

$$r^{\mathcal{S}_i}(R;B) := \inf\left\{\|\Delta_R\| \;\middle|\; \Delta_R \in \mathcal{S}_i(R,B),\ \Lambda\big((J-R)Q - (B\Delta_R B^H)Q\big) \cap i\mathbb{R} \neq \emptyset\right\}.$$

Using Lemma 3.1 and Lemma 2.8, following the lines of the proof of Theorem 4.2 we get

$$r^{\mathcal{S}_i}(R;B) = \inf\left\{\|\Delta_R\| \;\middle|\; \Delta_R \in \mathcal{S}_i(R,B),\ \Delta_R(B^H Qx) = -B^\dagger RQx \text{ for some } x \in \Omega\right\}.$$

Since $\mathcal{S}_i(R,B) \subseteq \mathrm{Herm}(r)$, we obtain

$$r^{\mathcal{S}_i}(R;B) \geq \inf\left\{\|\Delta_R\| \;\middle|\; \Delta_R \in \mathbb{C}^{r,r},\ \Delta_R^H = \Delta_R,\ \Delta_R(B^H Qx) = -B^\dagger RQx \text{ for some } x \in \Omega\right\}. \tag{4.9}$$

If $\Omega \neq \emptyset$, then the finiteness of the right hand side in (4.9) follows from Theorem 2.1 as there exist $\Delta \in \mathrm{Herm}(r)$ such that $\Delta B^H Qx = -B^\dagger RQx$ if and only if $x^H Q^H BB^\dagger RQx \in \mathbb{R}$. This condition is satisfied, because of the fact that $BB^\dagger RQx = RQx$ and because R is Hermitian.

If $r^{\mathcal{S}_i}(R;B)$ is finite, then $\Omega \neq \emptyset$, because otherwise the right hand side of (4.9) would be infinite. Then, using mappings of minimal spectral norm from Theorem 2.1, we obtain

$$\begin{aligned} r^{\mathcal{S}_i}(R;B) &\geq \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathrm{Herm}(r),\ \Delta(B^H Qx) = -B^\dagger RQx \text{ for some } x \in \Omega\right\} \\ &= \inf\left\{\frac{\|B^\dagger RQx\|}{\|B^H Qx\|} \;\middle|\; x \in \Omega\right\} = \frac{\|B^\dagger RQ\widetilde{x}\|}{\|B^H Q\widetilde{x}\|}, \end{aligned} \tag{4.10}$$

for some $\widetilde{x} \in \Omega$ (again using a compactness argument as in the proof of Theorem 4.2). This proves 2) and the inequality $\geq$ in (4.7). It remains to show that equality holds in (4.10) when $R > 0$. This would prove (4.7), and also that the non-emptyness of Ω implies that $r^{\mathcal{S}_i}(R;B)$ is finite.

Thus, assume that $R > 0$ and let $\widetilde{\Delta}_R \in \mathrm{Herm}(r)$ be such that

$$\widetilde{\Delta}_R(B^H Q\widetilde{x}) = -B^\dagger RQ\widetilde{x} \quad \text{with} \quad \|\widetilde{\Delta}_R\| = \frac{\|B^\dagger RQ\widetilde{x}\|}{\|B^H Q\widetilde{x}\|}. \tag{4.11}$$

We show that $(R + B\widetilde{\Delta}_R B^H) \geq 0$, because this implies that $\widetilde{\Delta}_R$ is an element of the set $\mathcal{S}_i(R,B)$.

The matrix $\widetilde{H}$ in (2.1) from Theorem 2.1 has at most one negative eigenvalue, since it either is a matrix of rank one or two, and if it has rank two, then it is easy to check that $y \pm (\|y\|/\|x\|)x$ are eigenvectors of $\widetilde{H}$ associated with the eigenvalues $\pm\|y\|/\|x\|$, respectively. This implies that also $B\widetilde{\Delta}_R B^H$ has at most one negative eigenvalue. By using (4.11), we furthermore obtain

$$(R + B\widetilde{\Delta}_R B^H)Q\widetilde{x} = RQ\widetilde{x} - BB^\dagger RQ\widetilde{x} = RQ\widetilde{x} - RQ\widetilde{x} = 0,$$

because $\widetilde{x}$ satisfies $BB^\dagger RQ\widetilde{x} = RQ\widetilde{x}$. This implies that $R + B\widetilde{\Delta}_R B^H$ is singular and thus Lemma 4.4 yields that $(R + B\widetilde{\Delta}_R B^H) \geq 0$ as desired. ∎

Numerical experiments suggest that the lower bound in (4.8) is actually equal to the structured stability radius $r^{\mathcal{S}_i}(R;B)$. We make the following conjecture.

CONJECTURE 4.6. *Consider an asymptotically stable DH system of the form* (1.3)*, let $B \in \mathbb{C}^{n,r}$ have full rank r, and let $z \in \mathbb{C}^n \setminus \{0\}$ be an eigenvector of JQ such that $RQz \neq 0$ and $BB^\dagger RQz = RQz$. Define $x := \frac{B^H Qz}{\|B^H Qz\|}$ and $y := -\frac{B^\dagger RQz}{\|B^H Qz\|}$ and for this choice of x and y let $\widetilde{H}$ be defined by* (2.1)*. Then $R + B\widetilde{H}B^H$ is a Hermitian positive semidefinite matrix.*

So far we considered Hermitian perturbations to R, but we required that $R + B\Delta_R B^H \geq 0$, to preserve the property that we have a DH system. In Example 1.1, the perturbation that leads to disk brake squeal, however, is such that $R + B\Delta_R B^H$ is indefinite. Thus, while the symmetry structures are retained, the system is not DH anymore. It would be conceivable that in this case the stability radius $r(R;B,C)$ for general perturbations is the relevant quantity. However, as we will show in the next section, we may still get a larger distance.

4.3. The stability radius $r^{\mathrm{Herm}}(R;B)$. To derive an explicit formula for the distance

$$r^{\mathrm{Herm}}(R;B) = \inf_{\omega\in\mathbb{R}} \eta^{\mathrm{Herm}}(R;B,i\omega),$$

we use the backward error $\eta^{\mathrm{Herm}}(R;B,i\omega)$ which can be derived from [19, Theorem 6.2]. We only state the parts of that result that are necessary for this paper and remind the reader, that $\lambda_{\min}(H)$ stands for the smallest (possibly negative) eigenvalue of a Hermitian matrix H.

THEOREM 4.7 ([19]). *Let $H_0,\ H_1 \in \mathrm{Herm}(n)$. Then*

$$\inf\left\{y^H H_0 y \,\middle|\, y\in\mathbb{C}^n,\ \|y\|=1,\ y^H H_1 y = 0\right\} = \sup_{t\in\mathbb{R}} \lambda_{\min}(H_0 + tH_1).$$

In particular, this value is finite if and only if H_1 is not (positive or negative) definite.

We first employ this result to compute the eigenvalue backward error $\eta^{\mathrm{Herm}}(R;B,\lambda)$ under Hermitian perturbations to R. The following easy observation will be important when doing so.

REMARK 4.8. Let $W \in \mathbb{C}^{n,n}$ be nonsingular and let $B\in\mathbb{C}^{n,r}$ have full rank r. Then it follows easily by considering the singular value decomposition of B that the dimension of the kernel of $(I_n - BB^\dagger)W$ is r.

THEOREM 4.9. *Consider a DH system of the form (1.3), let $B\in\mathbb{C}^{n,r}$ have full rank r, and let $\lambda\in\mathbb{C}$ be such that $W := (J-R)Q - \lambda I_n$ is nonsingular. Furthermore, let the columns of $U\in\mathbb{C}^{n,r}$ form an orthonormal basis of the kernel of $(I_n - BB^\dagger)W$. Then B^HQU is invertible. Furthermore, let L be the Cholesky factor of U^HQBB^HQU and define the matrices*

$$\widetilde{H}_0 := B^\dagger W U L^{-H},\ \ \widetilde{H}_1 := L^{-1}U^HQWUL^{-H},$$

as well as

$$H_0^{(\lambda)} := \widetilde{H}_0^H \widetilde{H}_0, \quad H_1^{(\lambda)} := i(\widetilde{H}_1 - \widetilde{H}_1^H).$$

Then we have

$$\eta^{\mathrm{Herm}}(R;B,\lambda) = \sqrt{\sup_{t\in\mathbb{R}} \lambda_{\min}\big(H_0^{(\lambda)} + tH_1^{(\lambda)}\big)}. \tag{4.12}$$

In particular, $\eta^{\mathrm{Herm}}(R;B,\lambda)$ is finite if and only if $H_1^{(\lambda)}$ is not (positive or negative) definite.

Proof. The dependence on λ in the matrices $H_0^{(\lambda)}$ and $H_1^{(\lambda)}$ has been highlighted for the ease of future reference only, so in the proof, we will use the abbreviations H_0 and H_1, respectively. By definition, we have

$$\begin{aligned}
&\eta^{\mathrm{Herm}}(R;B,\lambda)\\
&= \inf\left\{\|\Delta\| \,\middle|\, \Delta\in\mathrm{Herm}(r),\ \lambda\in\Lambda\big(JQ-(R+B\Delta B^H)Q\big)\right\}\\
&= \inf\left\{\|\Delta\| \,\middle|\, \Delta\in\mathrm{Herm}(r),\, x\in\mathbb{C}^n\setminus\{0\},\ \big(JQ-(R+B\Delta B^H)Q\big)x = \lambda x\right\}\\
&= \inf\left\{\|\Delta\| \,\middle|\, \Delta\in\mathrm{Herm}(r),\, x\in\mathbb{C}^n\setminus\{0\},\ B\Delta B^HQx = Wx\right\}\\
&= \inf\left\{\|\Delta\| \,\middle|\, \Delta\in\mathrm{Herm}(r),\, x\in\mathbb{C}^n\setminus\{0\},\ (I_n - BB^\dagger)Wx = 0,\ \Delta B^HQx = B^\dagger Wx\right\}\\
&= \inf\left\{\|\Delta\| \,\middle|\, \Delta\in\mathrm{Herm}(r),\, \alpha\in\mathbb{C}^r\setminus\{0\},\ \Delta B^HQU\alpha = B^\dagger WU\alpha\right\},
\end{aligned}$$

where we have used Lemma 2.8 in the second last equality.

By using the minimal spectral norm Hermitian mapping from Theorem 2.1 and the fact that for any $x,\,y\in\mathbb{C}^n\setminus\{0\}$ there exist $\Delta\in\mathrm{Herm}(r)$ such that $\Delta x = y$ if and only if $x^Hy\in\mathbb{R}$, it follows that

$$\big(\eta^{\mathrm{Herm}}(R;B,\lambda)\big)^2 = \inf\left\{\frac{\|B^\dagger WU\alpha\|^2}{\|B^HQU\alpha\|^2} \,\middle|\, \alpha\in\mathbb{C}^r\setminus\{0\},\ \alpha^HU^HQ^HWU\alpha\in\mathbb{R}\right\}. \tag{4.13}$$

Note that B^HQU is invertible. Indeed, if we would have $B^HQU\alpha = 0$ for some $\alpha \in \mathbb{C}^r \setminus \{0\}$, then $B^\dagger WU\alpha = \Delta B^HQU\alpha = 0$ which, together with $(I_n - BB^\dagger)WU\alpha = 0$, implies $WU\alpha = 0$ in contradiction to the fact that W is nonsingular. Setting $y = L^H\alpha$ in (4.13), we get

$$\left(\eta^{\mathrm{Herm}}(R; B, \lambda)\right)^2 = \inf\left\{ \frac{y^H \widetilde{H}_0^H \widetilde{H}_0 y}{y^H y} \,\middle|\, y \in \mathbb{C}^r \setminus \{0\},\ y^H \widetilde{H}_1 y \in \mathbb{R} \right\}, \tag{4.14}$$

where $\widetilde{H}_0 = B^\dagger WUL^{-H}$ and $\widetilde{H}_1 = L^{-1}U^HQWUL^{-H}$. Observe that in (4.14) we have $y^H\widetilde{H}_1 y \in \mathbb{R}$ if and only if $y^H H_1 y = 0$. Thus, we have

$$\begin{aligned}\left(\eta^{\mathrm{Herm}}(R; B, \lambda)\right)^2 &= \inf\left\{ \frac{y^H H_0 y}{y^H y} \,\middle|\, y \in \mathbb{C}^r \setminus \{0\},\ y^H H_1 y = 0 \right\}\\ &= \sup_{t\in\mathbb{R}} \lambda_{\min}\big(H_0 + tH_1\big),\end{aligned}$$

where the last equality follows from Theorem 4.7. ∎

Theorem 4.9 gives us the possibility to characterize the distance $r^{\mathrm{Herm}}(R; B)$ for the case that iJ is indefinite. (Note that this is the case when J is assumed to be real).

COROLLARY 4.10. *Consider an asymptotically stable DH system of the form* (1.3)*, where in addition iJ is indefinite. Let $B \in \mathbb{C}^{n,r}$ have full rank r. Then $r^{\mathrm{Herm}}(R; B)$ is finite and we have*

$$r^{\mathrm{Herm}}(R; B) = \inf_{\omega\in\mathbb{R}} \sqrt{\sup_{t\in\mathbb{R}} \lambda_{\min}\Big(H_0^{(i\omega)} + tH_1^{(i\omega)}\Big)}, \tag{4.15}$$

with $H_0^{(\lambda)}, H_1^{(\lambda)}$ as introduced in Theorem 4.9 *for a given value $\lambda \in \mathbb{C}$.*

Proof. The formula follows immediately from Theorem 4.9. Thus, it remains to show that $r^{\mathrm{Herm}}(R; B)$ is finite, which is the case if the supremum in (4.15) is finite for at least one value of ω. For this, we have to check that $H_1^{(i\omega)}$ is not definite for at least one value of ω. But this follows, because by Theorem 4.9, we have

$$H_1^{(i\omega)} = iL^{-1}U^H\big(QW - (QW)^H\big)UL^{-H} = L^{-1}U^H\big(2Q(iJ)Q + 2\omega Q\big)UL^{-H}.$$

Since iJ is assumed to be indefinite, it is clear that $H_1^{(i\omega)}$ is indefinite for $\omega = 0$. ∎

Having characterized the relevant distances under structured perturbations to R, in the next section we discuss perturbations to J.

5. Stability radii under structure-preserving perturbations of J. The analysis for the case that structure-preserving perturbations are carried out to the structure matrix J is somewhat simpler than in the case of the dissipation matrix R. We have the following theorem.

THEOREM 5.1. *Consider an asymptotically stable DH system of the form* (1.3)*, let $B \in \mathbb{C}^{n,r}$ have full rank r. Then $r^{\mathcal{S}}(J; B)$ is finite if and only if there exists a nontrivial intersection Ω of the kernel of $(I_n - BB^\dagger)(i\omega I_n - JQ)$ and the kernel of RQ for some $\omega \in \mathbb{R}$. If this is the case, then B^HQU has full rank and we have*

$$r^{\mathcal{S}}(J; B) = \inf_{\omega\in\mathbb{R}} \sigma_{\min}\big(G_J^{\mathcal{S}}(\omega)\big),$$

with $G_J^{\mathcal{S}}(\omega) := B^\dagger(i\omega I_n - JQ)UL^{-H}$, where the columns of U form an orthonormal basis for Ω and L is the Cholesky factor of U^HQBB^HQU.

Proof. By definition we have

$$r^{\mathcal{S}}(J; B) = \inf\Big\{ \|\Delta\| \,\Big|\, \Delta \in \mathrm{SHerm}(r),\ \Lambda\big((J + B\Delta B^H)Q - RQ\big) \cap i\mathbb{R} \neq \emptyset \Big\}.$$

Applying Lemma 3.1 it follows that

$$\begin{aligned} &r^{\mathcal{S}}(J; B)\\ &\quad= \inf\big\{\|\Delta\| \,\big|\Delta \in \mathrm{SHerm}(r),\ (J + B\Delta B^H)Qx = \lambda x,\ RQx = 0,\ \lambda \in i\mathbb{R},\ x \in \mathbb{C}^n \setminus \{0\}\big\}\\ &\quad= \inf\big\{\|\Delta\| \,\big|\Delta \in \mathrm{SHerm}(r),\ B\Delta B^HQx = (\lambda I_n - JQ)x,\ \lambda \in i\mathbb{R},\ RQx = 0,\ x \neq 0.\big\}. \end{aligned} \tag{5.1}$$

Clearly, if $R > 0$ then the kernel of RQ is $\{0\}$, and by Lemma 3.1 there does not exist $\Delta \in \mathrm{SHerm}(r)$ such that $\big((J + B\Delta B^H) - R\big)Q$ has an eigenvalue on the imaginary axis and hence $r^{\mathcal{S}}(J; B) = \infty$. Thus, for the remainder of the proof we assume that R is singular. For a fixed $\omega \in \mathbb{R}$ define the *eigenvalue backward error* under perturbations in J by

$$\eta^{\mathcal{S}}(J; B, i\omega) = \inf \big\{ \|\Delta\| \,\big|\, \Delta \in \mathrm{SHerm}(r),\ B\Delta B^H Qx = (i\omega I_n - JQ)x, \\ \text{for some } x \neq 0 \ \text{ with } \ RQx = 0 \big\}. \tag{5.2}$$

Inserting (5.2) into (5.1), we obtain

$$r^{\mathcal{S}}(J; B) = \inf_{\omega \in \mathbb{R}} \eta^{\mathcal{S}}(J; B, i\omega), \tag{5.3}$$

Note that $\Delta \in \mathrm{Herm}(r)$ if and only if $i\Delta \in \mathrm{SHerm}(r)$. Hence, in view of Lemma 2.7 and 2.8, for any $x \neq 0$ and $\omega \in \mathbb{R}$ there exist $\Delta \in \mathrm{SHerm}(r)$ such that $(B\Delta B^H)Qx = (i\omega I_n - JQ)x$ if and only if

$$\Delta B^H Qx = B^\dagger (i\omega I_n - JQ)x \ \text{ and } (I_n - BB^\dagger)(i\omega I_n - JQ)x = 0.$$

Indeed, the condition $x^H Q^H BB^\dagger (i\omega I_n - JQ)x \in i\mathbb{R}$ from Lemma 2.7 is satisfied, because

$$x^H Q^H BB^\dagger (i\omega I_n - JQ)x = x^H Q^H (i\omega I_n - JQ)x$$

and $Q^H(i\omega I_n - JQ)$ is skew-Hermitian. Using this in (5.2) we obtain

$$\begin{aligned} \eta^{\mathcal{S}}(J; B, i\omega) &= \inf \big\{ \|\Delta\| \,\big|\, \Delta \in \mathrm{SHerm}(r),\ x \in \Omega \setminus \{0\},\ \Delta B^H Qx = B^\dagger (i\omega I_n - JQ)x \big\} \\ &= \inf \left\{ \frac{\|B^\dagger (i\omega I_n - JQ)x\|}{\|B^H Qx\|} \,\middle|\, x \in \Omega \setminus \{0\} \right\}, \end{aligned} \tag{5.4}$$

where the last equality follows by using the minimal norm mappings from Theorem 2.1 which can be done since $\Delta \in \mathrm{Herm}(r)$ if and only if $i\Delta \in \mathrm{SHerm}(r)$. Thus, by (5.3) and (5.4), $r^{\mathcal{S}}(J; B)$ is finite if and only if $\eta^{\mathcal{S}}(J; B, i\omega)$ is finite for some $\omega \in \mathbb{R}$, i.e., $\Omega \neq \{0\}$ for some $\omega \in \mathbb{R}$.

If $\eta^{\mathcal{S}}(J; B, i\omega)$ is finite for some $\omega \in \mathbb{R}$, then let $\dim(\Omega) = k$ and let the columns of $U \in \mathbb{C}^{n,k}$ form an orthonormal basis for Ω. Then $x \in \Omega \setminus \{0\}$ implies that $x = U\alpha$ for some $\alpha \in \mathbb{C}^k \setminus \{0\}$. Using this in (5.4), we obtain

$$\eta^{\mathcal{S}}(J; B, i\omega) = \inf \left\{ \frac{\|B^\dagger (i\omega I_n - JQ)U\alpha\|}{\|B^H QU\alpha\|} \,\middle|\, \alpha \in \mathbb{C}^k \setminus \{0\} \right\}. \tag{5.5}$$

Note that $B^H QU$ is a full rank matrix, because $(J - R)Q$ has no eigenvalues on the imaginary axis. Indeed, if we would have $B^H QU\alpha = 0$ for some $\alpha \in \mathbb{C}^k \setminus \{0\}$ then from (5.2) we have $0 = B\Delta B^H QU\alpha = (i\omega I_n - JQ)U\alpha$ and this implies $(J - R)QU\alpha = i\omega U\alpha$, because $U\alpha \in \Omega$, which is a contradiction. Thus let L be the unique Cholesky factor of $U^H Q^H BB^H QU$, then by inserting $y = L^H \alpha$ in (5.5) we have

$$\begin{aligned} \eta^{\mathcal{S}}(J; B, i\omega) &= \inf \left\{ \frac{\|B^\dagger (i\omega I_n - JQ)UL^{-H} y\|}{\|y\|} \,\middle|\, y \in \mathbb{C}^k \setminus \{0\} \right\} \\ &= \sigma_{\min} \big(B^\dagger (i\omega I_n - JQ)UL^{-H} \big), \end{aligned}$$

and the assertion follows from (5.3). $\square$

Having obtained the stability radii for structure-preserving perturbations in R and J, in the next section we finally consider perturbations in Q.

6. Stability radii under structure-preserving perturbations of Q. The case of perturbations in Q needs somewhat more discussions than the other cases. Considering Example 1.2 which has the form $M\dot{x} = (J - R)x$ with $Q = I$, if M was positive definite, then we could make a change of basis and consider the system $\dot{\xi} = (J - R)Q\xi$, with $Q = M^{-1}$. However, since M is singular, this formulation can be made only in the restricted system of dynamical equations, i.e., the part of the system that corresponds to the variables v and i_L. The full system of Example 1.2 is automatically on the boundary of the stability region, since we can make M invertible and indefinite by an arbitrarily small perturbation, and then also $Q^{-1} = M$ would be indefinite. In the following, however, we do not consider this more general situation of descriptor systems, but defer this to a subsequent paper.

By definition we have the following relationships between the stability radii and the distances to singularity for Q:

$$r(Q;B) \le d(Q;B), \quad r^{\mathcal{S}_d}(Q;B) \le d^{\mathcal{S}_d}(Q;B) \quad \text{and} \quad r^{\mathcal{S}_i}(Q;B) \le d^{\mathcal{S}_i}(Q;B). \tag{6.1}$$

So let us first consider the singularity distances.

6.1. The distances to singularity. The following easy observation will be important in the following.

REMARK 6.1. Let $W \in \mathbb{C}^{n,n}$ be Hermitian positive definite and let $B \in \mathbb{C}^{n,r}$ have full rank r. Recall that by Remark 4.8 the dimension of the kernel of $(I_n - BB^\dagger)W$ is r. If the columns of $U \in \mathbb{C}^{n,r}$ form a basis of the kernel of $(I_n - BB^\dagger)W$, then $B^H U$ is invertible. Indeed, suppose that $B^H U\alpha = 0$ for some $\alpha \in \mathbb{C}^r$. Since $B^\dagger = (B^H B)^{-1} B^H$, we obtain $B^\dagger U\alpha = 0$ which in turn implies that

$$\alpha^H U^H W U\alpha = \alpha^H U^H (BB^\dagger) W U\alpha = \alpha^H U^H (BB^\dagger)^H W U\alpha = (BB^\dagger U\alpha)^H W U\alpha = 0,$$

where we have used that $U\alpha$ is in the kernel of $(I_n - BB^\dagger)W$. Since W is positive definite, this implies that $U\alpha = 0$ and thus $\alpha = 0$.

THEOREM 6.2. *Let $Q \in \mathrm{Herm(n)}$ with $Q > 0$, and let $B \in \mathbb{C}^{n,r}$ be such that $\mathrm{rank}(B) = r$. Let the columns of $U \in \mathbb{C}^{n,r}$ form an orthonormal basis of the kernel of $(I_n - BB^\dagger)Q$. Then we have*

$$d^{\mathcal{S}_d}(Q;B) = \left(\sigma_{\min}(B^\dagger Q U L^{-H})\right)^2,$$

where L is the Cholesky factor of $U^H Q U$, and

$$d(Q;B,B^H) = d^{\mathcal{S}_i}(Q;B) = \sigma_{\min}(B^\dagger Q U \widetilde{L}^{-H}),$$

where $\widetilde{L}$ is the Cholesky factor of $U^H B B^H U$.

Proof. In the following we will denote the kernel of $(I - BB^\dagger)Q$ by Ω.

Concerning the first part of the theorem, we have by definition that

$$\begin{aligned} d^{\mathcal{S}_d}(Q;B) &= \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathcal{S}_d(Q,B),\ \det(Q + B\Delta B^H) = 0\right\} \\ &= \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathcal{S}_d(Q,B),\ x \in \mathbb{C}^n \setminus \{0\},\ (Q + B\Delta B^H)x = 0\right\} \\ &= \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathcal{S}_d(Q,B),\ x \in \mathbb{C}^n \setminus \{0\},\ B\Delta B^H x = -Qx\right\} \\ &\ge \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathrm{Herm}(r),\ \Delta \le 0,\ x \in \mathbb{C}^n \setminus \{0\},\ B\Delta B^H x = -Qx\right\}, \end{aligned} \tag{6.2}$$

where the last inequality holds, because $\mathcal{S}_d(Q,B) \subseteq \{\Delta \in \mathrm{Herm}(r) \mid \Delta \le 0\}$. By Lemma 2.8 we have $B\Delta B^H x = -Qx$ if and only if $\Delta B^H x = -B^\dagger Qx$ and $BB^\dagger Qx = Qx$. Note that the latter condition is already sufficient for the existence of a matrix $\Delta^H = \Delta \le 0$ such that $\Delta B^H x = -B^\dagger Qx$. Indeed, the necessary condition $-x^H BB^\dagger Qx < 0$ in Lemma 2.6 is automatically satisfied because $BB^\dagger Qx = Qx$ and because Q is Hermitian positive definite. Using this observation in (6.2), we obtain

$$\begin{aligned} d^{\mathcal{S}_d}(Q;B) \ge \inf\Big\{\|\Delta\| \;\Big|\; &\Delta \in \mathrm{Herm}(r),\ \Delta \le 0,\ x \in \mathbb{C}^n \setminus \{0\}, \\ &\Delta B^H x = -B^\dagger Qx,\ BB^\dagger Qx = Qx\Big\} \end{aligned} \tag{6.3}$$

$$= \inf\left\{\frac{\|(B^\dagger Qx)(B^\dagger Qx)^H\|}{\|x^H Qx\|} \;\middle|\; x \in \Omega \setminus \{0\}\right\}, \tag{6.4}$$

where the last equality follows by using minimal spectral norm mappings from Theorem 2.3.

We aim to show that equality holds in (6.3). To this end, note again that x can be scaled to norm one, and thus a compactness argument shows that the infimum in (6.4) is actually a minimum. Thus, let $\widehat{x} \in \Omega$ be such that

$$\frac{\|(B^\dagger Q\widehat{x})(B^\dagger Q\widehat{x})^H\|}{\|\widehat{x}^H Q\widehat{x}\|} = \inf\left\{ \frac{\|(B^\dagger Qx)(B^\dagger Qx)^H\|}{\|x^H Qx\|} \;\middle|\; x \in \Omega \setminus \{0\} \right\}$$

and define

$$\widehat{\Delta} := -\frac{(B^\dagger Q\widehat{x})(B^\dagger Q\widehat{x})^H}{\widehat{x}^H Q\widehat{x}},$$

so that we have $\widehat{\Delta} B^H \widehat{x} = -B^\dagger Q\widehat{x}$. Equality holds in (6.3) if we show that $(Q + B\widehat{\Delta}B^H) \geq 0$, because this will imply that $\widehat{\Delta} \in \mathcal{S}_d(Q,B)$. Since $B\widehat{\Delta}B^H$ is of rank one, it has at most one negative eigenvalue. Furthermore, we have

$$(Q + B\widehat{\Delta}B^H)\widehat{x} = Q\widehat{x} - BB^\dagger Q\widehat{x} = Q\widehat{x} - Q\widehat{x} = 0,$$

and thus, $(Q + B\widehat{\Delta}B^H) \geq 0$ by Lemma 4.4. Therefore we have equality in (6.3), i.e.,

$$d^{\mathcal{S}_d}(Q;B) = \inf\left\{ \frac{\|B^\dagger Qx\|^2}{x^H Qx} \;\middle|\; x \in \Omega \setminus \{0\} \right\} = \inf\left\{ \frac{\|B^\dagger QU\alpha\|^2}{\alpha^H U^H QU\alpha} \;\middle|\; \alpha \in \mathbb{C}^r \setminus \{0\} \right\}, \tag{6.5}$$

where the columns of $U \in \mathbb{C}^{n,r}$ form an orthonormal basis for Ω. Let L be the unique Cholesky factor of $U^H QU > 0$, then inserting $y = L^H\alpha$ in (6.5), we get

$$d^{\mathcal{S}_d}(Q;B) = \inf\left\{ \frac{\|B^\dagger QUL^{-H}y\|^2}{\|y\|^2} \;\middle|\; y \in \mathbb{C}^r \setminus \{0\} \right\} = \left(\sigma_{\min}(B^\dagger QUL^{-H})\right)^2.$$

which proves the first part of the assertion.

For the second part, again by definition, we have

$$d^{\mathcal{S}_i}(Q;B) := \inf\left\{ \|\Delta\| \;\middle|\; \Delta \in \mathcal{S}_i(Q,B),\ \det(Q + B\Delta B^H) = 0 \right\}.$$

Following the steps of the first part, we have

$$\begin{aligned} d^{\mathcal{S}_i}(Q;B) &\geq \inf\left\{ \|\Delta\| \;\middle|\; \Delta \in \mathrm{Herm}(r),\ x \in \mathbb{C}^n \setminus \{0\},\ \Delta B^H x = -B^\dagger Qx,\ BB^\dagger Qx = Qx \right\} \\ &= \inf\left\{ \frac{\|B^\dagger Qx\|}{\|B^H x\|} \;\middle|\; x \in \Omega \setminus \{0\} \right\}, \end{aligned} \tag{6.6}$$

where the last equality follows by using the minimal spectral norm mappings of Theorem 2.1. Let $\widetilde{x} \in \Omega \setminus \{0\}$ be such that

$$\frac{\|B^\dagger Q\widetilde{x}\|}{\|B^H\widetilde{x}\|} = \inf\left\{ \frac{\|B^\dagger Qx\|}{\|B^H x\|} \;\middle|\; 0 \neq x \in \Omega \right\}$$

(again the infimum is a minimum and thus attained) and let $\widetilde{\Delta} \in \mathrm{Herm}(r)$ be the corresponding minimal norm mapping such that $\|\widetilde{\Delta}\| := \frac{\|B^\dagger Q\widetilde{x}\|}{\|B^H\widetilde{x}\|}$ and $\widetilde{\Delta}B^H\widetilde{x} = -B^\dagger Q\widetilde{x}$. Then we have equality in (6.6) if we show that $(Q + B\widetilde{\Delta}B^H) \geq 0$, because this will imply that $\widetilde{\Delta} \in \mathcal{S}_i(Q,B)$. But this follows from Lemma 4.4, because $\widetilde{\Delta}$ is either a matrix of rank one or an indefinite matrix of rank two. Thus,

$$d^{\mathcal{S}_i}(Q;B) = \inf\left\{ \frac{\|B^\dagger Qx\|}{\|B^H x\|} \;\middle|\; x \in \Omega \setminus \{0\} \right\} = \inf\left\{ \frac{\|B^\dagger QU\alpha\|}{\|B^H U\alpha\|} \;\middle|\; \alpha \in \mathbb{C}^r \setminus \{0\} \right\}, \tag{6.7}$$

where columns of $U \in \mathbb{C}^{n,r}$ form an orthonormal basis for Ω. By Remark 6.1 the matrix $B^H U$ is invertible. Thus, let $\widetilde{L}$ be the Cholesky factor of $U^H B B^H U$. By inserting $y = \widetilde{L}^H \alpha$ in (6.7), we obtain

$$d^{\mathcal{S}_i}(Q;B) = \inf \left\{ \frac{\|B^\dagger Q U \widetilde{L}^{-H} y\|}{\|y\|} \;\middle|\; y \in \mathbb{C}^k \setminus \{0\} \right\} = \sigma_{\min}(B^\dagger Q U \widetilde{L}^{-H}).$$

It remains to show that $d(Q;B,B^H) = d^{\mathcal{S}_i}(Q;B)$. By definition, we have

$$\begin{aligned} d(Q;B,B^H) &= \inf \left\{ \|\Delta\| \;\middle|\; \Delta \in \mathbb{C}^{r,r},\ \det(Q + B\Delta B^H) = 0 \right\} \\ &= \inf \left\{ \|\Delta\| \;\middle|\; \Delta \in \mathbb{C}^{r,r},\ x \in \mathbb{C}^n \setminus \{0\},\ (Q + B\Delta B^H)x = 0 \right\} \\ &= \inf \left\{ \|\Delta\| \;\middle|\; \Delta \in \mathbb{C}^{r,r}, x \in \Omega \setminus \{0\},\ \Delta B^H x = -B^\dagger Q x \right\}, \end{aligned} \tag{6.8}$$

where the last equality holds due to Lemma 2.8. Note that by [39], for any $x,\, y \in \mathbb{C}^n$, $x \neq 0$ we have

$$\inf \left\{ \|\Delta\| \big|\ \Delta x = y \right\} = \frac{\|y\|}{\|x\|}.$$

Using this in (6.8), we obtain

$$d(Q;B,B^H) = \inf \left\{ \frac{\|B^\dagger Q x\|}{\|B^H x\|} \;\middle|\; x \in \Omega \setminus \{0\} \right\} = d^{\mathcal{S}_i}(Q;B),$$

where the last equality follows from (6.7). Therefore we have

$$d(Q;B,B^H) = d^{\mathcal{S}_i}(Q;B) = \sigma_{\min}(B^\dagger Q U \widetilde{L}^{-H}). \qquad \square$$

After characterizing the distances to singularity, in the next subsections we characterize the stability radii.

6.2. The stability radius $r^{\mathcal{S}_i}(Q;B)$. For the characterization of $r^{\mathcal{S}_i}(Q;B)$ we have the following theorem.

THEOREM 6.3. *Consider an asymptotically stable DH system of the form* (1.3)*, let $B \in \mathbb{C}^{n,r}$ have full rank r, and let $d^{\mathcal{S}_i}(Q;B)$ be as in Theorem* 6.2*. If $R > 0$, then*

$$r^{\mathcal{S}_i}(Q;B) = d^{\mathcal{S}_i}(Q;B).$$

Furthermore, if $R \geq 0$ is singular, let $\mathcal{R}$ be the set of all $\omega \in \mathbb{R} \setminus \{0\}$ such that the intersection of the kernel Ω_R of R and the kernel Ω_ω of $(I - BB^\dagger)(i\omega I_n - JQ)$ is not the zero space, and for each $\omega \in \mathcal{R}$ let the columns of V_ω form an orthonormal basis for $\Omega_R \cap \Omega_\omega$. Then $B^H J V_\omega$ has full column rank for all $\omega \in \mathcal{R}$. If

$$\inf_{\omega \in \mathcal{R}} \sigma_{\min}\big(B^\dagger (i\omega I_n - QJ) V_\omega L_\omega^{-H}\big),$$

is attained for some $\widehat{\omega} \in \mathcal{R}$, where L_ω is the Cholesky factor of $V_\omega^H J^H B B^H J V_\omega$, then

$$r^{\mathcal{S}_i}(Q;B) = \min \left\{ d^{\mathcal{S}_i}(Q;B),\, \inf_{\omega \in \mathcal{R}} \sigma_{\min}\big(B^\dagger (i\omega I_n - QJ) V_\omega L_\omega^{-H}\big) \right\}. \tag{6.9}$$

Proof. If $R > 0$, then by definition

$$r^{\mathcal{S}_i}(Q;B) = \inf \left\{ \|\Delta\| \;\middle|\; \Delta \in \mathcal{S}_i(Q,B),\, \Lambda\big((J-R)(Q + B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset \right\}.$$

Observe that in this case zero is the only choice to move an eigenvalue of $(J-R)Q$ to the imaginary axis, and the way to achieve this is to make Q singular, because for any $W \in \mathrm{Herm}(r)$ if $Q + W$

is nonsingular, then $R(Q+W)x \neq 0$ for any $x \in \mathbb{C}^n \setminus \{0\}$. Thus by Lemma 3.1, $(J-R)(Q+W)$ cannot have any nonzero eigenvalues on the imaginary axis. With this observation we obtain

$$\begin{aligned} r^{\mathcal{S}_i}(Q;B) &= \inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathcal{S}_i(Q,B),\ \Lambda\big((J-R)(Q+B\Delta B^H)\big) \cap \{0\} \neq \emptyset\Big\} \\ &= \inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathcal{S}_i(Q,B),\ \det(Q+B\Delta B^H) = 0\Big\}, \end{aligned} \tag{6.10}$$

where the last equality holds, because $J-R$ is invertible, since $(J-R)Q$ is nonsingular, because it has no eigenvalues on the imaginary axis. Therefore, from (6.10) we have

$$r^{\mathcal{S}_i}(Q;B) = d^{\mathcal{S}_i}(Q;B).$$

Next, we consider the case that $R \geq 0$ is singular. By definition of $\mathcal{S}_i(Q,B)$ we have

$$\mathcal{S}_i(Q,B) = \big\{\Delta \in \mathrm{Herm}(r) \,\big|\, (Q+B\Delta B^H) \geq 0\big\},$$

and $(Q+B\Delta B^H) \geq 0$ if and only if either $(Q+B\Delta B^H) \geq 0$ and $\det(Q+B\Delta B^H) = 0$, or $(Q+B\Delta B^H) > 0$. Thus, we can write

$$\begin{aligned} r^{\mathcal{S}_i}(Q;B) = \min\Big\{ &\inf\big\{\|\Delta\| \,\big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) \geq 0, \det(Q+B\Delta B^H) = 0, \\ &\qquad \Lambda\big((J-R)(Q+B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\big\}, \\ &\inf\big\{\|\Delta\| \,\big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) > 0, \\ &\qquad \Lambda\big((J-R)(Q+B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\big\}\Big\}. \end{aligned} \tag{6.11}$$

For the first of the two infima in (6.11), we have

$$\begin{aligned} &\inf\big\{\|\Delta\| \,\big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) \geq 0, \det(Q+B\Delta B^H) = 0, \\ &\qquad \Lambda\big((J-R)(Q+B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\big\} \\ = \ &\inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) \geq 0, \det(Q+B\Delta B^H) = 0\Big\} \\ = \ &d^{\mathcal{S}_i}(Q;B), \end{aligned}$$

because $\Lambda\big((J-R)(Q+B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset$ is automatically satisfied if $\det(Q+B\Delta B^H) = 0$. For the second of the two infima in (6.11), using Lemma 3.1 we obtain that

$$\begin{aligned} &\inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) > 0,\ \Lambda\big((J-R)(Q+B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\Big\} \\ &= \inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) > 0,\ \omega \in \mathbb{R} \setminus \{0\},\ x \in \mathbb{C}^n \setminus \{0\}, \\ &\qquad J(Q+B\Delta B^H)x = i\omega x \ \text{ and } \ R(Q+B\Delta B^H)x = 0\Big\} \\ &= \inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) > 0, \omega \in \mathbb{R} \setminus \{0\},\ y \in \mathbb{C}^n \setminus \{0\}, \\ &\qquad (Q+B\Delta B^H)Jy = i\omega y \ \text{ and } \ Ry = 0\Big\} \\ &= \inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) > 0, \omega \in \mathbb{R} \setminus \{0\},\ y \in \mathbb{C}^n \setminus \{0\}, \\ &\qquad B\Delta B^H Jy = (i\omega I_n - QJ)y \ \text{ and } \ Ry = 0\Big\} \\ &= \inf_{\omega \in \mathcal{R}} \inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q+B\Delta B^H) > 0,\ y \in \mathbb{C}^n \setminus \{0\}, \\ &\qquad B\Delta B^H Jy = (i\omega I_n - QJ)y \ \text{ and } \ Ry = 0\Big\} \\ &\geq \inf_{\omega \in \mathcal{R}} \inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ y \in \Omega_R \setminus \{0\},\ B\Delta B^H Jy = (i\omega I_n - QJ)y\Big\}, \end{aligned} \tag{6.12}$$

where in the second equality, we replaced $(Q + B\Delta B^H)x$ with y using the nonsingularity of $Q + B\Delta B^H$, and for the last step we used $\{\Delta \in \mathrm{Herm}(r) \,|\, (Q + B\Delta B^H) > 0\} \subseteq \mathrm{Herm}(r)$. By Lemma 2.7, for any $y \in \mathbb{C}^n \setminus \{0\}$ there exist $\Delta \in \mathrm{Herm}(r)$ such that $B\Delta B^H Jy = (i\omega I_n - QJ)y$ if and only if $BB^\dagger(i\omega I_n - QJ)y = (i\omega I_n - QJ)y$. Indeed, note that the condition

$$y^H J^H BB^\dagger(i\omega I_n - QJ)y \in \mathbb{R} \tag{6.13}$$

in Lemma 2.7 is automatically satisfied due to the fact that $BB^\dagger(i\omega I_n - QJ)y = (i\omega I_n - QJ)y$ and that $J^H(i\omega I_n - QJ)$ is Hermitian. Using this in (6.12) and applying Lemma 2.8, we get

$$\begin{aligned}
&\inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q + B\Delta B^H) > 0,\ \Lambda\big((J-R)(Q + B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\Big\} \\
&\quad\geq \inf_{\omega\in\mathcal{R}} \inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ \ y \in \Omega_R \setminus \{0\},\ \Delta B^H Jy = B^\dagger(i\omega I_n - QJ)y, \\
&\qquad\qquad BB^\dagger(i\omega I_n - QJ)y = (i\omega I_n - QJ)y\Big\} \qquad (6.14)\\
&\quad= \inf_{\omega\in\mathcal{R}} \inf\left\{\frac{\|B^\dagger(i\omega I_n - QJ)y\|}{\|B^H Jy\|} \,\middle|\, y \in (\Omega_R \cap \Omega_\omega) \setminus \{0\}\right\} \\
&\quad= \inf_{\omega\in\mathcal{R}} \inf\left\{\frac{\|B^\dagger(i\omega I_n - QJ)V_\omega\alpha\|}{\|B^H JV_\omega\alpha\|} \,\middle|\, \alpha \in \mathbb{C}^{k_\omega} \setminus \{0\}\right\}, \qquad (6.15)
\end{aligned}$$

where for $\omega \in \mathcal{R}$, i.e., $\omega \neq 0$ and $\Omega_R \cap \Omega_\omega \neq \{0\}$, the columns of $V_\omega \in \mathbb{C}^{n,k_\omega}$ form an orthonormal basis of $\Omega_R \cap \Omega_\omega$, and we have set $y = V_\omega\alpha$ for some $\alpha \in \mathbb{C}^{k_\omega} \setminus \{0\}$. Note that for the second last equality we have used the minimal spectral norm mappings from Theorem 2.1.

Furthermore, we have $B^H Jy \neq 0$ for all $y \in (\Omega_R \cap \Omega_\omega) \setminus \{0\}$ and therefore also $B^H JV_\omega\alpha \neq 0$ for all $\alpha \in \mathbb{C}^{k_\omega} \setminus \{0\}$, and thus $B^H JV_\omega$ has full rank. Indeed, if $B^H Jy = 0$ for some $y \in \Omega_R \cap \Omega_\omega$ then by (6.14) we have $B^\dagger(i\omega I_n - QJ)y = 0$ and thus also $BB^\dagger(i\omega I_n - QJ)y = 0$. Since $y \in \Omega_\omega$, i.e., y is in the kernel of $(I - BB^\dagger)(i\omega I_n - JQ)$, it follows that $0 = (i\omega I_n - QJ)y = \big(i\omega I_n - Q(J-R)\big)y$, which is a contradiction to the fact that $(J-R)Q$ has no eigenvalues on the imaginary axis. Therefore, $V_\omega^H J^H BB^H JV_\omega$ has a unique Cholesky factor L_ω, and setting $\beta = L_\omega^H\alpha$ in (6.15), we obtain

$$\begin{aligned}
&\inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q + B\Delta B^H) > 0,\ \Lambda\big((J-R)(Q + B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\Big\} \\
&\quad\geq \inf_{\omega\in\mathcal{R}} \inf\left\{\frac{\|B^\dagger(i\omega I_n - QJ)V_\omega L_\omega^{-H}\beta\|}{\|\beta\|} \,\middle|\, \beta \in \mathbb{C}^k \setminus \{0\}\right\} \\
&\quad= \inf_{\omega\in\mathcal{R}} \sigma_{\min}\big(B^\dagger(i\omega I_n - QJ)V_\omega L_\omega^{-H}\big). \qquad (6.16)
\end{aligned}$$

By assumption, the infimum in (6.16) is attained for some $\widehat{\omega} \in \mathcal{R}$. Thus, writing $\widehat{V} = V_{\widehat{\omega}}$ and $\widehat{L} = L_{\widehat{\omega}}$ we have that

$$\widehat{\sigma} := \sigma_{\min}\left(B^\dagger(i\widehat{\omega} I_n - QJ)\widehat{V}\widehat{L}^{-H}\right) = \inf_{\omega\in\mathcal{R}} \sigma_{\min}\left(B^\dagger(i\omega I_n - QJ)V_\omega L_\omega^{-H}\right).$$

Let u be a right singular vector of $B^\dagger(i\widehat{\omega} I_n - QJ)\widehat{V}\widehat{L}^{-H}$ corresponding to the minimal singular value $\widehat{\sigma}$ and set $z := \widehat{V}\widehat{L}^{-H}u$. Furthermore, applying Theorem 2.1 let $\widehat{\Delta} \in \mathrm{Herm}(r)$ be such that $\widehat{\Delta}B^H Jz = B^\dagger(i\widehat{\omega} I_n - QJ)z$ and $\|\widehat{\Delta}\| = \sigma_{\min}\big(B^\dagger(i\widehat{\omega} I_n - QJ)\widehat{V}\widehat{L}^{-H}\big)$. Inserting this in (6.16) we have

$$\inf\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q + B\Delta B^H) > 0,\ \Lambda\big((J-R)(Q + B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\Big\} \geq \|\widehat{\Delta}\|.$$

Then to show (6.9), we consider two cases depending on further properties of the matrix $\widehat{\Delta}$.

If $(Q + B\widehat{\Delta}B^H) > 0$ then we have equality in (6.16) and therefore

$$r^{\mathcal{S}_i}(Q; B) = \|\widehat{\Delta}\| = \sigma_{\min}\left(B^\dagger(i\widehat{\omega} I_n - QJ)\widehat{V}\widehat{L}^{-H}\right).$$

If $(Q + B\widehat{\Delta}B^H)$ has some nonpositive eigenvalues, then

$$\begin{aligned}\inf &\Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r),\ (Q + B\Delta B^H) > 0,\ \Lambda\big((J-R)(Q+B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\Big\}\\ &\geq \inf \Big\{\|\Delta\| \,\Big|\, \Delta \in \mathrm{Herm}(r), (Q + B\Delta B^H) \geq 0, \det(Q + B\Delta B^H) = 0,\\ &\qquad\qquad \Lambda\big((J-R)(Q+B\Delta B^H)\big) \cap i\mathbb{R} \neq \emptyset\Big\}.\end{aligned}$$

This follows by using the fact that eigenvalues depend continuously on the entries of a matrix and therefore there exist $t \in (0,1]$ such that $(Q + tB\widehat{\Delta}B^H) \geq 0$ and $(Q + tB\widehat{\Delta}B^H)$ is singular. But this implies that $\|\widehat{\Delta}\| \geq \|t\widehat{\Delta}\| \geq d^{\mathcal{S}_i}(Q;B)$. Thus, we have $r^{\mathcal{S}_i}(Q;B) = d^{\mathcal{S}_i}(Q;B)$ in that case which finishes the proof. ∎

6.3. The stability radius $r^{\mathcal{S}_d}(Q;B)$. As final case we obtain the following formula for the structured restricted stability radius $r^{\mathcal{S}_d}(Q;B)$.

THEOREM 6.4. *Consider an asymptotically stable DH system of the form* (1.3)*, let $B \in \mathbb{C}^{n,r}$ have full rank r, and let $d^{\mathcal{S}_d}(Q;B)$ be as in Theorem* 6.2*.*

If $R > 0$, then

$$r^{\mathcal{S}_d}(Q;B) = d^{\mathcal{S}_d}(Q;B).$$

If, however, $R \geq 0$ is singular, then let $\widetilde{\mathcal{R}}$ be the set of all $\omega \in \widetilde{\mathcal{R}}\backslash\{0\}$ such that the intersection of the kernel Ω_R of R and the kernel Ω_ω of $(I - BB^\dagger)(i\omega I_n - QJ)$ is not the zero space. Assume that for every $\omega \in \widetilde{\mathcal{R}}$ we have $y^H J^H (iwI_n - QJ)y < 0$ for all vectors $y \in (\Omega_R \cup \Omega_\omega) \setminus \{0\}$. Moreover, for each $\omega \in \widetilde{\mathcal{R}}$ let the columns of V_ω form an orthonormal basis for $\Omega_R \cap \Omega_\omega$. If

$$\inf_{\omega \in \widetilde{\mathcal{R}}} \left(\sigma_{\min}\big(B^\dagger(i\omega I_n - QJ)V_\omega L_\omega^{-H}\big)\right)^2,$$

is attained for some $\widehat{\omega} \in \widetilde{\mathcal{R}}$, where L_ω is the Cholesky factor of $V_\omega^H J^H (QJ - i\omega I_n)V_\omega$, then

$$r^{\mathcal{S}_d}(Q;B) = \min\left\{d^{\mathcal{S}_d}(Q;B), \inf_{\omega \in \widetilde{\mathcal{R}}} \left(\sigma_{\min}\big(B^\dagger(i\omega I_n - QJ)V_\omega L_\omega^{-H}\big)\right)^2\right\}.$$

Proof. The proof is analogous to the proof of Theorem 6.3, by using Theorem 2.3 instead of Theorem 2.1. The difference is that condition (6.13) becomes $y^H J^H (i\omega I_n - QJ)y < 0$ which is no longer automatically satisfied and thus, leads to a further assumption in the definition of the set $\widetilde{\mathcal{R}}$. ∎

7. Numerical experiments. In this section, we present some numerical experiments to illustrate the results of this paper and to show that the stability radii are indeed larger when structure-preserving perturbations are considered instead of general ones, and this difference can be significant. To compute the distances, in all cases we used the function `fminsearch` in Matlab Version No. 7.8.0 (R2009a) to solve the associated optimization problems, except for the computation of $r^{\mathrm{Herm}}(R;B)$, where we first used the software package CVX [12] for the inner supremum and then the function `fminsearch` for the outer infimum.

In all our numerical experiments, we chose random matrices $J,\ R,\ Q,\ B \in \mathbb{C}^{n,n}$ for different values of $n \leq 9$ with $J^H = -J$, $R^H = R \geq 0$ and $Q^H = Q > 0$ and B of full rank, such that all restricted stability radii were finite.

The computed stability radii $r^{\mathcal{S}_d}(R;B)$ and $r^{\mathcal{S}_i}(R;B)$ with respect to structured restricted perturbations to R are as obtained in Theorem 4.2 and 4.5, respectively, $r^{\mathcal{S}}(J;B)$ with respect to structured restricted perturbations to J is as obtained in Theorem 5.1, and $r^{\mathcal{S}_i}(Q;B)$ and $r^{\mathcal{S}_d}(Q;B)$ with respect to structured restricted perturbations to Q are as obtained in Theorem 6.3 and 6.4, respectively. The structured distances to singularity $d^{\mathcal{S}_i}(Q;B)$, $d^{\mathcal{S}_d}(Q;B)$ are as obtained in Theorem 6.2.

size n	$r(R;B,B^H)$	$r^{\mathrm{Herm}}(R;B)$	$r^{\mathcal{S}_i}(R;B)$	$r^{\mathcal{S}_d}(R;B)$
3	0.2985	0.2985	2.7154	11.3693
4	0.0826	0.4704	1.1527	1.9571
5	3.3358	3.4302	4.0866	6.0652
6	0.5919	0.6020	2.6858	6.4674
7	0.0675	0.1048	0.9606	6.0310
8	1.5933	1.6193	14.7738	20.5105
9	1.4520	1.4524	16.4666	29.0445

TABLE 7.1
Unstructured and various structured stability radii when perturbing only R.

The unstructured and various structured stability radii with respect to structure-preserving restricted perturbations to R are depicted in Table 7.1. As mentioned in Section 4 earlier, Conjecture 4.6 holds in all our numerical experiments, so the lower bounds in Theorem 4.5 indeed gave the values of $r^{\mathcal{S}_i}(R;B)$.

In Table 7.2, we compare various stability radii. The results illustrate that stability radii with respect to perturbations to only Q are much smaller than the other stability radii. In some cases the stability radius $r^{\mathcal{S}_i}(Q;B)$ is even smaller than the stability radius $r(J;B,B^H)$. Table 7.2 also exhibits the difference between the stability radii $r^{\mathcal{S}}(J;B)$ and $r^{\mathcal{S}_i}(R;B)$.

size n	$r(Q;B,B^H)$	$r^{\mathcal{S}_i}(Q;B)$ $= d^{\mathcal{S}_i}(Q;B)$	$r(J;B,B^H)$ $= r(R;B,B^H)$	$r^{\mathcal{S}}(J;B)$	$r^{\mathcal{S}_i}(R;B)$
3	0.1608	0.1751	0.3427	2.5123	0.9028
4	0.0101	0.0119	0.2054	2.2756	1.6206
5	0.0485	0.1191	0.1391	2.2292	1.2025
6	0.0129	0.0132	0.1763	2.2127	5.1237
7	0.0736	0.0813	0.0747	1.8043	4.1788
8	0.0021	0.0025	0.1154	2.2579	1.6838
9	0.0074	0.0076	0.0913	3.0741	2.8177

TABLE 7.2
Comparison of various structured stability radii.

In our numerical experiments we found several instances of randomly generated matrices J, R, Q and B for which structured stability radii with respect to restricted perturbations to Q are significantly smaller than the structured restricted distances to singularity for Q. The values for a few such cases are reported in Table 7.3.

size n	$r(Q;B,B^H)$	$r^{\mathcal{S}_i}(Q;B)$	$d^{\mathcal{S}_i}(Q;B)$	$r^{\mathcal{S}_d}(Q;B)$	$d^{\mathcal{S}_d}(Q;B)$
3	0.0031	0.1310	0.2079	0.1988	0.2079
4	0.0776	0.2308	0.2571	0.2404	0.2571
5	0.0014	0.0649	0.1287	0.1104	0.1287
6	0.0262	0.0731	0.1068	0.0923	0.1068
7	0.0071	0.0276	0.0355	0.0290	0.0355
8	0.0022	0.0163	0.0257	0.0224	0.0257
9	0.0055	0.0263	0.0424	0.0371	0.0424

TABLE 7.3
Unstructured and various structured stability radii while perturbing only Q.

EXAMPLE 7.1. To illustrate the distance to instability in the case that R is perturbed to be indefinite, consider again Example 1.1 and the first order formulation (1.4) and write $R = R_1 + R_2$, where

$$R_1 := \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \; R_2 := \begin{bmatrix} 0 & \frac{1}{2}N \\ \frac{1}{2}N^H & 0 \end{bmatrix}.$$

Note that R_1 moves the spectrum of the pencil to the left half of the complex plane, so R_2 is solely responsible for making the system unstable. To illustrate this, we constructed a small scale complex example with

$$G = \begin{bmatrix} -0.3782i & -0.0681 + 1.3361i \\ 0.0681 + 1.336i & 1.9397i \end{bmatrix}, \; M = \begin{bmatrix} 0.4822 & -0.9341 - 0.3521i \\ -0.9341 + 0.3521i & 6.0622 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.7419 & 1.1228 + 0.4116i \\ 1.1228 - 0.4116i & 2.9265 \end{bmatrix}, \; D = \begin{bmatrix} 3.8881 & -4.1385 + 0.2047i \\ -4.1385 - 0.2047i & 7.2071 \end{bmatrix},$$

$$N = \begin{bmatrix} 0.7164 + 0.695i & -0.4012 - 0.2008i \\ -0.4281 + 0.1007i & 0.2716 + 0.0373i \end{bmatrix},$$

so that the matrix $(J-R)Q$ has an eigenvalue $0.0130i$ on the imaginary axis, but $(J-R_1)Q$ has no eigenvalues on the imaginary axis. By using the various stability radii formulae from Section 4.1, we obtain the following results for the restriction matrices $B = I_4$ and $C = I_4$:

$r(R_1; B, C)$	$r^{\text{Herm}}(R_1; B)$	$r^{\mathcal{S}_i}(R_1; B)$	$r^{\mathcal{S}_d}(R_1; B)$
0.0308	0.0308	3.0389	5.6149

This implies that the stability radius of the matrix triple (J, R, Q), while perturbing only R_1 with a symmetric indefinite perturbation of the form

$$\Delta_{R_1} = \begin{bmatrix} 0 & \Delta \\ \Delta^H & 0 \end{bmatrix}$$

is bounded below by 0.0308 and bounded above by $\|R_2\| = 0.4970$. The structured stability radii $r^{\mathcal{S}_i}(R_1; B)$ and $r^{\mathcal{S}_d}(R_1; B)$ preserving the semidefiniteness of R_1 are significantly larger than the upper bound 0.4970 of the stability radius with respect to symmetric indefinite perturbations of the form Δ_{R_1} to R_1. Again, by setting $B = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$ and $C = B^H$, where e_i is the i^{th} standard unit vector of $\mathbb{C}^4$, we can perturb only the damping matrix D. For this choice of B and C the corresponding stability radii are given as follows:

$r(R_1; B, C)$	$r^{\text{Herm}}(R_1; B)$	$r^{\mathcal{S}_i}(R_1; B)$	$r^{\mathcal{S}_d}(R_1; B)$
1.5041	1.5970	3.3346	5.6149

One can see that in this case the smallest Hermitian perturbation to R_1 is larger than the stability radius $r(R_1; B, C)$, thus it makes sense to study this distance separately.

8. Conclusions and outlook. We have derived explicit formulas for stability radii of dissipative Hamiltonian systems under structure-preserving perturbations, when the three factors J, R, Q are perturbed individually. The results show that the restriction to structure-preserving perturbations leads to much more robustness in the sense that much larger perturbations have to be applied to move an eigenvalue to the imaginary axis. The stability radii are in the form of minima that still require optimization techniques to compute the radii. To construct efficient optimization techniques is a topic of our current research. It should be noted that the situation may change if we perturb all three terms J, Q, R at the same time. It is an open problem to derive stability radii for this case. Also, if all coefficient matrices are real than it is natural to also restrict the perturbation matrices to be real. Another important topic is the extension of these results to descriptor systems such as models for electrical circuits and power grids, where as an extra difficulty we have eigenvalues at infinity and the perturbations are restricted even further. All these questions are currently under investigation or subject to future research.

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